CIRM Research School

Modeling and Control of Open Quantum Systems[1]

15-20 April 2018

Spin/spring composite quantum systems: usual notations and some useful formulae[2]

[1] https://conferences.cirm-math.fr/1732.html
https://sites.google.com/view/mcqs2018/cirm-school

[2] Slides based on a Master course given by M. Mirrahimi and P. Rouchon:
http://cas.ensmp.fr/~rouchon/MasterUPMC/index.html

Some books

1. Cohen-Tannoudji, C.; Diu, B. & Laloë, F.: Mécanique Quantique. Hermann, Paris, 1977, I& II (*quantum physics: a well known and tutorial textbook*)
2. S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006. (*quantum physics: spin/spring systems, decoherence, Schrödinger cats, entanglement*)
3. C. Gardiner, P. Zoller: The Quantum World of Ultra-Cold Atoms and Light I& II. Imperial College Press, 2009. (*quantum physics: quantum optics view-point on quantum science and technologies, quantum stochastic process and continuous measurement*)
4. Barnett, S. M. & Radmore, P. M.: Methods in Theoretical Quantum Optics Oxford University Press, 2003. (*mathematical physics: many useful operator formulae for spin/spring systems*)
5. E. Davies: Quantum Theory of Open Systems. Academic Press, 1976. (*mathematical physics: functional analysis aspects when the Hilbert space is of infinite dimension*)
6. Gardiner, C. W.: Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences [3rd ed], Springer, 2004. (*tutorial introduction to probability, Markov processes, stochastic differential equations and Ito calculus*)
7. M. Nielsen, I. Chuang: Quantum Computation and Quantum Information. Cambridge University Press, 2000. (*tutorial introduction with a computer science and communication view point*)

Outline

Dirac notations, spectral decomposition and trace

- Vectors of the Hilbert $\mathcal{H}$ are denoted by ket: $|\psi\rangle$, $|g\rangle$, $|e\rangle$, $|n\rangle$, $|\alpha\rangle$. Usually their norms are equal to one. $\bar{\psi} = |\psi\rangle^\dagger$ with † for Hermitian conjugate. If $(|n\rangle)$ is an Hilbert basis $|\psi\rangle = \sum_n \psi_n |n\rangle$ with $\psi_n \in \mathbb{C}$, $\langle\psi| = \sum_n \psi_n^* \langle n|$ and $\langle n'|n\rangle = \delta_{n'n}$.
- Spectral decomposition of an Hermitian operator $\boldsymbol{H} = \boldsymbol{H}^\dagger$ of $\mathcal{H}$: $\boldsymbol{H} = \sum_\nu \lambda_\nu \boldsymbol{P}_\nu$ where ν is the label attached to the real eigenvalue λ_ν and $\boldsymbol{P}_\nu$ is the orthogonal projector on the eigen-space attached to λ_ν ($\sum_\nu \boldsymbol{P}_\nu = \boldsymbol{I}$ and $\lambda_\nu \neq \lambda_{\nu'}$ for $\nu \neq \nu'$).
- A unitary operator $\boldsymbol{U}$ preserves the Hermitian product: $\boldsymbol{U}^\dagger \boldsymbol{U} = \boldsymbol{U}\boldsymbol{U}^\dagger = \boldsymbol{I}$. Typically $e^{-it\boldsymbol{H}/\hbar}$ for $\boldsymbol{H}$ Hermitian and t real is unitary. Usually $\hbar$ is set to 1 and $\boldsymbol{H}$ is in pulsation unit ($2\pi\times$ frequency unit).
- Function of Hermitian operator: take $\lambda \mapsto f(\lambda)$ a real function. For any Hermitian operator $\boldsymbol{H} = \sum_\nu \lambda_\nu \boldsymbol{P}_\nu$ one has $f(\boldsymbol{H}) \triangleq \sum_\nu f(\lambda_\nu)\boldsymbol{P}_\nu$. For any Hermitian operator $\boldsymbol{H}$, unitary operator $\boldsymbol{U}$ and function f: $\boldsymbol{U}^\dagger f(\boldsymbol{H})\boldsymbol{U} \equiv f(\boldsymbol{U}^\dagger \boldsymbol{H}\boldsymbol{U})$.
- Take a (trace-class) operator $\boldsymbol{M}$ on $\mathcal{H}$: $\text{Tr}\,(\boldsymbol{M}) = \sum_n \langle n|\boldsymbol{M}|n\rangle$ independent of the chosen Hilbert basis $(|n\rangle)$. For operators $\boldsymbol{M}_{AB} : \mathcal{H}_A \mapsto \mathcal{H}_B$ and $\boldsymbol{M}_{BA} = \mathcal{H}_B \mapsto \mathcal{H}_A$, $\text{Tr}\,(\boldsymbol{M}_{AB}\boldsymbol{M}_{BA}) = \text{Tr}\,(\boldsymbol{M}_{BA}\boldsymbol{M}_{AB})$. In particular $\text{Tr}\,(|\psi\rangle\langle\psi|) \equiv \langle\psi|\psi\rangle = \||\psi\rangle\|^2$.
- Density operator: an Hermitian operator ρ is a density operator on $\mathcal{H}$ if, and only if, it is trace-class (important when $\mathcal{H}$ is of infinite dimension), non negative and of trace one. The spectral decomposition, $\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$ with $(|\psi_n\rangle)$ an Hilbert basis of $\mathcal{H}$, $p_n \geq 0$ and $\text{Tr}\,(\rho) = \sum_n p_n = 1$, shows that ρ is a statistical mixture of orthogonal states $|\psi_n\rangle$ (here p_n is not necessarily different from $p_{n'}$ when $n \neq n'$). ρ is said pure when $p_n = 0$ except for $n = \bar{n}$: the pure state $\rho = |\psi_{\bar{n}}\rangle\langle\psi_{\bar{n}}|$ corresponds to the wave function $|\psi_{\bar{n}}\rangle$.

