Fluctuations for linear statistics for biorthogonal ensembles

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March 2, Luminy

Based on :

M. D. and R. Kozhan, *Relative Szegö Asymptotics for Toeplitz determinants, arXiv*

M. D., *On global fluctuations for non-colliding processes,* arXiv

J. Breuer and M.D., *Central Limit Theorem for biorthogonal ensembles and asymptotics of recurrence coefficients,* Jour. Amer. Math. Soc. 2016

J. Breuer and M.D., *The Nevai condition and a local law of large numbers for orthogonal polynomial ensembles,* Adv. in Math. 2014

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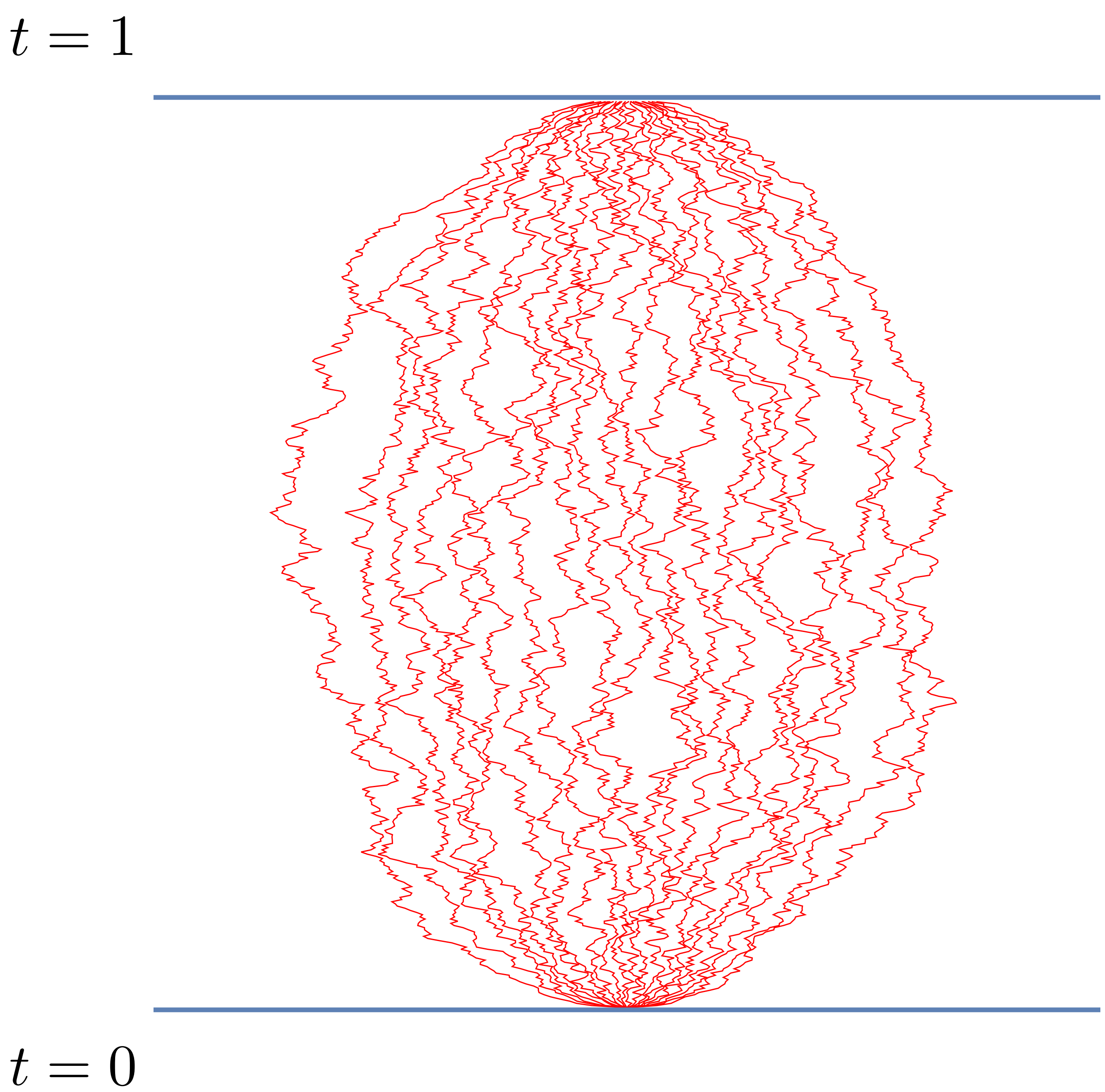
Non-colliding brownie bridges



