Infinite Birkhoff Sums: Rates of Growth and Dynamical Borel-Cantelli Theorems.

Meagan Carney and Matthew Nicol (UH)

Partly supported by NSF-DMS 1600780

CIRM

February 24, 2017

Suppose (T, X, μ) is an ergodic probability measure preserving transformation and $\phi : X \in \mathbb{R}$ is a non-integrable non-negative function.
Let

$$S_n := \sum_{i=1}^{n} \phi \circ T^i$$

be the Birkhoff sum.

A natural question is the rate of growth of Birkhoff sums of non-integrable functions.

Proposition (Aaronson (1977))

For any sequence $b(n) > 0$*, if* $\lim_{n\to\infty} \frac{b(n)}{n} = \infty$ *then either*

$$\limsup \frac{S_n}{b(n)} = \infty \ a.e.$$

or

$$\liminf \frac{S_n}{b(n)} = 0 \ a.e.$$

A useful result, due again to Aaronson, states:

Proposition

If $a(x)$ is increasing, $\lim_{x\to\infty} \frac{a(x)}{x} = 0$ *and*

$$\int a(\phi(x))d\mu < \infty$$

then for μ a.e. x

$$\lim_{n\to\infty} \frac{a(S_n)}{n} = 0$$

This result gives bounds on $\limsup S_n$.

The relation between Birkhoff sums and extreme values, such as the maxima, is investigated in the topic of trimmed Birkhoff sums. In this approach the time series $\{\phi(x), \phi(Tx), \phi(T^2x), \ldots, \phi(T^nx)\}$ is rearranged into increasing order $\{\phi(T^{i_0}x) \leq \phi(T^{i_1}x) \leq \phi(T^{i_2}x) \leq \ldots \phi(T^{i_n}x)\}$ so that $\phi(T^{i_n}x) = M_n(x)$.

We will this denote this rearrangement by $\{M_0^n(x), M_1^n(x), \ldots, M_n^n(x)\}$ so that $M_n(x) = M_n^n(x)$.

Almost sure limit theorems for trimmed sums involve two sequences of constants $a(n)$,$b(n)$ so that the scaled truncated sum

$$\frac{1}{a(n)} \sum_{j=0}^{n-b(n)} M_j^n$$

satisfies a strong law of large numbers.
Recent work of Aaronson and Nakada (2003) and Schindler (2015), give precise information on the limiting behavior and choice of constants $a(n)$,$b(n)$ is given for certain dynamical systems.
Trimmed results make clear the relations between large extremal values of the time series and the behavior of the Birkhoff sum.

Aaronson, Kosloff and Weiss (2016) and Kosloff (2016) have interesting recent results on trimmed symmetric Birkhoff sums in the setting of infinite ergodic theory (when the underlying probability space has infinite measure).

Let

$$M_n := \max\{\phi(x), \phi(Tx), \phi(T^2x), \ldots, \phi(T^nx)\}$$

be the sequence of successive maxima $M_n(x)$ observed along a trajectory.
In extreme value theory literature the observables considered are often of form

$$\phi(x) = -\log d(x, q)$$

or

$$\phi(x) = d(x, q)^{-k} \ (k > 0)$$

.
Dynamical Borel-Cantelli lemmas have been used to give information on the almost sure behavior of the maxima M_n for certain classes of observables on a variety of chaotic dynamical systems (T, X, μ).

For many hyperbolic dynamical systems there is no almost sure limit for $\frac{M_n}{a(n)}$, in the case $\phi(x) = d(x, q)^{-k}$, $k > 0$, even if k is such that ϕ is integrable (so that a strong law of large numbers does hold for the Birkhoff sum).

Proposition (Holland, N., Török (2016))

Suppose that (T, X, μ) has an invariant ergodic measure μ which is absolutely continuous with respect to Lebesgue measure m. Suppose there exists $C > 0$ and $0 < \theta < 1$ such that for all ϕ of bounded variation and all $\psi \in L^1(m)$ we have:

$$\left| \int \phi \cdot \psi \circ f^j d\mu - \int \phi d\mu \int \psi d\mu \right| \leq C\theta^j \|\phi\|_{BV} \|\psi\|_{L^1(m)},$$

Assume for a point $q \in X$, $0 < \frac{d\mu}{dm}(q) < \infty$. Then if $\phi(x) = d(x, q)^{-k}$ ($k > 0$) for any monotone sequence $b(n) \to \infty$, either

$$\limsup_{n\to\infty} \frac{M_n(x)}{b(n)} = 0 \ a.e., \ or \ \limsup_{n\to\infty} \frac{M_n(x)}{b(n)} = \infty \ a.e. \tag{1}$$

We consider the observable

$$\phi(x) = d(x, q)^{-k}$$

over chaotic dynamical systems (T, X, μ) for values of k which ensure that $\int \phi d\mu = \infty$.
Most of our results generalize in an obvious way to a wider class of functions, for example those for which $\mu(\phi > t) = \frac{L(t)}{t^\gamma}$ where $0 < \gamma < 1$ and $L(t)$ is a slowly varying function, as long as the sets $(\phi > t)$ for large t correspond to sets which are topological balls.