Tensor products, composite systems and partial trace

- Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ of the bipartite system (A, B) with $\mathcal{H}_A$ (resp. $\mathcal{H}_B$) Hilbert space for sub-system A (resp. B): $\mathcal{H} \ni |\psi\rangle = \sum_{n_A, n_B} \psi_{n_A n_B} |n_A n_B\rangle$ with $(|n_A\rangle)$ and $(|n_B\rangle)$ Hilbert basis for A and B, $(|n_A n_B\rangle \triangleq |n_A\rangle \otimes |n_B\rangle)$ Hilbert basis of $\mathcal{H}$: $\langle n'_A n'_B | n_A n_B\rangle = \langle n'_A | n_A\rangle \langle n'_B | n_B\rangle = \delta_{n'_A n_A} \delta_{n'_B n_B}$. With $|\phi\rangle = \sum_{n_A, n_B} \phi_{n_A n_B} |n_A n_B\rangle$ the Hermitian product $\langle\psi|\phi\rangle = \sum_{n_A, n_B} \psi^*_{n_A n_B} \phi_{n_A n_B}$ is independent of the chosen Hilbert basis for A and B. If $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ and $|\phi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$ then $\langle\psi|\phi\rangle = \langle\psi_A|\phi_A\rangle \langle\psi_B|\phi_B\rangle$.
- $|\psi\rangle \in \mathcal{H}$ is said entangled if and only $\forall |\psi_A\rangle \in \mathcal{H}_A$ and $\forall |\psi_B\rangle \in \mathcal{H}_B$ $|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$ (similar to functions of two variables $f(x_A, x_B)$ that are not the product of two functions of one variables, $f(x_A, x_B) \neq f_A(x_A) f_B(x_B)$ for all f_A and f_B). Otherwise $|\psi\rangle$ is said separable.
- For operators $\boldsymbol{A}$ on $\mathcal{H}_A$ and $\boldsymbol{B}$ on $\mathcal{H}_B$, their tensor product $\boldsymbol{A} \otimes \boldsymbol{B}$ is an operator of $\mathcal{H}$ defined by $\boldsymbol{A} \otimes \boldsymbol{B} |\psi\rangle = \sum_{n_A, n_B} \psi_{n_A n_B} \boldsymbol{A} \otimes \boldsymbol{B}(|n_A n_B\rangle)$ with $\boldsymbol{A} \otimes \boldsymbol{B}(|n_A n_B\rangle) = \boldsymbol{A}|n_A\rangle \otimes \boldsymbol{B}|n_B\rangle$. Thus $\boldsymbol{A} \otimes \boldsymbol{B}(|\psi_A\rangle \otimes |\psi_B\rangle) = \boldsymbol{A}|\psi_A\rangle \otimes \boldsymbol{B}|\psi_B\rangle$. By a slight abuse of notation, $\boldsymbol{A} \otimes \boldsymbol{B}$ is denoted by $\boldsymbol{AB}$.
- Any operator $\boldsymbol{A}$ on $\mathcal{H}_A$ admits an direct extension on $\mathcal{H}$, corresponding to $\boldsymbol{A} \otimes \boldsymbol{I}_B$. Thus $\boldsymbol{A} \otimes \boldsymbol{I}_B(|\psi_A\rangle \otimes |\psi_B\rangle) = (\boldsymbol{A}|\psi_A\rangle) \otimes |\psi_B\rangle$. Most of the time, , $\boldsymbol{A} \otimes \boldsymbol{I}_B$ is denoted by $\boldsymbol{A}$.
- Partial trace: for any (trace-class) operator ρ on $\mathcal{H}$ is attached a (trace-class) operator ρ_B on $\mathcal{H}_B$, denoted by $\mathrm{Tr}_A(\rho)$, obtained by tracing over A and characterized by the identity $\mathrm{Tr}_{(A,B)}(\boldsymbol{I}_A \otimes \boldsymbol{B}\rho) \equiv \mathrm{Tr}_B(\boldsymbol{B}\rho_B)$ for any (bounded) operator $\boldsymbol{B}$ on $\mathcal{H}_B$. The map (super-operator) $\mathrm{Tr}_A(\,) : \ \rho \mapsto \rho_B$ is linear, trace preserving and completely positive (quantum channel see Nielsen-Chuang book).

Models of open quantum systems are based on three features[3]

1. Schrödinger: wave funct. $|\psi\rangle \in \mathcal{H}$ or density op. $\boldsymbol{\rho} \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -\tfrac{i}{\hbar}\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\boldsymbol{\rho} = -\tfrac{i}{\hbar}[\boldsymbol{H},\boldsymbol{\rho}], \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1$$

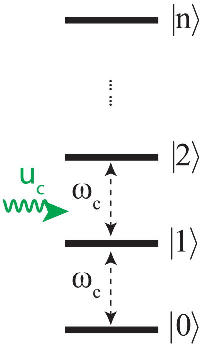
2. Entanglement and tensor product for composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.
3. Randomness and irreversibility induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle = \text{Tr}\left(\boldsymbol{\rho}\boldsymbol{P}_\mu\right)$ depending on $|\psi\rangle$, $\boldsymbol{\rho}$ just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}, \quad \boldsymbol{\rho} \mapsto \boldsymbol{\rho}_+ = \frac{\boldsymbol{P}_y\boldsymbol{\rho}\boldsymbol{P}_y}{\text{Tr}\left(\boldsymbol{\rho}\boldsymbol{P}_y\right)}$$

[3] S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006.

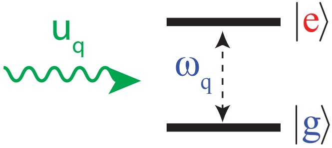
Summary: quantum harmonic oscillator (spring system)

- Hilbert space:
  $\mathcal{H} = \left\{ \sum_{n\geq 0} \psi_n |n\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathbb{D} = \{\rho \in \mathcal{L}(\mathcal{H}), \rho^\dagger = \rho, \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n}\ |\boldsymbol{n\text{-}1}\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a} f(\boldsymbol{N}) = f(\boldsymbol{N}+\boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^\dagger \boldsymbol{a}}$, $\boldsymbol{D}_{-\alpha} \boldsymbol{a} \boldsymbol{D}_\alpha = \boldsymbol{a} + \alpha \boldsymbol{I}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = \imath \boldsymbol{I}/2$.
- Hamiltonian: $\boldsymbol{H}/\hbar = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c x - \sqrt{2}u_c$).
- Classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}:\ |\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}} e^{\imath\sqrt{2}x\Im\alpha} e^{-\frac{(x-\sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.