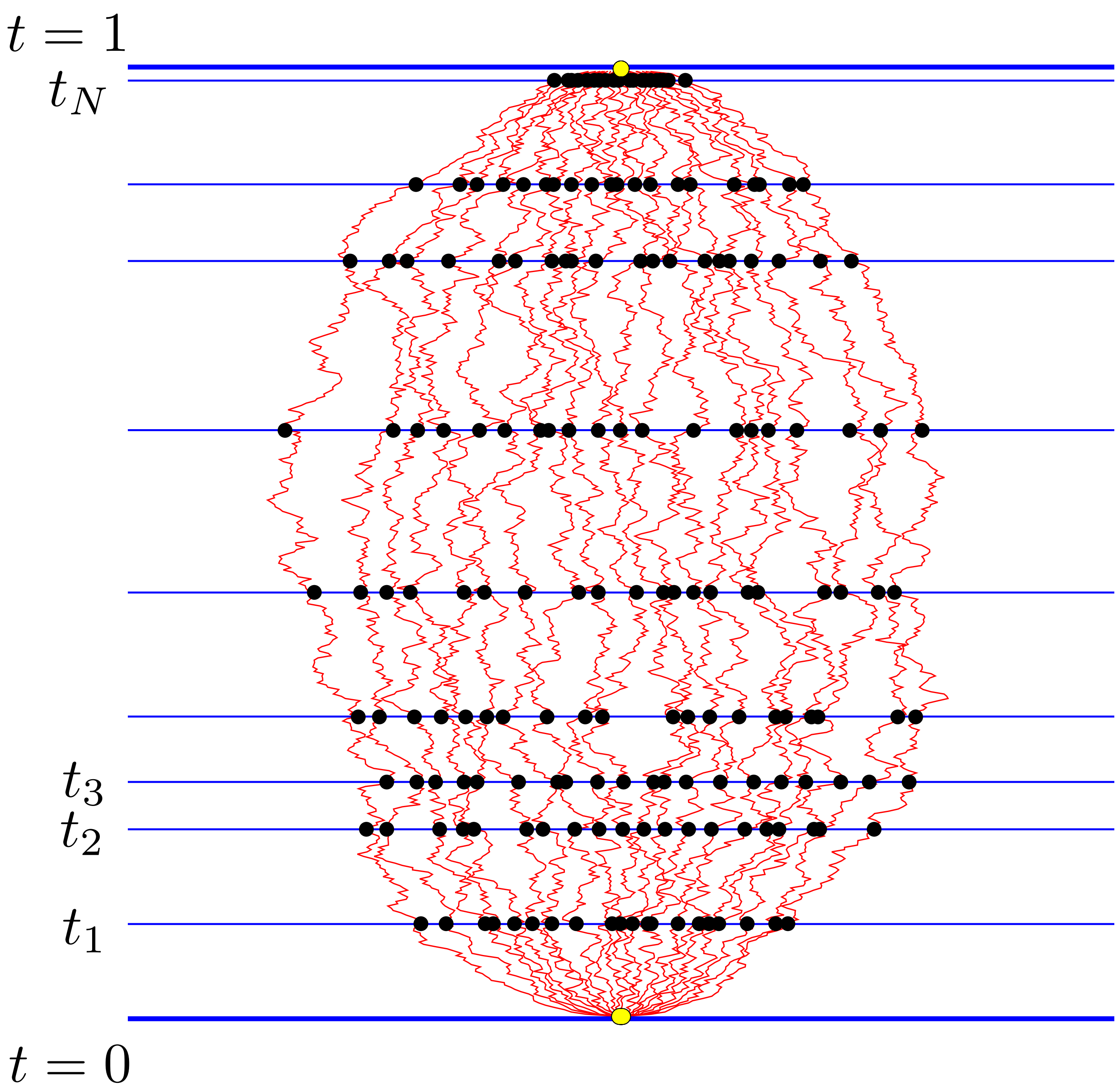
Non-colliding brownian motions have deteminantal correlations.

Paths are level lines for a random surface.

Fluctuations of the surface around its mean are governed by the **Gaussian Free Field**

This is a universal phenomenon.

