On the stability and the applications of interacting particle systems

P. Del Moral

INRIA Bordeaux Sud Ouest

PDE/probability interactions - CIRM, April 2017

Synthesis $\subset$ **joint works** *with A.N. Bishop, J. Garnier, A. Guionnet, A. Kurtzmann, J. Jacod M. Ledoux, L. Miclo, J. Tugaut, L. Wu*

Nonlinear Markov chains

Mean field particle methods

Some stochastic perturbation tools

Feynman-Kac models ($\supset$ filtering, rare events,...)
- A brief review on genetic type models
- Non commutative models
- Lyapunov weighted dynamics model

Particle Gibbs-Glauber dynamics
- Many body Feynman-Kac (particle) models
- Duality formulae + Taylor expansions
- Branching proc. and Interacting MCMC

Continuous time models
- Feynman-Kac models - Diffusion Monte Carlo
- Gradient type diffusions
- Ensemble Kalman-Bucy filter
- Stability + Unif. prop. chaos

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Continuous time $\rightsquigarrow$ Discrete time on time mesh

- Drift + Diffusion : Euler type scheme
- Jumps + Exponential rates : Geometric/Bernoulli type schemes

$$\eta_n = \Phi_n(\eta_{n-1}) \qquad \approx_{dt\downarrow 0} \quad \text{nonlinear integro-diff equation}$$

Nonlinear Markov models

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$$\Updownarrow$$

Always $\exists$ $K_{n,\eta}$ Markov transitions s.t.

$$\Phi_n(\eta_{n-1}) = \eta_{n-1} K_{n,\eta_{n-1}} = \mathrm{Law}\left(\overline{X}_n\right) = \eta_n$$

i.e. :

$$\mathbb{P}(\overline{X}_n \in dx_n \mid \overline{X}_{n-1}) = K_{n,\eta_{n-1}}(\overline{X}_{n-1}, dx_n)$$

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Non unique Markov interpretations,...

McKean measures (historical process)

$$\mathbb{P}\left(\left(\overline{X}_0, \ldots, \overline{X}_n\right) \in d(x_0, \ldots, x_n)\right)$$

$$= \eta_0(dx_0)\ K_{1,\eta_0}(x_0, dx_1)\ \ldots\ K_{n,\eta_{n-1}}(x_{n-1}, dx_n)$$

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Mean field particle interpretation

Markov chain $\xi_n = (\xi_n^1, \xi_n^2, \ldots, \xi_n^N) \in S_n^N$

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim K_{n,\eta_{n-1}^N}(\xi_{n-1}^i, dx_n) \quad \text{with} \quad \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} := \eta_n^N \; \simeq_{N\uparrow\infty} \; \eta_n$$

Same models for the McKean measures

$$\boldsymbol{\xi_n^i} = (\xi_0^i, \ldots, \xi_n^i) \quad \text{mean field particle of} \quad \overline{\boldsymbol{X}}_{\boldsymbol{n}} := \left(\overline{X}_0, \ldots, \overline{X}_n\right)$$

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$\neq$ Interpretations:

Stochastic adaptive grid, stochastic linearization, microscopic particle model, Sequential Monte Carlo method,...

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General fluctuation theorem

local fluctuation random fields

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$\forall f$ **/Multivariate/Functional/Donsker theorems**
(dp+Guionnet [99],Miclo [00],Ledoux [00],Jacod[02],...,Wu [11,13])

$$\left(V_n^N\right)_{n\geq 0} \quad \longrightarrow_{N\to\infty} \quad \left(V_n\right)_{n\geq 0}$$

with independent centered Gaussian fields V_n s.t.

$$\mathbb{E}\left([V_n(f)]^2\right) = \eta_{n-1}\left[K_{n,\eta_{n-1}}[f - K_{n,\eta_{n-1}}(f)]^2\right]$$

"Intuitive picture" $\rightsquigarrow$ nonlinear sg :

$$\eta_n = \Phi_n(\eta_{n-1}) = \Phi_{p,n}(\eta_p) = \eta_n$$

Local fluctuations transport formulation :

$$\begin{array}{ccccccccc}
\eta_0 & \to & \eta_1 = \Phi_1(\eta_0) & \to & \eta_2 = \Phi_{0,2}(\eta_0) & \to & \cdots & \to & \Phi_{0,n}(\eta_0) \\
\Downarrow & & & & & & & & \\
\eta_0^N & \to & \Phi_1(\eta_0^N) & \to & \Phi_{0,2}(\eta_0^N) & \to & \cdots & \to & \Phi_{0,n}(\eta_0^N) \\
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Local fluctuation errors

$$\eta_n^N = \Phi_n(\eta_{n-1}^N) + \frac{1}{\sqrt{N}}\ V_n^N$$

Key decomposition formulae

For any "time-window"

$$\eta_n^N - \eta_n = \underbrace{\eta_n^N - \Phi_{n-p,n}(\eta_{n-p}^N)}_{\simeq\ e^{c_1\,p}/\sqrt{N}} + \underbrace{\Phi_{n-p,n}(\eta_{n-p}^N) - \Phi_{n-p,n}(\eta_{n-p})}_{\simeq\ e^{-c_2\,p}}$$

then $c = c_1 + c_2$

$$p = (2c)^{-1}\log N \Longrightarrow e^{c_1\,p}/\sqrt{N} = e^{-c_2\,p} = \left(\frac{1}{\sqrt{N}}\right)^{c_2/(c_1+c_2)}$$

More refined analysis

$$\eta_n^N - \eta_n \quad = \quad \sum_{q=0}^{n} \left[\Phi_{q,n}(\eta_q^N) - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))\right]$$

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More refined analysis

$$\begin{aligned}
\eta_n^N - \eta_n &= \sum_{q=0}^{n} [\Phi_{q,n}(\eta_q^N) - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))] \\
&= \sum_{q=0}^{n} \left[\Phi_{q,n}\left(\Phi_q(\eta_{q-1}^N) + \frac{1}{\sqrt{N}}\ V_n^N\right) - \Phi_{q,n}\left(\Phi_q(\eta_{q-1}^N)\right)\right]
\end{aligned}$$

Semigroup/Stability/Uniform propagation of chaos/. . .
dp+Guionnet CRAS [99], and IHP [00], Miclo [00], [02], [03], Rousset [05],. . .