Summary: 2-level system, i.e. a qubit (half-spin system)

- Hilbert space:
  $\mathcal{H} = \mathbb{C}^2 = \Big\{ c_g|g\rangle + c_e|e\rangle,\ c_g, c_e \in \mathbb{C} \Big\}.$
- Quantum state space:
  $\mathbb{D} = \{\rho \in \mathcal{L}(\mathcal{H}), \rho^\dagger = \rho, \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{\sigma_-} = |g\rangle\langle e|$, $\boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^\dagger = |e\rangle\langle g|$
  $\boldsymbol{\sigma_x} = \boldsymbol{\sigma_-} + \boldsymbol{\sigma_+} = |g\rangle\langle e| + |e\rangle\langle g|$;
  $\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_-} - i\boldsymbol{\sigma_+} = i|g\rangle\langle e| - i|e\rangle\langle g|$;
  $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+} = |e\rangle\langle e| - |g\rangle\langle g|$;
  $\boldsymbol{\sigma_x}^2 = \boldsymbol{I}$, $\boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}$, $[\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}$, ...



- Hamiltonian: $\boldsymbol{H}/\hbar = \omega_q \boldsymbol{\sigma_z}/2 + \boldsymbol{u_q}\boldsymbol{\sigma_x}/2$.
- Bloch sphere representation:
  $\mathbb{D} = \Big\{ \frac{1}{2}\big(\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}\big) \ \Big|\ (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1 \Big\}$
  With $\vec{M} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ Schrödinger eq. reads $\frac{d}{dt}\vec{M} = (\boldsymbol{u_q}\vec{\imath} + \omega_q\vec{k}) \times \vec{M}$.

Summary: spin/spring composite system

Hilbert space

$\mathcal{H} = \left\{ \sum_{n\geq 0} \psi_{gn}|gn\rangle + \psi_{en}|en\rangle,\ \sum_n |\psi_{gn}|^2 + |\psi_{en}|^2 < +\infty \right\} \equiv \mathbb{C}^2 \otimes L^2(\mathbb{R},\mathbb{C})$.

Dispersive Hamiltonian ($\omega_{\text{eg}} \neq \omega_c$)

$$\boldsymbol{H}_{\text{disp}}/\hbar = \frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega_c \boldsymbol{N} - \frac{\chi}{2}\ \boldsymbol{\sigma_z}\boldsymbol{N}$$

and propagator ($i\hbar\frac{d}{dt}\boldsymbol{U} = \boldsymbol{H}\boldsymbol{U}$ with $\boldsymbol{U}(0) = \boldsymbol{I}$):

$$e^{-it\boldsymbol{H}_{\text{disp}}/\hbar} = \boldsymbol{U}(t) = e^{i\omega_{\text{eg}}t/2}|g\rangle\langle g| \otimes \exp\left(-i(\omega_c + \tfrac{\chi}{2})t\boldsymbol{N}\right) + e^{-i\omega_{\text{eg}}t/2}|e\rangle\langle e| \otimes \exp\left(-i(\omega_c - \tfrac{\chi}{2})t\boldsymbol{N}\right)$$

Resonant Hamiltonian (Jaynes-Cummings) ($\omega_{\text{eg}} = \omega_c = \omega$):

$$\boldsymbol{H}_{\text{JC}}/\hbar = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\boldsymbol{N} + i\frac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}).$$

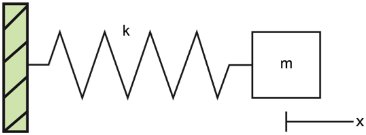
with $\boldsymbol{U}(t) = e^{-it\boldsymbol{H}_{\text{JC}}/\hbar} = e^{-i\omega t\left(\frac{\boldsymbol{\sigma_z}}{2}+\boldsymbol{N}\right)} e^{\frac{\Omega t}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a})}$ where ($\theta = \frac{\Omega t}{2}$)

$$e^{\theta(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a})} = |g\rangle\langle g| \otimes \cos(\theta\sqrt{\boldsymbol{N}}) + |e\rangle\langle e| \otimes \cos(\theta\sqrt{\boldsymbol{N}+\boldsymbol{I}}) - \boldsymbol{\sigma_+} \otimes \boldsymbol{a}\frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}} + \boldsymbol{\sigma_-} \otimes \frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}}\boldsymbol{a}^\dagger$$

Classical Hamiltonian formulation of $\frac{d^2}{dt^2}x = -\omega^2 x$

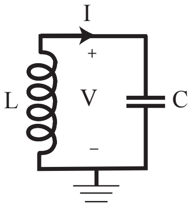
$$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Mechanical oscillator



Frictionless spring: $\frac{d^2}{dt^2}x = -\frac{k}{m}x.$

Electrical oscillator:



LC oscillator:

$$\frac{d}{dt}I = \frac{V}{L}, \frac{d}{dt}V = -\frac{I}{C}, \quad \left(\frac{d^2}{dt^2}I = -\frac{1}{LC}I\right).$$

Harmonic oscillator[4]: quantization and correspondence principle

$$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Quantization: probability wave function $|\psi\rangle_t \sim (\psi(x,t))_{x\in\mathbb{R}}$ with $|\psi\rangle_t \sim \psi(.,t) \in L^2(\mathbb{R},\mathbb{C})$ obeys to the Schrödinger equation ($\hbar = 1$ in all the sequel)

$$i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle, \quad \boldsymbol{H} = \omega(\boldsymbol{P}^2 + \boldsymbol{X}^2) = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$$

where $\boldsymbol{H}$ results from $\mathbb{H}$ by replacing x by position operator $\sqrt{2}\boldsymbol{X}$ and p by momentum operator $\sqrt{2}\boldsymbol{P} = -i\frac{\partial}{\partial x}$. $\boldsymbol{H}$ is a Hermitian operator on $L^2(\mathbb{R},\mathbb{C})$, with its domain to be given.