Multi-time fluctuations



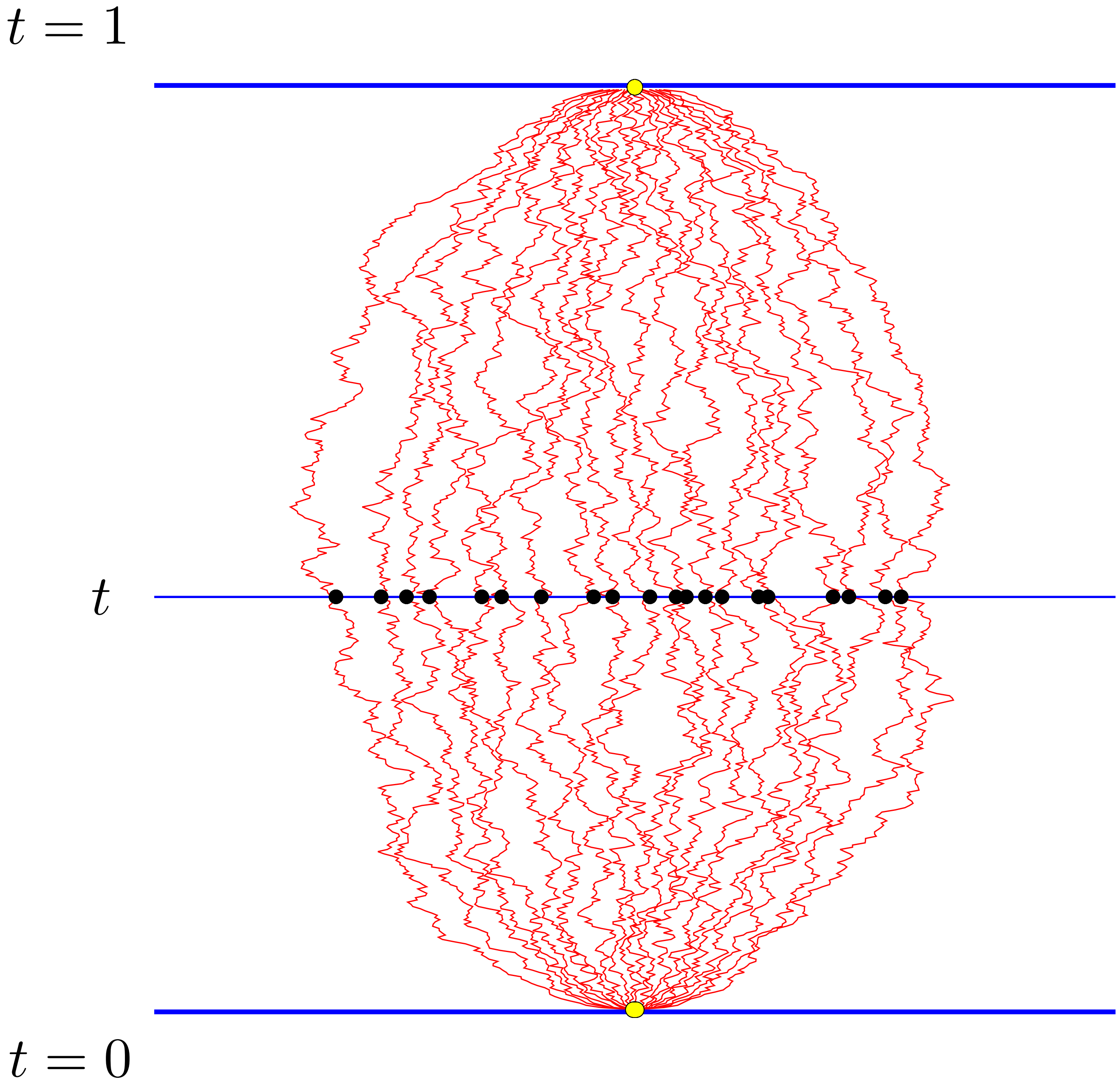
A way to capture the GFF correlations is by studying multi-time linear statistics

$$X_n(f) = \sum_{m=1}^{N} \sum_{j=1}^{n} f(m, x_j(t_m))$$

and ask whether there is a CLT

$$X_n(f) - \mathbb{E}X_n(f) \to N(0, \sigma_{MT}(f)^2)$$

Single time fluctuations



We will explain the method by looking at single time fluctuations mainly.

Then we take a smooth test function and consider the linear statistic

$$X_n(f) = \sum_{j=1}^{n} f(x_j(t))$$

Linear statistic for determinantal point processes satisfy CLT

$$X_n(f) - \mathbb{E}X_n(f) \to N(0, \sigma(f)^2)$$

where the variance is a half Sobolev norm.