Another tool $I_q^N :=$ q-multi-indexes with $\neq$ values

$$\eta_n^{(N,q)}(f) := m(\xi_n)^{\odot q} := \frac{1}{(N)_q} \sum_{c \in I_q^N} \delta_{\left(\xi_n^{c(1)},\ldots,\xi_n^{c(q)}\right)}$$

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$\exists$ combinatorial/transport/coalescent tree formulae between $m(\xi_n)^{\odot q}$ and $m(\xi_n)^{\otimes q}$ and

$$\|m(\xi_n)^{\otimes q} - m(\xi_n)^{\odot q}\|_{tv} \leq (q-1)^2/N$$

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$\Rightarrow$ Propagation of chaos = Bias

$$\mathbb{E}\left(f(\xi_n^1, \ldots, \xi_n^q) \mid \xi_{n-1}\right) \quad = \quad \mathbb{E}\left(\eta_n^{(N,q)}(f) \mid \xi_{n-1}\right) = \eta_{n-1}^{(N,q)} K_{m(\xi_{n-1})}^{\otimes q}(f)$$

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Non asymp. Taylor exp.+ Uniform propagation of chaos q-particles

$\subset$ dp Springer [04], dp+Peters `SIAM [06]`, dp+Patras+Rubenthaler AoAP `[06]`, dp+Miclo SAA[07] Chan+dp+Jasra `Arxiv [14]`

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Feynman-Kac measures

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with the historical process

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Mean field particle $\rightsquigarrow$ **Genetic/Branching process** $\xi_n = (\xi_n^i)_i$

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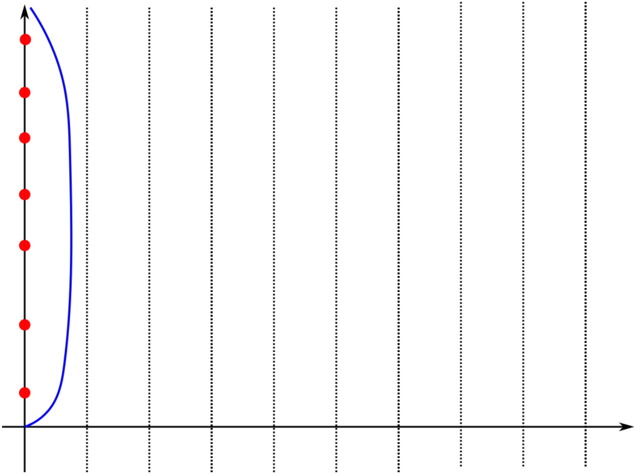
Graphical illustration : $\eta_n \simeq \eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i}$



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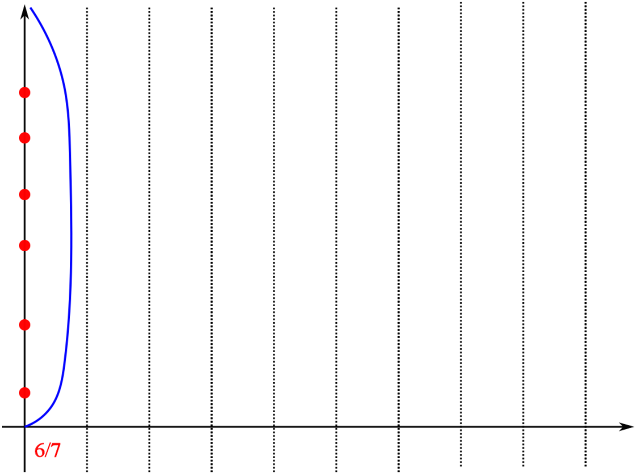

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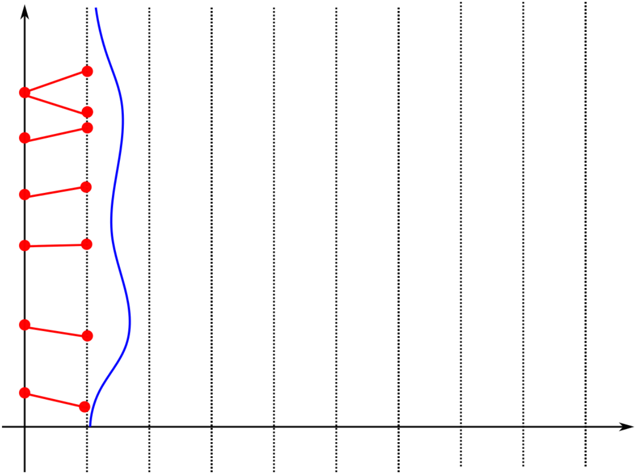

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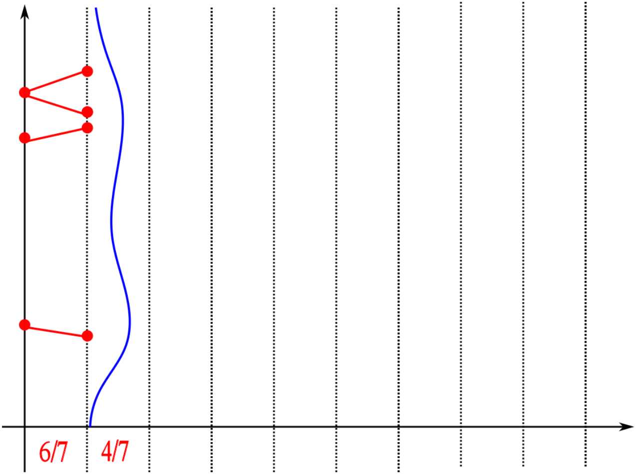

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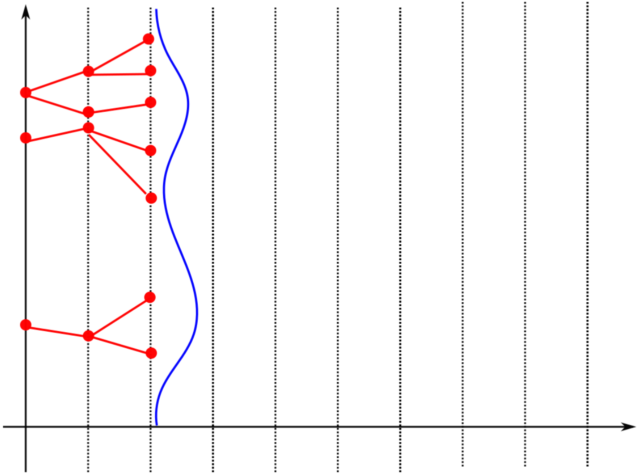