PDE model: $i\frac{\partial \psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2 \psi}{\partial x^2}(x,t) + \frac{\omega}{2}x^2\psi(x,t), \quad x \in \mathbb{R}.$

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[4] Two references: C. Cohen-Tannoudji, B. Diu, and F. Laloë. *Mécanique Quantique*, volume I& II. Hermann, Paris, 1977.
M. Barnett and P. M. Radmore. *Methods in Theoretical Quantum Optics*. Oxford University Press, 2003.

Average position $\langle \boldsymbol{X} \rangle_t = \langle \psi | \boldsymbol{X} | \psi \rangle$ and momentum $\langle \boldsymbol{P} \rangle_t = \langle \psi | \boldsymbol{P} | \psi \rangle$:

$$\langle \boldsymbol{X} \rangle_t = \tfrac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} x |\psi|^2 dx, , \quad \langle \boldsymbol{P} \rangle_t = -\tfrac{i}{\sqrt{2}} \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx.$$

Annihilation $\boldsymbol{a}$ and creation operators $\boldsymbol{a}^\dagger$ (domains to be given):

$$\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \tfrac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right), \quad \boldsymbol{a}^\dagger = \boldsymbol{X} - i\boldsymbol{P} = \tfrac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right)$$

Commutation relationships:

$$[\boldsymbol{X}, \boldsymbol{P}] = \tfrac{i}{2} \boldsymbol{I}, \quad [\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}, \quad \boldsymbol{H} = \omega(\boldsymbol{P}^2 + \boldsymbol{X}^2) = \omega \left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2} \right).$$

Set $\boldsymbol{X}_\lambda = \frac{1}{2} \left(e^{-i\lambda} \boldsymbol{a} + e^{i\lambda} \boldsymbol{a}^\dagger \right)$ for any angle λ:

$$\left[\boldsymbol{X}_\lambda, \boldsymbol{X}_{\lambda + \frac{\pi}{2}} \right] = \tfrac{i}{2} \boldsymbol{I}.$$

Harmonic oscillator: spectral decomposition and Fock states

Spectrum of Hamiltonian $\boldsymbol{H} = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$:

$$E_n = \omega(n+\frac{1}{2}), \;\; \psi_n(x) = \left(\frac{1}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x), \;\; H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Spectral decomposition of $\boldsymbol{a}^\dagger \boldsymbol{a}$ using $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$:

- If $|\psi\rangle$ is an eigenstate associated to eigenvalue λ, $\boldsymbol{a}|\psi\rangle$ and $\boldsymbol{a}^\dagger|\psi\rangle$ are also eigenstates associated to $\lambda - 1$ and $\lambda + 1$.
- $\boldsymbol{a}^\dagger \boldsymbol{a}$ is semi-definite positive.
- The ground state $|\psi_0\rangle$ is necessarily associated to eigenvalue 0 and is given by the Gaussian function $\psi_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2)$.

Harmonic oscillator: spectral decomposition and Fock states

$[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$: spectrum of $\boldsymbol{a}^\dagger \boldsymbol{a}$ is non-degenerate and is $\mathbb{N}$.

Fock state with n photons (phonons): the eigenstate of $\boldsymbol{a}^\dagger \boldsymbol{a}$ associated to the eigenvalue n ($|n\rangle \sim \psi_n(x)$):

$$\boldsymbol{a}^\dagger \boldsymbol{a}|n\rangle = n|n\rangle, \quad \boldsymbol{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \quad \boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

The ground state $|0\rangle$ is called 0-photon state or vacuum state.

The operator $\boldsymbol{a}$ (resp. $\boldsymbol{a}^\dagger$) is the annihilation (resp. creation) operator since it transfers $|n\rangle$ to $|n-1\rangle$ (resp. $|n+1\rangle$) and thus decreases (resp. increases) the quantum number n by one unit.

Hilbert space of quantum system: $\mathcal{H} = \{\sum_n c_n|n\rangle \mid (c_n) \in l^2(\mathbb{C})\} \sim L^2(\mathbb{R}, \mathbb{C})$.
Domain of $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$: $\{\sum_n c_n|n\rangle \mid (c_n) \in h^1(\mathbb{C})\}$.
Domain of $\boldsymbol{H}$ ot $\boldsymbol{a}^\dagger \boldsymbol{a}$: $\{\sum_n c_n|n\rangle \mid (c_n) \in h^2(\mathbb{C})\}$.

$$h^k(\mathbb{C}) = \{(c_n) \in l^2(\mathbb{C}) \mid \sum n^k |c_n|^2 < \infty\}, \qquad k = 1, 2.$$

Quantization of $\frac{d^2}{dt^2}x = -\omega^2 x - \omega\sqrt{2}u$, ($\mathbb{H} = \frac{\omega}{2}(p^2 + x^2) + \sqrt{2}ux$)

$$\boldsymbol{H} = \omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger).$$

The associated controlled PDE

$$i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \left(\tfrac{\omega}{2}x^2 + \sqrt{2}ux\right)\psi(x,t).$$

Glauber displacement operator $\boldsymbol{D}_\alpha$ (unitary) with $\alpha \in \mathbb{C}$:

$$\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^* \boldsymbol{a}} = e^{2i\Im\alpha \boldsymbol{X} - 2i\Re\alpha \boldsymbol{P}}$$

From Baker-Campbell Hausdorf formula, for all operators $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$e^{\boldsymbol{A}}\boldsymbol{B}e^{-\boldsymbol{A}} = \boldsymbol{B} + [\boldsymbol{A},\boldsymbol{B}] + \tfrac{1}{2!}[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] + \tfrac{1}{3!}[\boldsymbol{A},[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]]] + \ldots$$

we get the Glauber formula[5] when $[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] = [\boldsymbol{B},[\boldsymbol{A},\boldsymbol{B}]] = 0$:

$$e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}\ e^{\boldsymbol{B}}\ e^{-\frac{1}{2}[\boldsymbol{A},\boldsymbol{B}]}.$$

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[5] Take s derivative of $e^{s(\boldsymbol{A}+\boldsymbol{B})}$ and of $e^{s\boldsymbol{A}}\ e^{s\boldsymbol{B}}\ e^{-\frac{s^2}{2}[\boldsymbol{A},\boldsymbol{B}]}$.