Biorthogonal Ensembles

- We consider probability measures of the form

$$\frac{1}{n!}\det\left(\phi_j(x_k)\right)_{j,k=1}^{n}\det\left(\psi_j(x_k)\right)_{j,k=1}^{n}\mathrm{d}\mu(x_1)\ldots\mathrm{d}\mu(x_n)$$

where $d\mu$ is a background measure and $\{\phi_j(x)\}_{j=1}^{\infty}$ and $\{\psi_k(x)\}_{k=1}^{\infty}$ are biorthogonal families

$$\int \phi_j(x)\psi_k(x)\mathrm{d}\mu(x) = \delta_{jk}$$

- We will study fluctuations of the linear statistic $X_n(f) = \sum_{j=1}^{n} f(x_j)$ around its mean and ask whether we have, as $n \to \infty$

$$X_n(f) - \mathbb{E}X_n(f) \to N(0, \sigma(f)^2)$$

Examples

- Orthogonal polynomials on the real line wrt $d\mu$
  - varying weight
    GUE, LUE, general UE with potential V,…
  - non-varying weight
    JacUE, UE,…
  - Discrete weight
    Tiling models,…
- Multiple orthogonal polynomials
  External source random matrices, two-matrix models, multiple speed dynamics on interlacing particles, …
- Orthogonal polynomials on the unit circle
  CUE and other unitary ensembles on the circle,..

The matrix $\mathcal{J}$

- It is too much to ask that a CLT holds for any biorthogonal ensemble, so we need to narrow our class. We do this by assuming that there is a recurrence relation for the biorthogonal family.

- We start by defining the matrix $\mathcal{J}$ by

$$\mathcal{J}_{k,\ell} = \int x \phi_k(x) \psi_\ell(x) \mathrm{d}\mu(x)$$

- In other words:

$$x \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \end{pmatrix} = \mathcal{J} \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \end{pmatrix}$$

Main Assumption

Assumption:

We assume that $\mathcal{J}$ is well-defined and ***banded***

In other words, we assume there exists a recurrence for the biorthogonal family

$$x\phi_j(x) = \sum_{|k|\leq\rho} \mathcal{J}_{j,j+k}\phi_{j+k}(x)$$

- In case of orthogonal polynomials on the real line, $\mathcal{J}$ is called the Jacobi operator associated to the background measure $d\mu$ and is a symmetric tridiagonal matrix with positive off-diagonal entries.

- In case of the circle, we can take $\mathcal{J}$ to be the CMV matrix which is a pentadiagonal unitary matrix.

Andreiéf's identity

Lemma

The characteristic function for the linear statistic $X_n(f) = \sum_{j=1}^{n} f(x_j)$ is given by

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}tX_n(f)}] \;=\; \det P_n \mathrm{e}^{\mathrm{i}tf(\mathcal{J})} P_n$$

Proof:

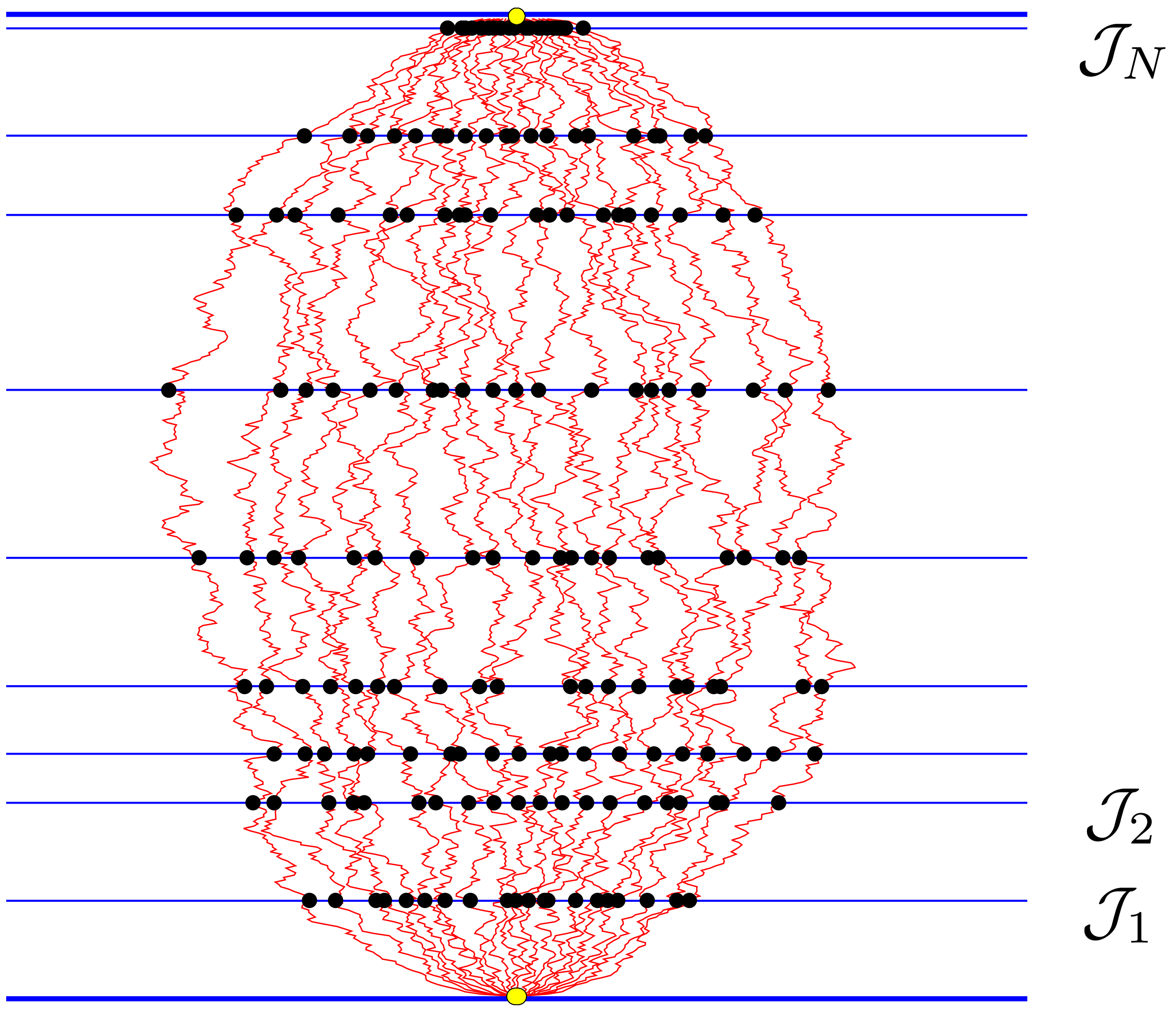
$$\frac{1}{n!}\int\cdots\int e^{\mathrm{i}t\sum_{j=1}^{n} f(x_j)} \det\left(\phi_j(x_k)\right)_{j,k=1}^{n} \det\left(\psi_j(x_k)\right)_{j,k=1}^{n} \mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n)$$

$$= \quad \frac{1}{n!}\int\cdots\int \det\left(\mathrm{e}^{\mathrm{i}tf(x_k)}\phi_j(x_k)\right)_{j,k=1}^{n} \det\left(\psi_j(x_k)\right)_{j,k=1}^{n} \mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n)$$

$$= \quad \det\left(\int \mathrm{e}^{\mathrm{i}tf(x)}\phi_j(x)\psi_k(x)\mathrm{d}\mu(x)\right)_{j,k=1}^{n}$$

$\square$

Non-colliding processes



We define a matrix $\mathcal{J}_m$ for each time t_m and assume that they are all banded.

Then in *D'15* the starting point was to show that the linear statistic

$$X_n(f) = \sum_{m=1}^{N} \sum_{j=1}^{n} f(m, x_j(t_m))$$

has the moment generating function

$$\mathbb{E}[\mathrm{e}^{itX_n(f)}] = \det P_n \mathrm{e}^{itf(1,\mathcal{J}_1)} \mathrm{e}^{itf(2,\mathcal{J}_2)} \cdots \mathrm{e}^{itf(N,\mathcal{J}_N)} P_n$$

Simplification

- So the question of CLTs for single and multi-time linear statistics boils down to asking whether the following objects have gaussian limits:

- Biorthogonal ensembles

$$\det P_n \mathrm{e}^{tB} P_n \qquad\qquad B = \mathrm{i} f(\mathcal{J})$$

- Non-colliding processes

$$\det P_n \mathrm{e}^{tB_1} \mathrm{e}^{tB_2} \cdots \mathrm{e}^{tB_N} P_n \qquad\qquad B_m = \mathrm{i} f(m, \mathcal{J}_m)$$