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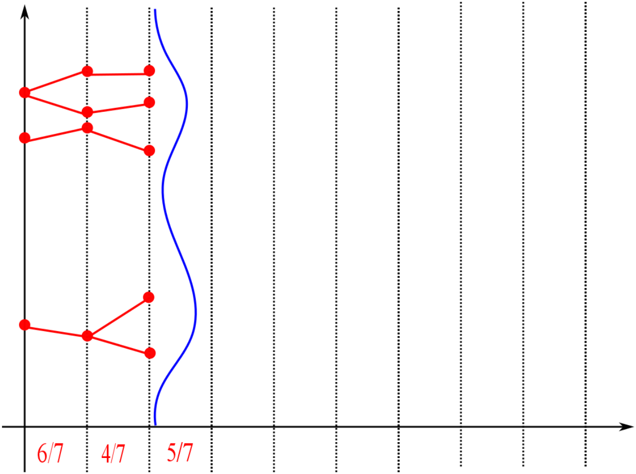

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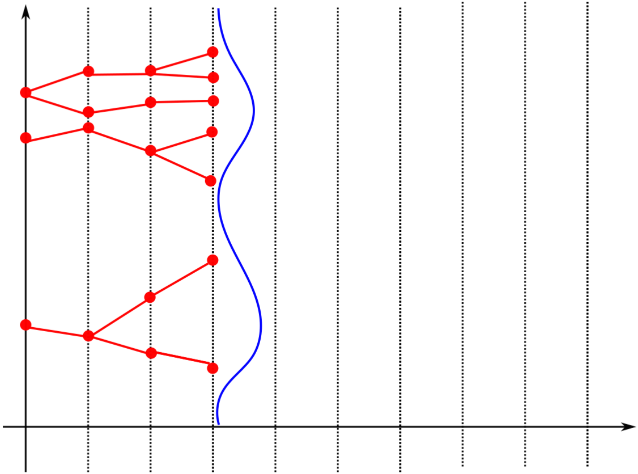

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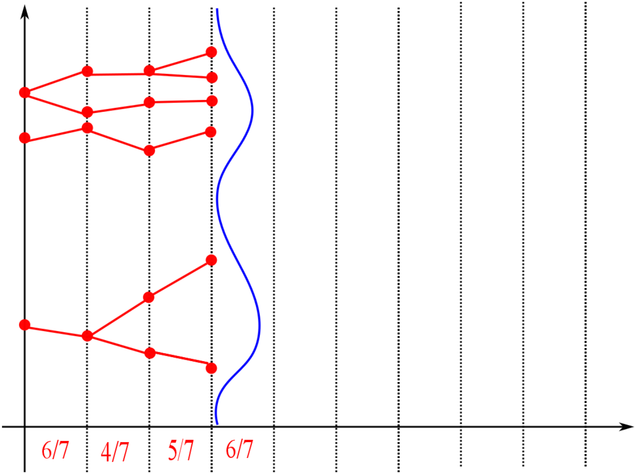

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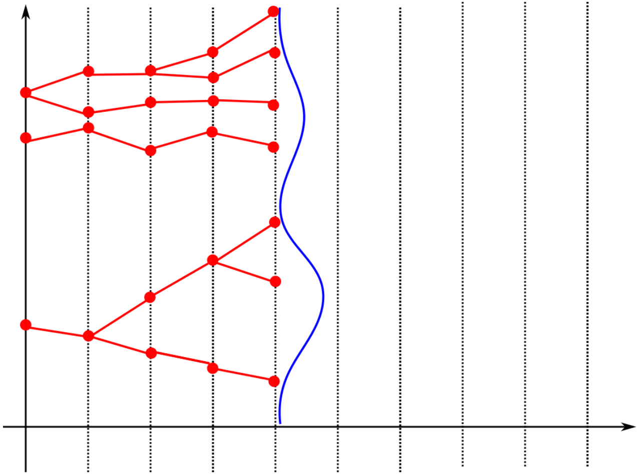

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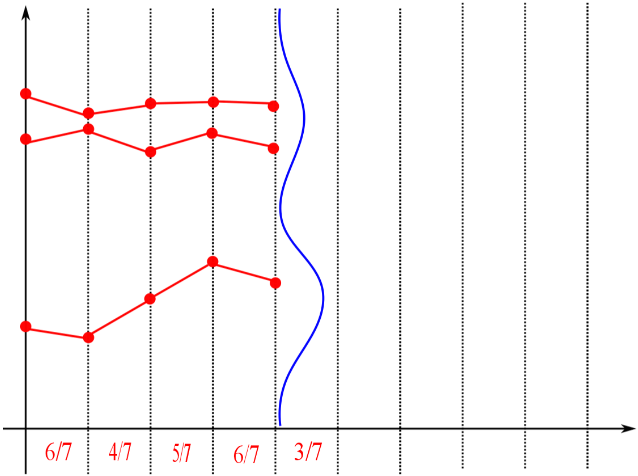

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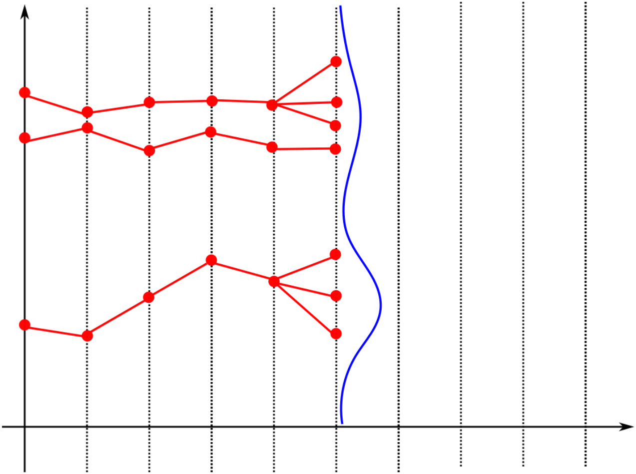