With $\boldsymbol{A} = \alpha \boldsymbol{a}^\dagger$ and $\boldsymbol{B} = -\alpha^* \boldsymbol{a}$, Glauber formula gives:

$$\boldsymbol{D}_\alpha = e^{-\frac{|\alpha|^2}{2}} \; e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} = e^{+\frac{|\alpha|^2}{2}} \; e^{-\alpha^* \boldsymbol{a}} e^{\alpha \boldsymbol{a}^\dagger}$$

$$\boldsymbol{D}_{-\alpha} \boldsymbol{a} \boldsymbol{D}_\alpha = \boldsymbol{a} + \alpha \boldsymbol{I} \quad \text{and} \quad \boldsymbol{D}_{-\alpha} \boldsymbol{a}^\dagger \boldsymbol{D}_\alpha = \boldsymbol{a}^\dagger + \alpha^* \boldsymbol{I}.$$

With $\boldsymbol{A} = 2i\Im\alpha \boldsymbol{X} \sim i\sqrt{2}\Im\alpha x$ and $\boldsymbol{B} = -2\imath\Re\alpha \boldsymbol{P} \sim -\sqrt{2}\Re\alpha \frac{\partial}{\partial x}$, Glauber formula gives[6]:

$$\boldsymbol{D}_\alpha = e^{-i\Re\alpha\Im\alpha} \; e^{i\sqrt{2}\Im\alpha x} e^{-\sqrt{2}\Re\alpha \frac{\partial}{\partial x}}$$

$$(\boldsymbol{D}_\alpha |\psi\rangle)_{x,t} = e^{-i\Re\alpha\Im\alpha} \; e^{i\sqrt{2}\Im\alpha x} \psi(x - \sqrt{2}\Re\alpha, t)$$

Exercise: Prove that, for any $\alpha, \beta, \epsilon \in \mathbb{C}$, we have

$$\boldsymbol{D}_{\alpha+\beta} = e^{\frac{\alpha^*\beta - \alpha\beta^*}{2}} \boldsymbol{D}_\alpha \boldsymbol{D}_\beta$$

$$\boldsymbol{D}_{\alpha+\epsilon} \boldsymbol{D}_{-\alpha} = \left(1 + \frac{\alpha\epsilon^* - \alpha^*\epsilon}{2}\right) \boldsymbol{I} + \epsilon \boldsymbol{a}^\dagger - \epsilon^* \boldsymbol{a} + \boldsymbol{O}(|\epsilon|^2)$$

$$\left(\frac{d}{dt} \boldsymbol{D}_\alpha\right) \boldsymbol{D}_{-\alpha} = \left(\frac{\alpha \frac{d}{dt}\alpha^* - \alpha^* \frac{d}{dt}\alpha}{2}\right) \boldsymbol{I} + \left(\frac{d}{dt}\alpha\right) \boldsymbol{a}^\dagger - \left(\frac{d}{dt}\alpha^*\right) \boldsymbol{a}.$$

[6] Note that the operator $e^{-r\partial/\partial x}$ corresponds to a translation of x by r.

Take $|\psi\rangle$ solution of the controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = \left(\omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger)\right)|\psi\rangle$. Set $\langle \boldsymbol{a}\rangle = \langle\psi|\boldsymbol{a}|\psi\rangle$. Then

$$\frac{d}{dt}\langle \boldsymbol{a}\rangle = -i\omega\langle \boldsymbol{a}\rangle - iu.$$

From $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P}$, we have $\langle \boldsymbol{a}\rangle = \langle \boldsymbol{X}\rangle + i\langle \boldsymbol{P}\rangle$ where $\langle \boldsymbol{X}\rangle = \langle\psi|\boldsymbol{X}|\psi\rangle \in \mathbb{R}$ and $\langle \boldsymbol{P}\rangle = \langle\psi|\boldsymbol{P}|\psi\rangle \in \mathbb{R}$. Consequently:

$$\frac{d}{dt}\langle \boldsymbol{X}\rangle = \omega\langle \boldsymbol{P}\rangle, \quad \frac{d}{dt}\langle \boldsymbol{P}\rangle = -\omega\langle \boldsymbol{X}\rangle - u.$$

Consider the change of frame $|\psi\rangle = e^{-i\theta_t}\boldsymbol{D}_{\langle \boldsymbol{a}\rangle_t}|\chi\rangle$ with

$$\theta_t = \int_0^t \left(\omega|\langle \boldsymbol{a}\rangle|^2 + u\Re(\langle \boldsymbol{a}\rangle)\right), \quad D_{\langle \boldsymbol{a}\rangle_t} = e^{\langle \boldsymbol{a}\rangle_t \boldsymbol{a}^\dagger - \langle \boldsymbol{a}\rangle_t^* \boldsymbol{a}},$$

Then $|\chi\rangle$ obeys to autonomous Schrödinger equation

$$i\frac{d}{dt}|\chi\rangle = \omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)|\chi\rangle.$$

The dynamics of $|\psi\rangle$ can be decomposed into two parts:

- a controllable part of dimension two for $\langle \boldsymbol{a}\rangle$
- an uncontrollable part of infinite dimension for $|\chi\rangle$.

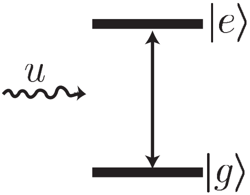
Coherent states

$$|\alpha\rangle = \boldsymbol{D}_\alpha|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad \alpha \in \mathbb{C}$$

are the states reachable from vacuum set. They are also the eigenstate of $\boldsymbol{a}$: $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$.
A widely known result in quantum optics[7]: classical currents and sources (generalizing the role played by u) only generate classical light (quasi-classical states of the quantized field generalizing the coherent state introduced here)
We just propose here a control theoretic interpretation in terms of reachable set from vacuum.

[7] See complement B_{III}, page 217 of C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and Atoms: Introduction to Quantum Electrodynamics*. Wiley, 1989.

2-level system (spin-1/2)



The simplest quantum system: a ground state $|g\rangle$ of energy ω_g; an excited state $|e\rangle$ of energy ω_e. The quantum state $|\psi\rangle \in \mathbb{C}^2$ is a linear superposition $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ and obey to the Schrödinger equation (ψ_g and ψ_e depend on t).