- In this talk we will focus on the single time case. But many statements and techniques have extensions to multi-time statistics for non-colliding processes

A cumulant type expansion ……

- Using $\log\det(1+A) = \mathrm{Tr}\log(1+A)$ and further expansions we find

$$\det P_n e^{tB} P_n \quad = \exp\left(\sum_{m=1}^{\infty} t^m C_m^{(n)}(B)\right)$$

where

$$C_m^{(n)}(B) = \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j} \sum_{\substack{\ell_1+\ldots\ell_j=m \\ \ell_i\geq 1}} \frac{\mathrm{Tr}\, P_n B^{\ell_1} \cdots P_n B^{\ell_j} P_n}{\ell_1!\ell_2!\cdots\ell_j!}$$

- For $m \geq 2$

$$\sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{\substack{\ell_1+\ldots+\ell_j=m \\ \ell_i \geq 1}} \frac{1}{\ell_1! \cdots \ell_j!} = 0$$

and thus, for $m \geq 2$,

$$C_m^{(n)}(B) = \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j} \sum_{\substack{\ell_1+\ldots\ell_j=m \\ \ell_i \geq 1}} \frac{\mathrm{Tr}\, P_n B^{\ell_1} \cdots P_n B^{\ell_j} P_n - \mathrm{Tr}\, P_n B^m P_n}{\ell_1! \ell_2! \cdots \ell_j!}$$

- Each term in the summand can be written as

$$\mathrm{Tr}\, P_n B^{\ell_1} \cdots P_n B^{\ell_j} P_n - \mathrm{Tr}\, P_n B^m P_n = \sum \mathrm{Tr}\, P_n B^{\ell_1} \cdots [B, P_n] B^k P_n \ldots [B, P_n]$$

(Sum of terms that all contain two factors $[B, P_n]$)

Bounding the "cumulants"

Theorem (Breuer-D. '14)
For $t \in \mathbb{C}$ sufficiently close to the origin, we have

$$\det P_n e^{tB} P_n = \exp\left(t \operatorname{Tr} P_n B P_n + \frac{t^2}{4} \operatorname{Tr}[B, P_n]^2 + \sum_{m=3}^{\infty} t^m C_m^{(n)}(B) \right)$$

where

$$|C_m^{(n)}(B)| \leq \quad e^m m^{3/2} \|B\|_\infty^{m-2} \|[P_n, B]\|_{H.S.}^2$$

This can be interpreted as a Bernstein-type inequality for biorthogonal ensembles and can be used define concentration inequalities for linear statistics. (which also work on mesoscopic scales)

Comparison Principle

Theorem (Breuer-D ’15)

Let $B^{(1)}$ and $B^{(2)}$ be two banded and bounded matrices such that

$$\forall k,\ell \in \mathbb{Z} \qquad B^{(1)}_{n+k,n+\ell} - B^{(2)}_{n+k,n+\ell} \to 0$$

as $n \to \infty$. Then, for $m \geq 2$ we have

$$C_m^{(n)}(B^{(1)}) - C_m^{(n)}(B^{(2)}) \to 0$$

as $n \to \infty$

Gaussian limits

Theorem (D-Kohzan '16 based on Breuer-D '15)

If B is a banded matrix such that

$$(B)_{n+k,n+\ell} - (\mathbb{B})_{n+k,n+\ell} \to 0$$

for a bounded matrix $\mathbb{B}$ that can be decomposed as $\mathbb{B} = \mathbb{B}_+ + \mathbb{B}_-$ such that

1) $\mathbb{B}_+$ lower triangular
2) $\mathbb{B}_-$ upper triangular
3) $[\mathbb{B}_+, \mathbb{B}_-]$ is of trace class.