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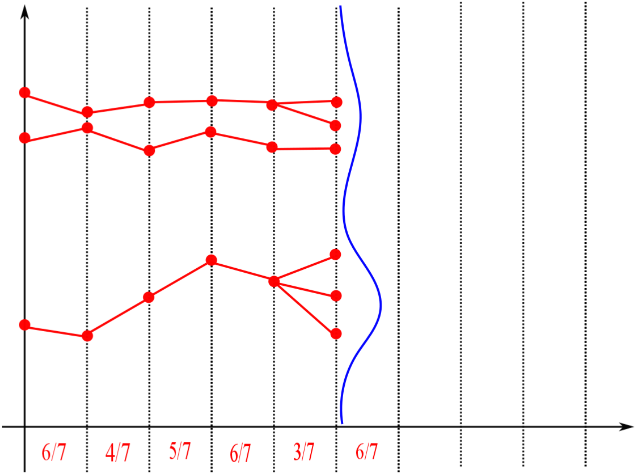

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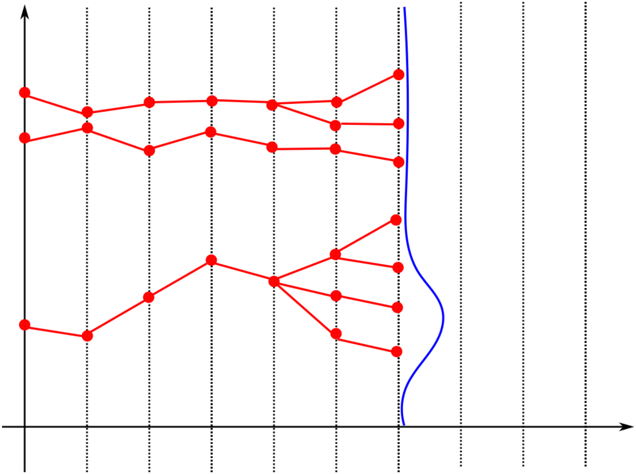

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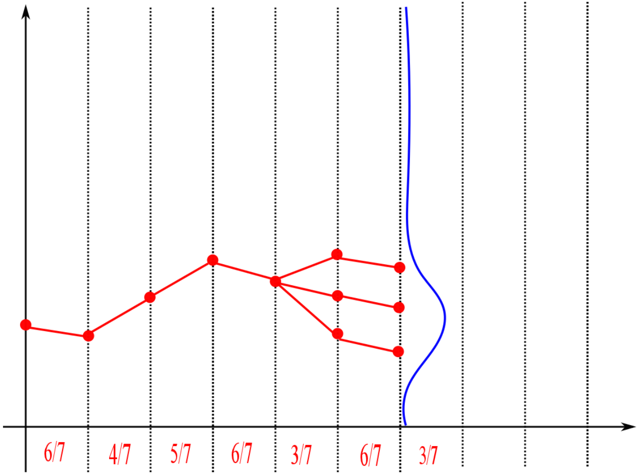

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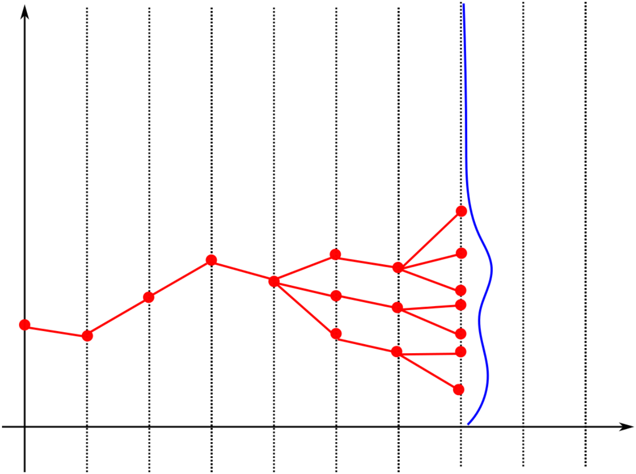

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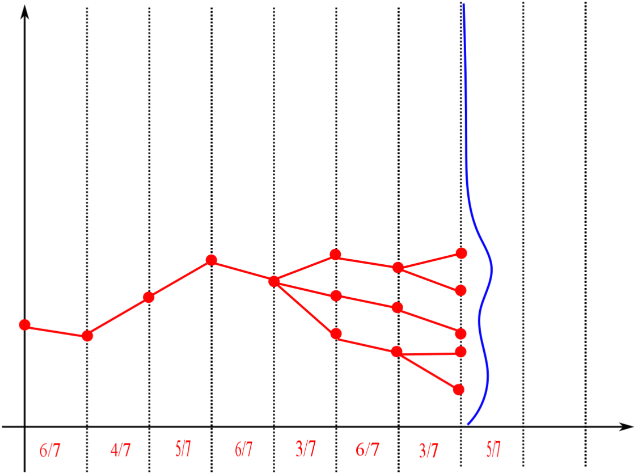

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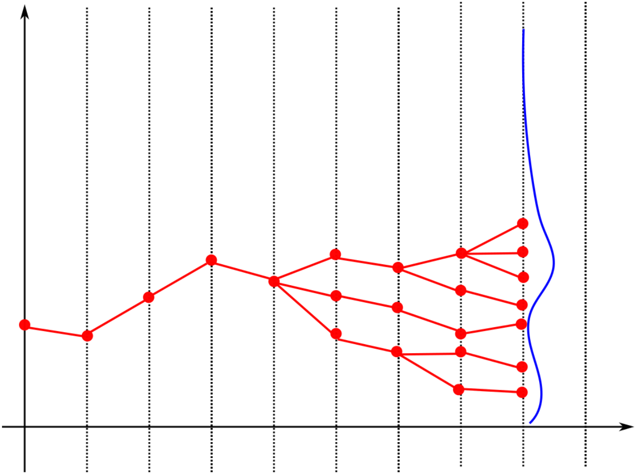

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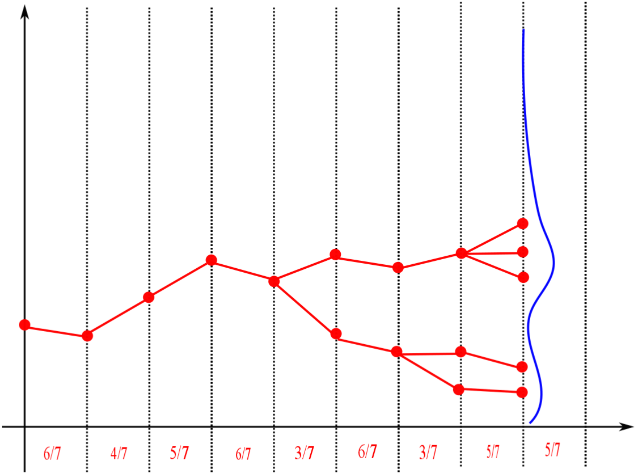