Dynamical Borel Cantelli Lemmas

Assume (T, X, μ) is an ergodic dynamical system and X is a measure and metric space with a Riemannian metric d.
Let $B(q, r) := \{x : d(q, x) < r\}$ denote the ball of radius r about a point q.
Suppose that B_j is a sequence of nested sets in X based about a point q. Define

$$E_n = \sum_{j=1}^{n} \mu(B_j)$$

We say that the Strong Borel Cantelli (SBC) property holds for (B_j) if for μ a.e. $x \in X$

$$\sum_{j=1}^{n} 1_{B_j} \circ T^j(x) = E_n + o(E_n)$$

In the literature we often have a better estimate of the error term and, for any $\delta > 0$,

$$\sum_{j=1}^{n} 1_{B_j} \circ T^j(x) = E_n + O(E_n^{1/2+\delta}) \ (*)$$

If $(*)$ holds we say that the sequence (B_j) satisfies the QSBC property, or quantitative Strong Borel Cantelli property.
If $T^j(x) \in B_j$ infinitely often for μ a.e. we say that the sequence (B_j) has the Borel-Cantelli property.

Examples of systems for which the QSBC property has been proved for balls nested at points q in phase space include Axiom A diffeomorphisms (Chernov and Kleinbock (2001)), uniformly partially hyperbolic systems preserving a volume measure with exponential decay of correlations (Dolgopyat (2004)), uniformly expanding C^2 maps of the interval (Phillipp (1967)), and Gibbs-Markov type maps of the interval (Kim (2007)).

For intermittent type maps with an absolutely continuous invariant probability measure the work of Kim (2007) and Gouëzel (2007) gives a fairly complete picture: the Borel-Cantelli property holds for nested balls except those based at the indifferent fixed point.

Other results on dynamical Borel-Cantelli lemmas include work on one-dimensional maps modeled by Young towers with exponential decay of correlations (Gupta , N. and Ott (2010)) and general systems under mixing conditions (Haydn, N., Persson and Vaienti (2013)).

Relevant work in hyperbolic settings includes that of Galatolo and Kim (2007),Luzia (2014), Maucourant (2006) and Jaerisch, Kesseböhmer and Stratmann (2013).

Theorem

Suppose that (T, X, μ) *is an ergodic dynamical system with* $dim(X) = D$*. Let* $\phi(x) = d(x, q)^{-k}$ *for some distinguished point* q*. Suppose* $0 < C_1 < \frac{d\mu}{dm}(q) < C_2$ *and that the QSBC property holds for nested balls about* q*.*
If $k > D$ *then for any* $\epsilon > 0$

$$(a) \limsup \frac{S_n}{n^{k/D}[\log(n)]^{k/D+\epsilon}} = 0$$

and for any $\epsilon > 0$

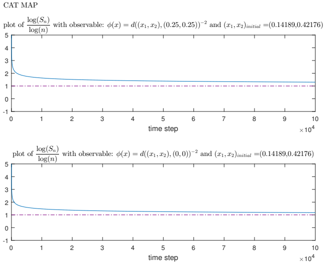
$$(b) \liminf \frac{S_n(e^{(\log n)^{\frac{1}{2}+\epsilon}})}{n^{k/D}} = \infty$$

Remark

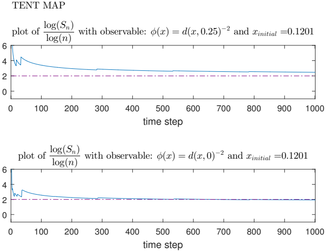
Recent work of Tanja Schindler (2015) on trimmed Birkhoff sums has shown that for Gibbs-Markov maps of the unit interval the limit infimum estimate d can be improved to

$$\liminf \frac{S_n(\log\log n)^{k-1+\epsilon}}{n^k} = \infty$$

.



Hyperbolic Toral Automorphism.



Tent Map.

Intermittent maps.