Schrödinger equation for the uncontrolled 2-level system ($\hbar = 1$) :

$$\imath \frac{d}{dt}|\psi\rangle = \boldsymbol{H}_0|\psi\rangle = \left(\omega_e|e\rangle\langle e| + \omega_g|g\rangle\langle g|\right)|\psi\rangle$$

where $\boldsymbol{H}_0$ is the Hamiltonian, a Hermitian operator $\boldsymbol{H}_0^\dagger = \boldsymbol{H}_0$. Energy is defined up to a constant: $\boldsymbol{H}_0$ and $\boldsymbol{H}_0 + \varpi(t)\boldsymbol{I}$ ($\varpi(t) \in \mathbb{R}$ arbitrary) are attached to the same physical system. If $|\psi\rangle$ satisfies $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}_0|\psi\rangle$ then $|\chi\rangle = e^{-i\vartheta(t)}|\psi\rangle$ with $\frac{d}{dt}\vartheta = \varpi$ obeys to $i\frac{d}{dt}|\chi\rangle = (\boldsymbol{H}_0 + \varpi\boldsymbol{I})|\chi\rangle$. Thus for any ϑ, $|\psi\rangle$ and $e^{-i\vartheta}|\psi\rangle$ represent the same physical system: The global phase of a quantum system $|\psi\rangle$ can be chosen arbitrarily at any time.

Take origin of energy such that ω_g (resp. ω_e) becomes $-\frac{\omega_e-\omega_g}{2}$ (resp. $\frac{\omega_e-\omega_g}{2}$) and set $\omega_{eg} = \omega_e - \omega_g$
The solution of $i\frac{d}{dt}|\psi\rangle = H_0|\psi\rangle = \frac{\omega_{eg}}{2}(|e\rangle\langle e| - |g\rangle\langle g|)|\psi\rangle$ is

$$|\psi\rangle_t = \psi_{g0} e^{\frac{i\omega_{eg} t}{2}}|g\rangle + \psi_{e0} e^{\frac{-i\omega_{eg} t}{2}}|e\rangle.$$

With a classical electromagnetic field described by $u(t) \in \mathbb{R}$, the coherent evolution the controlled Hamiltonian

$$\boldsymbol{H}(t) = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u(t)}{2}\boldsymbol{\sigma_x} = \frac{\omega_{eg}}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{u(t)}{2}(|e\rangle\langle g| + |g\rangle\langle e|)$$

The controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = (\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1)|\psi\rangle$ reads:

$$i\frac{d}{dt}\begin{pmatrix}\psi_e \\ \psi_g\end{pmatrix} = \frac{\omega_{eg}}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_e \\ \psi_g\end{pmatrix} + \frac{u(t)}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_e \\ \psi_g\end{pmatrix}.$$

The 3 Pauli Matrices[8]

$$\boldsymbol{\sigma_x} = |e\rangle\langle g| + |g\rangle\langle e|, \ \boldsymbol{\sigma_y} = -i|e\rangle\langle g| + i|g\rangle\langle e|, \ \boldsymbol{\sigma_z} = |e\rangle\langle e| - |g\rangle\langle g|$$

[8] They correspond, up to multiplication by i, to the 3 imaginary quaternions.

Pauli matrices and some formula

$$\boldsymbol{\sigma_x} = |e\rangle\langle g| + |g\rangle\langle e|, \ \boldsymbol{\sigma_y} = -i|e\rangle\langle g| + i|g\rangle\langle e|, \ \boldsymbol{\sigma_z} = |e\rangle\langle e| - |g\rangle\langle g|$$

$$\boldsymbol{\sigma_x}^2 = \boldsymbol{I}, \quad \boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}, \quad [\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}, \ \text{circular permutation} \ \ldots$$

- Since for any $\theta \in \mathbb{R}$, $e^{i\theta\boldsymbol{\sigma_x}} = \cos\theta + i\sin\theta\boldsymbol{\sigma_x}$ (idem for $\boldsymbol{\sigma_y}$ and $\boldsymbol{\sigma_z}$), the solution of $i\frac{d}{dt}|\psi\rangle = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}|\psi\rangle$ is

$$|\psi\rangle_t = e^{\frac{-i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\psi\rangle_0 = \left(\cos\left(\frac{\omega_{eg}t}{2}\right)\boldsymbol{I} - i\sin\left(\frac{\omega_{eg}t}{2}\right)\boldsymbol{\sigma_z}\right)|\psi\rangle_0$$

- For $\alpha, \beta = x, y, z$, $\alpha \neq \beta$ we have

$$\boldsymbol{\sigma_\alpha} e^{i\theta\boldsymbol{\sigma_\beta}} = e^{-i\theta\boldsymbol{\sigma_\beta}}\boldsymbol{\sigma_\alpha}, \qquad \left(e^{i\theta\boldsymbol{\sigma_\alpha}}\right)^{-1} = \left(e^{i\theta\boldsymbol{\sigma_\alpha}}\right)^{\dagger} = e^{-i\theta\boldsymbol{\sigma_\alpha}}.$$

  and also

$$e^{-\frac{i\theta}{2}\boldsymbol{\sigma_\alpha}}\boldsymbol{\sigma_\beta} e^{\frac{i\theta}{2}\boldsymbol{\sigma_\alpha}} = e^{-i\theta\boldsymbol{\sigma_\alpha}}\boldsymbol{\sigma_\beta} = \boldsymbol{\sigma_\beta} e^{i\theta\boldsymbol{\sigma_\alpha}}$$