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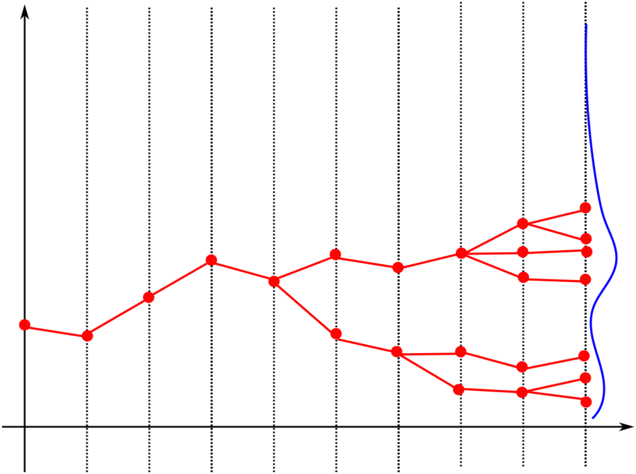

Equivalent particle algorithms

Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching-selection
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Accept-reject-recycle

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Many other lively buzzwords :
bootstrapping, spawning, cloning, pruning, replenish, splitting, enrichment, go with the winner, look-ahead, weighted dynamics, quantum teleportation,...

1947 $\leq$ Heuristic style algo. $\leq$ 1996 $\leq$ Particle Feynman-Kac models

Convergence analysis : Uniform propagation of chaos, concentration, stability, CLT, LDP, MDP, Empirical processes, ...

Feynman-Kac measures

Unnormalized measures

$$\gamma_n(f_n) = \eta_n(f_n) \times \prod_{0 \leq p < n} \eta_p(G_p)$$

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Backward Markov chain formula

$$G_{n-1}(x_{n-1}) M_n(x_{n-1}, dx_n) = H_n(x_{n-1}, x_n)\ \lambda_n(dx_n)$$

$$\Downarrow$$

$$\mathbb{Q}_n(d(x_0, \ldots, x_n)) = \eta_n(dx_n)\ \prod_{0 \leq q < n} \underbrace{\mathbb{M}_{q+1,\eta_q}(x_{q+1}, dx_q)}_{\propto\ \eta_q(dx_q)\ H_{q+1}(x_q, x_{q+1})}$$

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Application domains : Any conditional distributions, Bayesian prior/posterior, nonlinear filtering, Boltzmann-Gibbs measures, branching processes, Twisted importance sampling,...

Non commutative models

- $G_n(x_n) \in \mathbb{R}^{\mathbf{d}\times\mathbf{d}}$ s.t. $\forall u \in \mathbb{S}^{d-1} := \{\|u\| = 1\}$ we have $\|G_n(x)\ u\| > 0$
- $f_n(x_0, \ldots, x_n) \in \mathbb{R}^d$ and $\prod_{0\leq p\leq n} A_p = A_0 A_1 \ldots A_n$

$$\begin{aligned}\Gamma_n(f_n).u_0 \quad &:= \quad \mathbb{E}\left(f_n(X_0, \ldots, X_n) \prod_{0\leq p<n} G_p(X_p)\ .u_0 \right) \\ &= \quad \mathbb{E}\left(\boldsymbol{f_n(X_0, \ldots, X_n) \prod_{0\leq p<n} G_p(X_p)} \right)\end{aligned}$$

with

$$\boldsymbol{X_n} = (X_n, U_n) \in \left(E_n \times \mathbb{S}^{d-1}\right) \quad \text{and} \quad \boldsymbol{G_n(X_n)} = \|G_n(X_n).U_n\|$$

and the walk on the sphere model

$$U_{n+1} = \frac{G_n(X_n).U_n}{\|G_n(X_n).U_n\|}$$

$\bigtriangledown$ of $P_n(\varphi)(x) = \mathbb{E}_x(\varphi(Y_n))$ with $Y_{n+1} = F_n(Y_n, W_n)$

First variational equation

$$\begin{aligned} \mathrm{Jac}\,(Y_{n+1}) &= G_n(Y_n, W_n)\ \mathrm{Jac}\,(Y_n) && \text{with} \quad G_n^{i,j} = \partial_{x^j} F_n^i \\ &= \prod_{0\leq p\leq n} G_p(X_p) && \text{with} \quad X_n = (Y_n, W_n) \end{aligned}$$

$$\Rightarrow \bigtriangledown P_n(\varphi)(x)\,.\,u_0 \;=\; \mathbb{E}\left(\underbrace{\boldsymbol{f_n(X_n)}}_{=\bigtriangledown(\varphi)(Y_n)\,.\,U_n} \;\times \prod_{\boldsymbol{0\leq p<n}} \underbrace{\boldsymbol{G_p(X_p)}}_{=\|G_p(X_p)\,.\,U_p\|} \right)$$

$\Leftrightarrow$ FK model w.r.t. Y_n weighted with the directional Lyap. exp.

$$\prod_{\boldsymbol{0\leq p\leq n}} \boldsymbol{G_p(X_p)} = \|\mathrm{Jac}\,(Y_n)\ .\,u_0\| = \prod_{0\leq p\leq n} \frac{\|\mathrm{Jac}\,(Y_p)\ .\,u_0\|}{\|\mathrm{Jac}\,(Y_{p-1})\,.\,u_0\|}$$

Related Feynman-Kac model

$$\mathbf{X_n} = (X_n, X_{n+1}) \quad \text{and} \quad \mathbf{G_n(X_n)} = \|\mathrm{Jac}\,(X_{n+1})\|^{\alpha} / \|\mathrm{Jac}\,(X_n)\|^{\alpha}$$

$$\Downarrow$$

Feynman-Kac model = The Lyapunov weighted dynamics model

$$\mathbf{dQ_n} = \frac{1}{\mathcal{Z}_n} \; \|\mathrm{Jac}\,(X_n)\|^{\alpha} \; \mathbf{dP_n}$$

- $\alpha < 0 \Leftrightarrow \mathbf{dQ_n}$ favors low Lyapunov trajectories
- $\alpha > 0 \Leftrightarrow \mathbf{dQ_n}$ favors high Lyapunov trajectories

Hyper-references :

- T Laffargue, K.D. Nguyen Thu Lam, J. Kurchan, J. Tailleur LDP of Lyapunov exp. (2013)
- S. Tanese Nicola, J. Kurchan. Metastable states, transitions, basins and borders at finite temperature J. Stat. Phys. (2004).
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- J. Tailleur, J. Kurchan. Probing rare physical trajectories with Lyapunov weighted dynamics, Nature Physics (2007)
- $\subset$ Genealogical particle analysis of aare events (joint work with J. Garnier)AAP (2005).