Then $$\det P_n e^{tB} P_n = \exp\left(t \operatorname{Tr} P_n B P_n + \frac{t^2}{2} \operatorname{Tr}[\mathbb{B}_+, \mathbb{B}_-] \right) (1 + o(1))$$

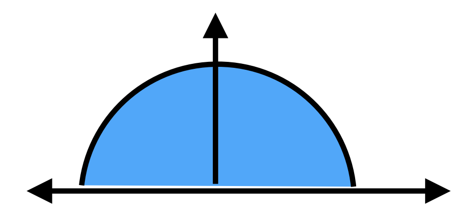
$$n \to \infty$$

Some remarks on the condition

- In certain examples (but certainly not all), the matrix $\mathbb{B}$ can be taken to be a Toeplitz matrix and then the condition **always** holds. *(Breuer-D '16)*

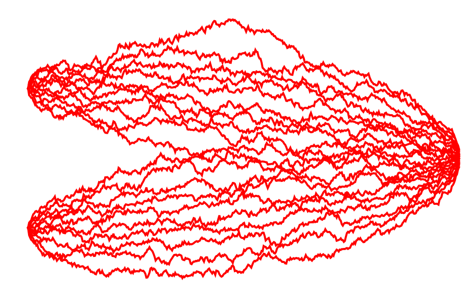
  - OPRL in Nevai class (beyond "solvable models")

  - OPRL for Unitary Ensembles in one-cut situation.



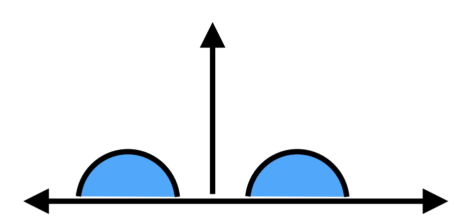
- In other examples the matrix $\mathbb{B}$ can be a **block** Toeplitz matrix and then the condition **sometimes** holds.

  - Multiple Hermite polynomials corresponding to external source models. *(D. unpublished)*



  - OPUC for measures supported on a single arc. *(D-Kozhan '16)*

  - OPRL in multi-cut cases it does not hold in general!



Unitary Ensembles on the circle

- Let μ be a Borel measure with an infinite number of points in its support. Consider the probability measure

$$\sim \prod_{1\le j<k\le n} |z_j - z_k|^2 d\mu(z_1)\cdots d\mu(z_n)$$

- Vandermonde determinant:

$$\sim \det\left(z_j^k\right)_{j,k=1}^n \det\left(z_j^{-k}\right)_{j,k=1}^n d\mu(z_1)\cdots d\mu(z_n)$$

- Which can be written as

$$\sim \det\left(z_j^{k-\lfloor\frac{n}{2}\rfloor}\right)_{j,k=1}^n \det\left(z_j^{-(k-\lfloor\frac{n}{2}\rfloor)}\right)_{j,k=1}^n d\mu(z_1)\cdots d\mu(z_n)$$

Orthogonalization

- Orthogonalization: Apply Gram-Schmidt to

$$\{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$$

  leading to a orthogonormal system

$$\{\phi_0(z), \phi_1(z), \phi_2(z), \ldots\}$$

- Then the probability measure can be written as

$$\frac{1}{n!} \quad \det\left(\phi_k(z_j)\right)_{j,k=1}^{n} \det\left(\overline{\phi_k(z_j)}\right)_{j,k=1}^{n} d\mu(z_1) \cdots d\mu(z_n)$$

CMV matrices

- Verblunsky coefficients: There exists $\{\alpha_n\}_{n=1}^{\infty}$ with $|\alpha_n| < 1$ such that

$$z \begin{pmatrix} \phi_0(z) \\ \phi_1(z) \\ \vdots \end{pmatrix} = \mathcal{C} \begin{pmatrix} \phi_0(z) \\ \phi_1(z) \\ \vdots \end{pmatrix}$$

where $\mathcal{C}$ is the CMV matrix *(Cantero-Moral-Valezques '03)*

$$\mathcal{C} = \begin{pmatrix} \bar{\alpha}_0 & \bar{\alpha}_1\rho_0 & \rho_0\rho_1 & 0 & 0 & 0 & \cdots \\ \rho_0 & -\alpha_0\bar{\alpha}_1 & -\alpha_0\rho_1 & 0 & 0 & 0 & \cdots \\ 0 & \bar{\alpha}_2\rho_1 & -\alpha_1\bar{\alpha}_2 & \bar{\alpha}_3\rho_2 & \rho_2\rho_3 & 0 & \cdots \\ 0 & \rho_1\rho_2 & -\alpha_1\rho_2 & -\alpha_2\bar{\alpha}_3 & -\alpha_2\rho_3 & 0 & \cdots \\ 0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\alpha_3\bar{\alpha}_4 & \bar{\alpha}_5\rho_4 & \cdots \\ 0 & 0 & 0 & \rho_3\rho_4 & -\alpha_3\rho_4 & -\alpha_4\bar{\alpha}_5 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$\rho_n = \sqrt{1 - |\alpha_n|^2}$$