We start from $|\psi\rangle$ that obeys $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle$. We consider the orthogonal projector on $|\psi\rangle$, $\rho = |\psi\rangle\langle\psi|$, called density operator. Then ρ is an Hermitian operator ≥ 0, that satisfies $\mathrm{Tr}\,(\rho) = 1$, $\rho^2 = \rho$ and obeys to the Liouville equation:

$$\frac{d}{dt}\rho = -i[\boldsymbol{H}, \rho].$$

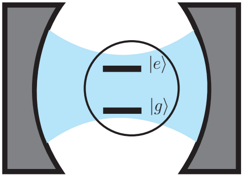
For a two level system $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ and

$$\rho = \frac{\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}}{2}$$

where $(x, y, z) = (2\Re(\psi_g\psi_e^*), 2\Im(\psi_g\psi_e^*), |\psi_e|^2 - |\psi_g|^2) \in \mathbb{R}^3$ represent a vector $\vec{M}$, the Bloch vector, that evolves on the unite sphere of $\mathbb{R}^3$, $\mathbb{S}^2$ called the Bloch Sphere since $\mathrm{Tr}\left(\rho^2\right) = x^2 + y^2 + z^2 = 1$. The Liouville equation with $\boldsymbol{H} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u}{2}\boldsymbol{\sigma_x}$ reads

$$\frac{d}{dt}\vec{M} = (u\vec{i} + \omega_{eg}\vec{k}) \times \vec{M}.$$

Composite system: 2-level and harmonic oscillator



2-level system lives on $\mathbb{C}^2$ with $\boldsymbol{H}_q = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}$
oscillator lives on $L^2(\mathbb{R},\mathbb{C}) \sim l^2(\mathbb{C})$ with

$$\boldsymbol{H}_c = -\frac{\omega_c}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega_c}{2}x^2 \sim \omega_c\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right)$$

$$\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a} \text{ and } \boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} \sim \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$$

The composite system lives on the tensor product
$\mathbb{C}^2 \otimes L^2(\mathbb{R},\mathbb{C}) \sim \mathbb{C}^2 \otimes l^2(\mathbb{C})$ with spin-spring Hamiltonian

$$\boldsymbol{H} = \frac{\omega_{eg}}{2}\,\boldsymbol{\sigma_z} \otimes \boldsymbol{I}_c + \omega_c\,\boldsymbol{I}_q \otimes \left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x} \otimes (\boldsymbol{a}^\dagger - \boldsymbol{a})$$

with the typical scales $\Omega \ll \omega_c, \omega_{eg}$ and $|\omega_c - \omega_{eg}| \ll \omega_c, \omega_{eg}$.
Shortcut notations:

$$\boldsymbol{H} = \underbrace{\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}}_{\boldsymbol{H}_q} + \underbrace{\omega_c\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right)}_{\boldsymbol{H}_c} + \underbrace{i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})}_{\boldsymbol{H}_{int}}$$

The Schrödinger system

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N}+\frac{\mathbf{I}}{2}\right) + i\tfrac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})\right)|\psi\rangle$$

corresponds to two coupled scalar PDE's:

$$i\frac{\partial\psi_e}{\partial t} = +\tfrac{\omega_{eg}}{2}\psi_e + \tfrac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_g$$

$$i\frac{\partial\psi_g}{\partial t} = -\tfrac{\omega_{eg}}{2}\psi_g + \tfrac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_e$$

since $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$, $\boldsymbol{a} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$ and $|\psi\rangle = (\psi_e(x,t), \psi_g(x,t))$, $\psi_g(.,t), \psi_e(.,t) \in L^2(\mathbb{R},\mathbb{C})$ and $\|\psi_g\|^2 + \|\psi_e\|^2 = 1$.

The spin-spring ODE's

The Schrödinger system

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})\right)|\psi\rangle$$

corresponds also to an infinite set of ODE's

$$i\frac{d}{dt}\psi_{e,n} = ((n+1/2)\omega_c + \omega_{eg}/2)\psi_{e,n} + i\frac{\Omega}{2}\left(\sqrt{n}\,\psi_{g,n-1} - \sqrt{n+1}\,\psi_{g,n+1}\right)$$

$$i\frac{d}{dt}\psi_{g,n} = ((n+1/2)\omega_c - \omega_{eg}/2)\psi_{g,n} + i\frac{\Omega}{2}\left(\sqrt{n}\,\psi_{e,n-1} - \sqrt{n+1}\,\psi_{e,n+1}\right)$$

where $|\psi\rangle = \sum_{n=0}^{+\infty}\psi_{g,n}|g,n\rangle + \psi_{e,n}|e,n\rangle$, $\psi_{g,n}, \psi_{e,n} \in \mathbb{C}$.

Dispersive case: approximate Hamiltonian for $\Omega \ll |\omega_c - \omega_{\text{eg}}|$.

$$\boldsymbol{H} \approx \boldsymbol{H}_{\text{disp}} = \tfrac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right) - \frac{\chi}{2}\,\boldsymbol{\sigma_z}\left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right) \quad \text{with } \chi = \tfrac{\Omega^2}{2(\omega_c - \omega_{\text{eg}})}$$

The corresponding PDE is :

$$i\frac{\partial \psi_e}{\partial t} = +\frac{\omega_{\text{eg}}}{2}\psi_e + \frac{1}{2}(\omega_c - \frac{\chi}{2})(x^2 - \frac{\partial^2}{\partial x^2})\psi_e$$
$$i\frac{\partial \psi_g}{\partial t} = -\frac{\omega_{\text{eg}}}{2}\psi_g + \frac{1}{2}(\omega_c + \frac{\chi}{2})(x^2 - \frac{\partial^2}{\partial x^2})\psi_g$$

The propagator, the t-dependant unitary operator $\boldsymbol{U}$ solution of $i\frac{d}{dt}\boldsymbol{U} = \boldsymbol{H}\boldsymbol{U}$ with $\boldsymbol{U}(0) = \boldsymbol{I}$, reads:

$$\boldsymbol{U}(t) = e^{i\omega_{\text{eg}}t/2}\exp\left(-i(\omega_c + \chi/2)t(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2})\right) \otimes |g\rangle\langle g| + e^{-i\omega_{\text{eg}}t/2}\exp\left(-i(\omega_c - \chi/2)t(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2})\right) \otimes |e\rangle\langle e|$$

Resonant case: approximate Hamiltonian for $\omega_c = \omega_{eg} = \omega$.