Theorem

Suppose $(T_\alpha, [0,1], \mu_\alpha)$ *is a Liverani-Saussol-Vaienti map with* $0 \le \alpha < 1$. *Let* $q \in [0,1]$ *and* $\phi(x) = d(x,q)^{-k}$ *with* $k \ge 1$. *Define* $S_n = \sum_{j=1}^{n} \phi \circ T^j$. *Then if* $q \neq 0$, *for any* $\epsilon > 0$

$$\liminf \frac{S_n(e^{(\log n)^{\frac{1}{2}+\epsilon}})}{n^k} = \infty$$

and

$$\limsup \frac{S_n}{n^k[\log(n)]^{k+\epsilon}} = 0$$

In particular

$$\lim_{n\to\infty} \frac{\log S_n}{\log n} = k$$

Theorem

If $q = 0$ *then for any* $\epsilon > 0$

$$\liminf \frac{S_n}{n^{k+\alpha-\epsilon}} = \infty$$

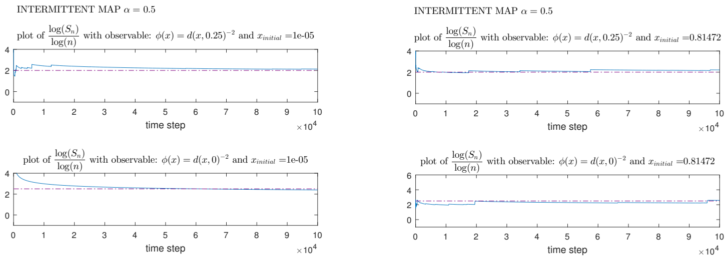
and

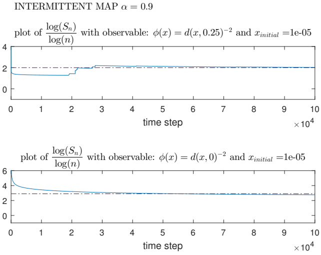
$$\limsup \frac{S_n}{n^{k+\alpha+\epsilon}} = 0$$

In particular

$$\lim_{n\to\infty} \frac{\log S_n}{\log n} = k + \alpha$$

Liverani-Saussol-Vaienti Map.





Predicted value of the limit is shown by a dotted line.

Exponential decay of correlations: hyperbolic systems.

Assumption (A): For all Lipschitz functions ϕ, ψ on X there exist constants $C, 0 < \theta < 1$ (independent of ϕ, ψ) such that

$$| \phi \, \psi \circ T^k d\mu - \int \phi d\mu \int \psi d\mu| < C\theta^k \|\phi\|_{\mathrm{Lip}} \|\psi\|_{\mathrm{Lip}}.$$

Under assumptions (A), if $\frac{d\mu}{dm} \in L^p$, $p > 1$, Haydn, N., Persson and Vaienti showed:
Suppose $\mu(B_i) \geq C \frac{\log^{\beta} i}{i}$ for some $\beta > 0$, then if $E_n = \sum_{j=1}^{n} \mu(B_j)$ for μ a.e. $x \in X$.

$$\sum_{j=1}^{n} 1_{B_j} \circ T^j(x) = E_n + O(E_n^{1/2+\epsilon})$$

for any $\epsilon > 0$.

Theorem

Suppose a dynamical system (T, X, μ) *satisfies (A),* $\frac{d\mu}{dm} \in L^p(m)$*, for* $p > 1$ *and* $q \in X$ *has density satisfying* $h(x) \sim Cd(q,x)^{-\beta}$*,* $1 > \beta > 0$*. Suppose also* $\dim(X) = D$*. Then if* $\phi(x) = d(x,q)^{-k}$*,* $k \geq D - \beta$*,*

$$\limsup \frac{S_n}{n^{k/(D-\beta)}[\log(n)]^{k+\epsilon}} = 0$$

and

$$\liminf \frac{S_n(e^{[\log(n)]^{\frac{1}{2}+\epsilon}})}{n^{k/(D-\beta)}} = \infty$$

for any $\epsilon > 0$*. Hence*

$$\lim n \to \infty \frac{\log S_n}{\log n} = \frac{k}{D-\beta}$$

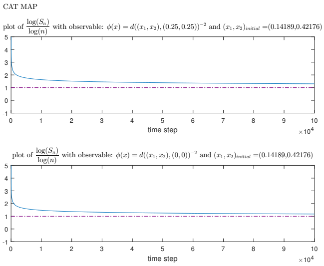
Corollary

Suppose $T(x) = 4x(1-x)$ *is a unimodal map of the interval* $[0, 1]$*. Let* $\phi(x) = d(x, q)^{-2}$*, then if* $q = 0$ *or* $q = 1$

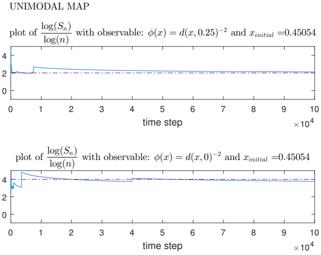
$$\lim_{n\to\infty} \frac{\log(S_n)}{\log n} = 4$$

while if $q \in (0, 1)$

$$\lim_{n\to\infty} \frac{\log(S_n)}{\log n} = 2$$



Hyperbolic Toral Automorphism.



Unimodal Map.

Predicted value of the convergence is marked in a dotted line.

Open problems.

Use techniques from trimmed sums on Gibbs-Markov maps to improve estimates, perhaps by Towers and induction?