Particle Feynman-Kac measures

Unbiased particle unnormalized measures (dp MPRF (96))

$$\gamma_n^N(f_n) := \eta_n^N(f_n) \times \prod_{0 \leq p < n} \eta_p^N(G_p)$$

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Particle backward model (dp+Doucet+Singh HAL (09), M2AN (10))

$$\mathbb{Q}_n^N(d(x_0,\ldots,x_n)) := \eta_n^N(dx_n) \prod_{0\leq q<n} \underbrace{\mathbb{M}_{q+1,\eta_q^N}(x_{q+1},dx_q)}_{\propto\ \eta_q^N(dx_q)\ H_{q+1}(x_q,x_{q+1})}$$

A first theorem

Theorem [dp+Kohn+Patras, Arxiv (14), CRAS (15)+IHP (16)]

$$\text{Law}\left(\mathbb{X}_n \mid (\xi_{k,k})_{0\leq k\leq n}\right)$$

$$= \text{Law of backward ancestral line } \mathbb{X}^\flat_n \text{ given all pop.}$$

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$$:= \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n, dx_{n-1})\cdots\mathbb{M}_{1,\eta_0^N}(x_1, dx_0)$$

with the occupation measures

$$\eta_k^N = m(\xi_{k,k}) := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi^i_{k,k}}$$

Nonlinear Markov chains

Mean field particle methods

Some stochastic perturbation tools

Feynman-Kac models ($\supset$ filtering, rare events,...)

Particle Gibbs-Glauber dynamics

- Many body Feynman-Kac (particle) models
- Duality formulae + Taylor expansions
- Branching proc. and Interacting MCMC

Continuous time models

Many body FK measures π_n

$$\mathcal{F}_n(\xi_n) := m(\xi_n)(f_n) \quad \text{and} \quad \mathcal{G}_n(\xi_n) := m(\xi_n)(G_n)$$

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$$\mathcal{Z}_n(\xi) := \prod_{0 \leq k < n} \mathcal{G}_k(\xi_k) \simeq \mathcal{Z}_n = \prod_{0 \leq k < n} \eta_k(G_k)$$

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Unbiasedness property in terms of many-body FK measures

$$\pi_n(\mathcal{F}_n) :\propto \mathbb{E}\left(\mathcal{F}_n(\xi_n)\ \mathcal{Z}_n(\xi)\right)$$

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(Note: the definition of π_n if for any symmetric functions $\mathcal{F}_n(\xi_n)$)

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Unbiasedness property in terms of ancestral lines

$$\mathbb{E}\left(f_n(\mathbb{X}_n)\ \mathcal{Z}_n(\xi)\right) = \mathbb{E}\left(f_n(\mathbb{X}_n^\flat)\ \mathcal{Z}_n(\xi)\right) = \mathbb{E}\left(f_n(X_n)\ Z_n(X)\right)$$

Second Extended Many body FK measures π_n

Definition: (Extended) Many-body FK measures

$$\pi_n(\mathcal{F}_n) :\propto \mathbb{E}\left(\mathcal{F}_n(\mathbb{X}_n, \boldsymbol{\xi_n})\ \mathcal{Z}_n(\xi)\right)$$

with historical process $\boldsymbol{\xi_n} = (\xi_0, \ldots, \xi_n)$ (ancestral lines evolution).

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Important observation:

$$\mathbb{X}_n \rightsquigarrow \overline{\mathbb{X}}_n \quad \text{is a Markov process on path space}$$

Duality formula

To sample the second PGD transition $\downarrow$.

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Theo.- Duality [dp+Kohn+Patras, `Arxiv (14)`, `CRAS (15)`+`IHP` (16)]

$$\forall \mathcal{F}_n \qquad \mathbb{E}\left(\mathcal{F}_n(\mathbb{X}_n, \boldsymbol{\xi_n})\ \mathcal{Z}_n(\xi)\right) = \mathbb{E}\left(\mathcal{F}_n(X_n, \boldsymbol{\xi_n^\star})\ Z_n(X)\right)$$

with the dual process:

$\boldsymbol{\xi_n^\star}$ defined as $\boldsymbol{\xi_n}$ but with a given frozen ancestral line X_n

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Corollary - Backward duality $\rightsquigarrow$ **Same duality** $\oplus$ **PGD when**

$$(\mathbb{X}_n, \boldsymbol{\xi_n}) \text{ is replaced by } (\mathbb{X}^{\flat}_n, \boldsymbol{\xi_{n,n}})$$

with the population historical process

$$\boldsymbol{\xi_{n,n}} = (\xi_{k,k})_{0 \le k \le n}$$

Taylor expansions

$\mathbb{X}_n \rightsquigarrow \overline{\mathbb{X}}_n$ Markov process on path space with η_n-reversible transition $\mathbb{K}_n$

$n =$ **FIXED** length/Duration of the ancestral trajectories
$N =$ **number of particles used in the genealogical trees**

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Theo. [dp+Kohn+Patras, Arxiv (14), CRAS (15)+IHP **(16)]**

$$\mathbb{K}_n(x, .) = \eta_n + \sum_{1 \leq k \leq l} \frac{1}{N^k} \; d^{(k)}\mathbb{K}_n(x, .) + \frac{1}{N^{l+1}} \; \partial^{l+1}\mathbb{K}_n(x, .)$$

$(\forall l \geq 1) \oplus$ Derivatives $\sim$ coalescent/colored trees.