The Hamiltonian becomes (Jaynes-Cummings Hamiltonian):

$$\boldsymbol{H} \approx \boldsymbol{H}_{JC} = \tfrac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{N} + \frac{\mathbf{I}}{2}\right) + i\tfrac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}).$$

The corresponding PDE is :

$$i\frac{\partial\psi_e}{\partial t} = +\frac{\omega}{2}\psi_e + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_e - i\tfrac{\Omega}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\psi_g$$

$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega}{2}\psi_g + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_g + i\tfrac{\Omega}{2\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\psi_e$$

Exercise: For $\boldsymbol{H}_{JC} = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a})$ show that the propagator, the t-dependant unitary operator $\boldsymbol{U}$ solution of $i\frac{d}{dt}\boldsymbol{U} = \boldsymbol{H}_{JC}\boldsymbol{U}$ with $\boldsymbol{U}(0) = \boldsymbol{I}$, reads $\boldsymbol{U}(t) = e^{-i\omega t\left(\frac{\boldsymbol{\sigma_z}}{2}+\boldsymbol{N}+\frac{1}{2}\right)} e^{\frac{\Omega t}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a})}$ where for any angle θ,

$$e^{\theta(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a})} = |g\rangle\langle g| \otimes \cos(\theta\sqrt{\boldsymbol{N}}) + |e\rangle\langle e| \otimes \cos(\theta\sqrt{\boldsymbol{N}+\boldsymbol{I}}) - \boldsymbol{\sigma_+} \otimes \boldsymbol{a}\,\frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}} + \boldsymbol{\sigma_-} \otimes \frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}}\,\boldsymbol{a}^\dagger$$

Hint: show that

$$\left[\frac{\boldsymbol{\sigma_z}}{2} + \boldsymbol{N}\,,\ \boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right] = 0$$

$$\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)^{2k} = (-1)^k\left(|g\rangle\langle g| \otimes \boldsymbol{N}^k + |e\rangle\langle e| \otimes (\boldsymbol{N}+\boldsymbol{I})^k\right)$$

$$\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)^{2k+1} = (-1)^k\left(\boldsymbol{\sigma_-} \otimes \boldsymbol{N}^k\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+} \otimes \boldsymbol{a}\boldsymbol{N}^k\right)$$

and compute de series defining the exponential of an operator.

Ito stochastic calculus

Consider a stochastic differential equation (SDE) of state $X \in \mathbb{R}^n$:

$$dX_t = F(X_t, t)dt + \sum_\nu G_\nu(X_t, t)dW_{\nu,t},$$

driven by independent Wiener processes $W_{\nu,t}$. Roughly speaking for each ν, $dW_{\nu,t}$ is a scalar random variable independent of X_t and t, following of Gaussian law of mean $\mathbb{E}\left(dW_{\nu,t}\right) = 0$ and standard deviation $\sqrt{dt}$, $(dW_{\nu,t})^2 \equiv dt$, $dW_{\nu,t}dW_{\nu',t} \equiv 0$ for $\nu \neq \nu'$ and $dW_{\nu,t}$ is of order $\sqrt{dt}$ ($dtdW_{\nu,t}$ is of order $dt^{3/2}$).
These heuristic recipes underly the following Ito's rigorous computation rules: for $f_t = f(X_t)$ a C^2 function of X, we have

$$df_t = \left(\frac{\partial f}{\partial X}\Big|_{X_t} F(X_t, t) + \frac{1}{2}\sum_\nu \frac{\partial^2 f}{\partial X^2}\Big|_{X_t}(G_\nu(X_t, t), G_\nu(X_t, t)) \right) dt + \sum_\nu \frac{\partial f}{\partial X}\Big|_{X_t} G_\nu(X_t, t)dW_{\nu,t}.$$

Furthermore

$$\frac{d}{dt}\mathbb{E}\left(f_t\right) = \mathbb{E}\left(\frac{\partial f}{\partial X}\Big|_{X_t} F(X_t, t) + \frac{1}{2}\sum_\nu \frac{\partial^2 f}{\partial X^2}\Big|_{X_t}(G_\nu(X_t, t), G_\nu(X_t, t)) \right).$$

Here df_t coincides with $f(X_t + dX_t) - f(X_t)$ up to terms of order strictly greater than one versus dt. For example, with $dx_t = -x_t dt + dW_t$ and $f(x) = x^2$ we have $df_t = (x_t + dx_t)^2 - x_t^2 + o(dt)$ and $(x_t + dx_t)^2 - x_t^2 = 2x_t dx_t + (dx_t)^2$. Since $(dx_t)^2 = (-x_t dt + dW_t)^2 = (dW_t)^2 + O(dt^{3/2}) = dt + O(dt^{3/2})$ and $2x_t dx_t = -2x_t^2 dt + 2x_t dW_t$, we have $df_t = (-2f_t + 1)dt + 2x_t dW_t$.

Assume that the density operator ρ on the Hilbert space $\mathcal{H}$ is governed by the following SME

$$d\boldsymbol{\rho}_t = \left(-\frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho}_t] + \boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger\boldsymbol{L})\right) dt$$
$$+ \sqrt{\eta}\left(\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \operatorname{Tr}\left((\boldsymbol{L} + \boldsymbol{L}^\dagger)\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t\right) dW_t,$$

with $\boldsymbol{H}$ Hermitian operator, $\boldsymbol{L}$ any measurement/decoherence operator and $\eta \in [0, 1]$ detection efficiency attached to the measurement classical signal $dy_t = dW_t + \sqrt{\eta}\operatorname{Tr}\left((\boldsymbol{L} + \boldsymbol{L}^\dagger)\boldsymbol{\rho}_t\right) dt$.
From Ito rules, one gets the equivalent formulation, useful for positivity preserving numerical scheme:

$$\boldsymbol{\rho}_{t+dt} = \frac{\boldsymbol{M}_{dy_t}\boldsymbol{\rho}_t\boldsymbol{M}_{dy_t}^\dagger + (1-\eta)\boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger dt}{\operatorname{Tr}\left(\boldsymbol{M}_{dy_t}\boldsymbol{\rho}_t\boldsymbol{M}_{dy_t}^\dagger + (1-\eta)\boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger dt\right)}$$

with

$$\boldsymbol{M}_{dy_t} = \boldsymbol{I} - (\tfrac{i}{\hbar}\boldsymbol{H} + \tfrac{1}{2}\boldsymbol{L}^\dagger\boldsymbol{L})dt + \sqrt{\eta}\,dy_t\boldsymbol{L}.$$