Example

- CUE: If $\mathrm{d}\mu(e^{\mathrm{i}\theta}) = \dfrac{d\theta}{2\pi}$ then $\alpha_n \equiv 0$

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

- **Theorem** *(Rakhmanov)*
  If $d\mu(e^{\mathrm{i}\theta}) = w(\theta)d\theta + d\mu_s(e^{\mathrm{i}\theta})$ and $w(\theta) > 0$ a.e. on $[0, 2\pi]$ then

$$\alpha_n \to 0, \qquad \text{as} \qquad n \to \infty$$

Thus, under that condition

$$(\mathcal{C})_{n+k,n+\ell} - (\mathcal{C}_{CUE})_{n+k,n+\ell} \to 0$$

Theorem (D-Kozhan 16)

Let $d\mu(e^{\mathrm{i}\theta}) = w(\theta)d\theta + d\mu_s(e^{\mathrm{i}\theta})$ with $w(\theta) > 0$ a.e. on $[0, 2\pi]$

Then for any $h(z) = \sum_{j=-\infty}^{\infty} h_j z^j$ such that $h_j = \overline{h_{-j}}$ and

$$\sum_j \sqrt{j}|h_j| < \infty$$

we have

$$X_n(h) - \mathbb{E}X_n(h) \to N\left(0, 2\sum_{j=1}^{\infty} |h_j|^2\right)$$

Some words on the proof

- We start with $h_N(z) = \sum_{j=-N}^{N} h_j z^j$
- Then $h_N(\mathcal{C})$ is bounded and banded

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}t\sum_{j=1}^{n} h_N(z_j)}] = \det P_n \mathrm{e}^{\mathrm{i}t h_N(\mathcal{C})} P_n$$

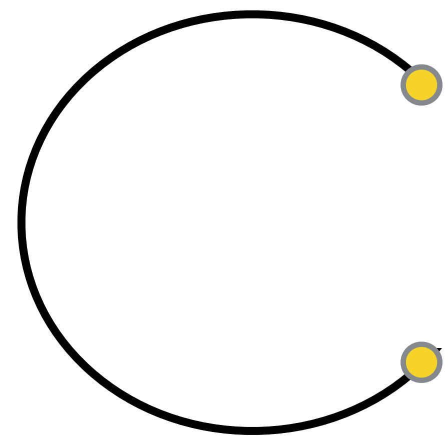
- Under the condition we know

$$(h_N(\mathcal{C}))_{n+k,n+\ell} - (h_N(\mathcal{C}_{CUE}))_{n+k,n+\ell} \to 0$$

- For $h_N(\mathcal{C}_{CUE})$ we use either the well-known results for CUE or apply the condition of our theorem and obtain the statement for h_N
- We can then get the result for $h = \lim_{N\to\infty} h_N$ by applying the concentration inequality.

Measure on an arc

- If $\alpha_n \equiv \alpha$ then the measure is a.c. and supported on an arc.



$$\left\{ \mathrm{e}^{\mathrm{i}\theta} \mid \theta \in [\phi, 2\pi - \phi) \right\},$$

$$\phi = 2 \arcsin |\alpha|.$$

- **Theorem** *(Bello-Lopez '98, Simon '05)*
  If $d\mu(e^{\mathrm{i}\theta}) = w(\theta)d\theta + d\mu_s(e^{\mathrm{i}\theta})$ has $\sigma_{ess}(\mu) = \Gamma_\phi$ and $w(\theta) > 0$ a.e. on $[0, 2\pi]$ then

$$\begin{cases} \lim_{n\to\infty} |\alpha_n| = |\alpha| \\ \lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1 \end{cases}$$

Theorem *(D.-Kohzan '16)*

Let $d\mu(e^{\mathrm{i}\theta}) = w(\theta)d\theta + d\mu_s(e^{\mathrm{i}\theta})$ such that $\sigma_{ess}(\mu) = \Gamma_\phi$ and $w(\theta) > 0$ a.e. on : Γ_ϕ

Then for any $h(z) = \sum_{j=-\infty}^{\infty} h_j z^j$ such that $h_j = \overline{h_{-j}}$ and

$$\sum_j \sqrt{j}|h_j| < \infty$$

we have

$$X_n(h) - \mathbb{E}X_n(h) \to N(0, Q_\alpha(h))$$

The function $Q_\alpha(h)$ is explicit, but slightly complicated.

Thank you for your attention!