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$(\forall l \geq 1) \oplus$ Derivatives $\sim$ coalescent/colored trees.

$$\left\| d^{(k)}\mathbb{K}_n \right\| \leq (cnk^2)^k \quad \text{and} \quad \left\| \partial^{(l+1)}\mathbb{K}_n \right\| \leq (cn(l+1)^2)^{l+1}$$

as soon as $N > cn(l+1)^2$, for some $c < \infty$.

A couple of direct corollaries

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Cor. 1: $\forall m \geq 1$ and $osc(f) \leq 1$

$$|\mathbb{K}_n^m(f)(x) - \eta_n(f)| \leq \beta\left(\mathbb{K}_n^m\right) \leq (cn/N)^m \quad \left(\longrightarrow_{\min(N,m)\to\infty} 0\right)$$

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Cor. 2: $\forall m \geq 1$, $\infty \geq p \geq 1$ and $\|f\| \leq 1$

$$\left| \|\mathbb{K}_n^m(f) - \eta_n(f)\|_{\mathbb{L}_p(\eta_n)} - N^{-m} \ \left\| \left[d^{(1)}\mathbb{K}_n\right]^m(f) \right\|_{\mathbb{L}_p(\eta_n)} \right| \leq (cn/N)^{m+1}$$

Some comments

- **Spatial branching processes** $\rightsquigarrow$ varying pop. size N_n

$$\text{First moments} \ \subset \ \text{FK-models} \ \Rightarrow \ \text{FK-particle} = \text{fixed pop.size}$$

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- Mutations $=$ MCMC with target $\eta_n \propto e^{-\beta_n V(x)}\ \lambda(dx)$ and branching rates/selection weights

$$e^{-(\beta_{n+1}-\beta_n)V(x)} \quad \beta_n \uparrow \quad \Longrightarrow \textbf{interacting MCMC/SMC/...}$$

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$$1_{A_{n+1}} \quad A_n \downarrow \quad \Longrightarrow \textbf{interacting MCMC/SMC/Multilevel splitting...}$$

Nonlinear Markov chains

Mean field particle methods

Some stochastic perturbation tools

Feynman-Kac models ($\supset$ filtering, rare events,...)

Particle Gibbs-Glauber dynamics

Continuous time models

- Feynman-Kac models - Diffusion Monte Carlo
- Gradient type diffusions
- Ensemble Kalman-Bucy filter
- Stability + Unif. prop. chaos

(weak sense) : infinitesimal generators $L_{t,\eta}$

$$\partial_t \eta_t(f) = \eta_t L_{t,\eta_t}(f) := \int_S \eta_t(dx)\ L_{t,\eta_t}(f)(x)$$

$L_{t,\eta}$ regular familly $\Rightarrow \exists\ \overline{X}_t$ ($\Leftrightarrow \exists$ McKean measure) $\rightsquigarrow$ Ex. jump-diff.

$$\begin{aligned} L_{t,\eta}(f)(x) &= a_t(x,\eta)\ f'(x) + \frac{1}{2}\ \sigma_t^2(x,\eta)\ f''(x) \\ &\quad + \lambda_t(x,\eta) \int \left(f(y) - f(x)\right)\ K_{t,\eta}(x,dy) \end{aligned}$$

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Mean field particle model

$$\xi_t^i \text{ with generator } L_{t,\eta_t^N} \text{ with } \quad \eta_t^N = N^{-1} \sum_{1 \le i \le N} \delta_{\xi_t^i}$$

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Local perturbation eq.

$$d\eta_t^N(f) = \eta_t^N L_{t,\eta_t^N}(f)\ dt + \frac{1}{\sqrt{N}}\ dM_t^N(f)$$

FKS-semigroups $\gamma_t = \gamma_s Q_{s,t}$

$$Q_{s,t}(f)(x) = \mathbb{E}\left(f(X_t) \exp\left\{ -\int_s^t V_u(X_u)du \right\} \mid X_s = x \right)$$

Normalization

$$\eta_t(f) \quad := \quad \gamma_t(f)/\gamma_t(1)$$

Jump-diffusion particles with

$$L_{t,\eta}(f)(x) = L^X(f)(x) + V(x) \int \left(f(y) - f(x)\right) \eta_t(dy)$$

Denormalization

$$\gamma_t(f) = \eta_t(f) \ \exp\left(-\int_0^t \eta_s(V_s)ds \right)$$

FKS-Stochastic analysis

- **Particle estimations**

$$\mathbb{E}\left(f(X_t)\exp\left[\int_0^t W(X_s)ds\right]\right) = \eta_t(f)\ \exp\left[\int_0^t \eta_s(W)ds\right]$$

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  (v.p. $\lambda \oplus$ ground state h (L μ-reversible))

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- **Asymptotic theory "$\sim$" discrete time models.**

  Some refs: dp+Miclo [00,02,03,07] $\oplus$ Mathias Rousset, Tony Lelièvre, and Gabriel Stoltz

  $\in$ MATHERIALS/MICMAC INRIA team.

Gradient diff. - Ex: $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) := \alpha\, \frac{x^2}{2}$

$$d\overline{X}_t = -\left[\nabla V + \nabla F * \eta_t\right]\left(\overline{X}_t\right) dt + \sigma dB_t$$

Applications : *micro-macro-dilute physical models*

$$N = 10^{25} \rightsquigarrow N^2 = 10^{50} \text{particles "algorithm" ?}$$

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- V non convex and σ too small $\Rightarrow$ non uniqueness invariant measures $\rightsquigarrow$ pionnering work by Tugaut [10], σ large enough and $\alpha > 1 \rightsquigarrow$ Uniform propagation of chaos dp+Tugaut [13].

Kalman-Bucy filter

Linear+Gaussian filtering problem

$$\left\{\begin{array}{rcl} dX_t & = & A\,X_t\,dt \;+\; R^{1/2}\,dW_t \\ dY_t & = & C\,X_t\,dt \;+\; \Sigma^{1/2}\,dV_t \end{array}\right. \qquad \rightsquigarrow \mathcal{F}_t := \sigma(Y_s,\ s \leq t).$$

Optimal $\mathbb{L}_2$-filter = Kalman-Bucy filter

$$\widehat{X}_t := \mathbb{E}(X_t \mid \mathcal{F}_t) \quad \text{and} \quad P_t := \mathbb{E}\left((X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))\,(X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))'\right)$$

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$$\Downarrow$$

$$d\widehat{X}_t = A\,\widehat{X}_t\,dt + P_t\,C'\Sigma^{-1}\left(dY_t - C\widehat{X}_t\,dt\right)$$

with the Riccati equation

$$\partial_t P_t = \mathrm{Ricc}(P_t) := AP_t + P_tA' - P_tSP_t + R \quad \text{with} \quad S := C'\Sigma C$$

Nonlinear Kalman-Bucy diffusion

$$d\overline{X}_t = A\,\overline{X}_t dt \;+\; R^{1/2}\, d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\;\left[dY_t - \left(C\overline{X}_t dt + \Sigma^{1/2}\; d\overline{V}_t\right)\right]$$

with the covariance matrices

$$\mathcal{P}_{\eta_t} = \eta_t\left[(e-\eta_t(e))(e-\eta_t(e))'\right] \quad \text{with} \quad e(x) := x.$$

$$\Downarrow$$

$$\mathbb{E}\left(\overline{X}_t \mid \mathcal{F}_t\right) = \widehat{X}_t \qquad \text{and} \qquad \mathcal{P}_{\eta_t} = P_t.$$

Ensemble Kalman-Bucy filter
$=$ **Interacting particles with empirical covariance**

$$p_t := \mathcal{P}_{\eta_t^N} \simeq P_t \quad \text{and} \quad m_t = \eta_t^N(e) \simeq \mathbb{E}(X_t \mid \mathcal{F}_t)$$

Nonlinear models

Extended Kalman-Bucy-filters

$$d\widehat{X}_t = A(\widehat{X}_t)\ dt + P_t C'\ \Sigma^{-1}\ \left[dY_t - C\widehat{X}_t\ dt\right]$$

with the "stochastic" Riccati equation:

$$\partial_t P_t = \partial A(\widehat{X}_t)P_t + P_t\ \partial A(\widehat{X}_t)' + R - P_t S P_t$$

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McKean-Vlasov interpretation

$$\begin{aligned} d\overline{X}_t &= \mathcal{A}\left(\overline{X}_t, \mathbb{E}[\overline{X}_t \mid \mathcal{G}_t]\right)\ dt + R^{1/2}\ d\overline{W}_t \\ &\qquad + \mathcal{P}_{\eta_t} C' R_2^{-1}\ \left[dY_t - \left(C\overline{X}_t\ dt + \Sigma^{1/2}\ d\overline{V}_t\right)\right] \end{aligned}$$

with the drift function

$$\mathcal{A}(x, m) := A\,[m] + \partial A\,[m]\ (x - m).$$

Some illustrations

Langevin type signal processes

$$R = \sigma^2 \; Id \quad \text{and} \quad (A, \partial A) = (-\partial \mathcal{V}, -\partial^2 \mathcal{V})$$

Non quadratic potential $(q \in \mathbb{R}^r, \mathcal{Q}_1, \mathcal{Q}_2 \geq 0)$

$$\mathcal{V}(x) = \frac{1}{2} \, \langle \mathcal{Q}_1 x, x \rangle + \langle q, x \rangle + \frac{1}{3} \, \langle \mathcal{Q}_2 x, x \rangle^{3/2}$$

Interacting diffusion gradient flows

$$\mathcal{V}(x) = \sum_{1 \leq i \leq r} \mathcal{U}_1(x_i) + \sum_{1 \leq i \neq j \leq r} \mathcal{U}_2(x_i, x_j)$$

for some convex confining potential $\mathcal{U}_i : \mathbb{R}^i \mapsto [0, \infty[$

Regularity conditions

Full observation $S = s\ Id$ and

$$-\lambda_{\partial A} \quad := \quad \sup_{x \in \mathbb{R}^{r_1}} \lambda_{max}(\partial A(x) + \partial A(x)') < 0$$

$$\|\partial A(x) - \partial A(y)\| \quad \leq \quad \kappa_{\partial A}\ \|x - y\|$$

Examples: Langevin signal-diffusion

$$(\lambda_{\partial A}, \kappa_{\partial A}) = \beta\ \left(2^{-1}\lambda_{min}(\mathcal{Q}_1),\ 2\lambda_{max}^{3/2}(\mathcal{Q}_2)\right).$$

more generally $\partial^2\mathcal{V} \geq v\ Id \oplus$ *Lipschitz condition*

Stability theorem

$$(\overline{X}_t, \overline{Z}_t) = \text{McKean-Vlasov starting at} \quad (\overline{X}_0, \overline{Z}_0)$$

$$\Downarrow$$

Theo [dp-Kurtzmann-Tugaut]

When $\lambda_{\partial A}$ is sufficiently large we have

$$\mathbb{W}_2(\text{Law}(\overline{X}_t), \text{Law}(\overline{Z}_t)) \leq c \ \exp\left[-t \ \lambda\right] \quad \text{for some} \quad \lambda > 0.$$

$\exists$ *more explicit description in terms of* $(R, S, \kappa_{\partial A})$.

Propagation of chaos

$$\mathbb{P}_t^N := \mathrm{Law}(m_t, p_t) \qquad \mathbb{P}_t := \mathrm{Law}(\widehat{X}_t, P_t)$$

and

$$\mathbb{Q}_t^N := \mathrm{Law}(\xi_t^1) \qquad \mathbb{Q}_t := \mathrm{Law}(\overline{X}_t)$$

$$\Downarrow$$

Theo SIAM Control & Opt [16] [dp-Kurtzmann-Tugaut]

When $\lambda_{\partial A}$ is sufficiently large, $\exists \beta \in]0, 1/2]$ s.t.

$$\sup_{t\geq 0} \mathbb{W}_2\left(\mathbb{P}_t^N, \mathbb{P}_t\right) \vee \sup_{t\geq 0} \mathbb{W}_2\left(\mathbb{Q}_t^N, \mathbb{Q}_t\right) \leq c\ N^{-\beta}$$

Linear models $\rightsquigarrow$ unif. propagation of chaos with $\beta = 1/2$ for any stable drift matrix.

- Arxiv 1 (Stability KBF)+ A.N. Bishop
- Arxiv 2 (Perturbation KBF)+ A.N. Bishop
- Arxiv 3 (Unif. EnKBF)+ J. Tugaut.
- Arxiv 4 (Stability EKBF)+ A. Kurtzmann, J. Tugaut.