Novikov's problem: when Interval Exchange Transformations are powerless

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Novikov's problem: the statement

In 1982 S. P. Novikov posed a problem of asymptotic behavior of plane sections of triply periodic surface.

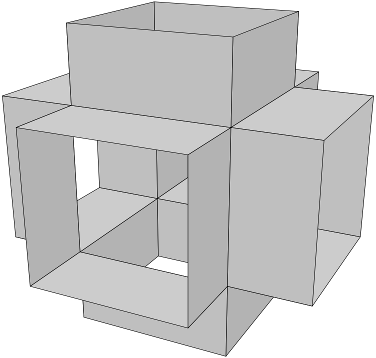
- $\hat{M}$ - triply periodic surface in $\mathbb{R}^3$;
- $\alpha_c : h_1x_1 + h_2x_2 + h_3x_3 = c$ - family of planes orthogonal to some fixed direction $H = (h_1, h_2, h_3)$;

Question: asymptotic behavior of unbounded components of $\hat{M} \cap \alpha_c$?

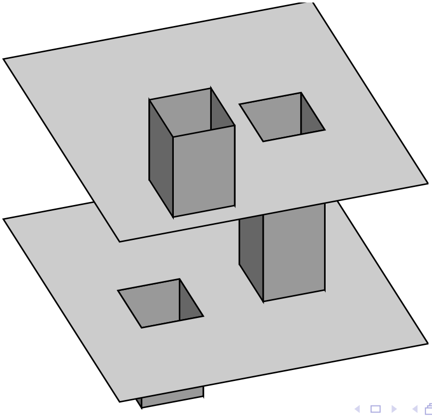
The same problem can be presented in terms of the measured foliations on the surface:

- $\pi : \mathbb{R}^3 \to \mathbb{T}^3$;
- $M = \pi(\hat{M})$ and F_ω is a foliation on M induced by 1-form $\omega = h_1dx_1 + h_2dx_2 + h_3dx_3$.

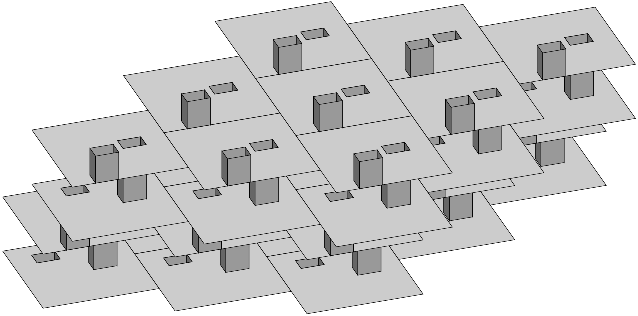
Triply periodic surface: fundamental domain - 1



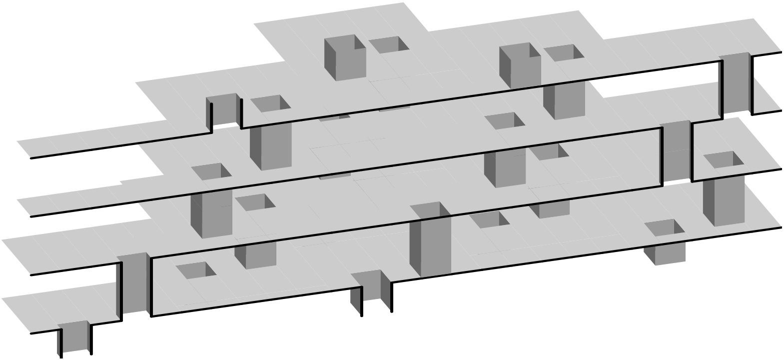
Triply periodic surface: fundamental domain - 2



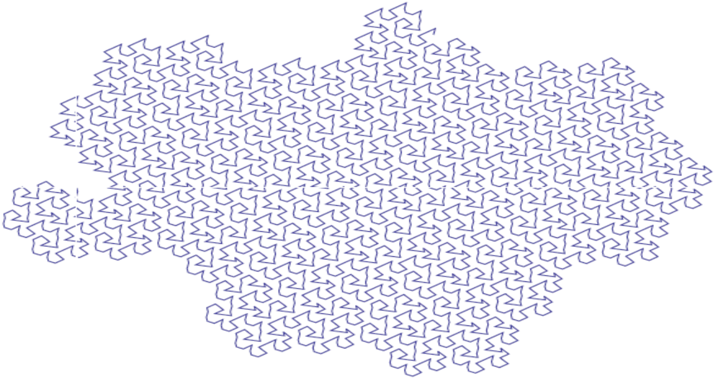
Triply periodic surface itself



Plane sections



Chaotic sections

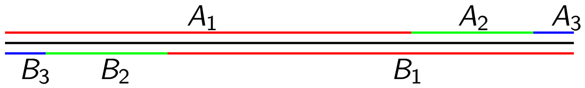


Interval Exchange Transformations: Definition

Definition

Let $I \subset \mathbb{R}$ be an interval and $\{A_i : i = 1 \cdots k\}$ be a partition of I. And *interval exchange transformation* is a bijective map $\phi : I \to I$ which is a translation of each A_i.

Restriction of map on each A_i is ϕ_i and $\phi_i(A_i) = B_i$. So, $\{B_i : i = 1 \cdots k\}$ is also a partition of I.
Two points $x, y \in I$ belong to the same *orbit* of IET if there exists a word consisting of ϕ_j and ϕ_j^{-1} that sends x to y.



Interval Exchange Transformations: Main Results

Definition

IET is *minimal* if every orbit is everywhere dense.

Generic IET is minimal (M. Keane, 1975)

Definition

IET is *uniquely ergodic* if it admits exactly one invariant probability measure.

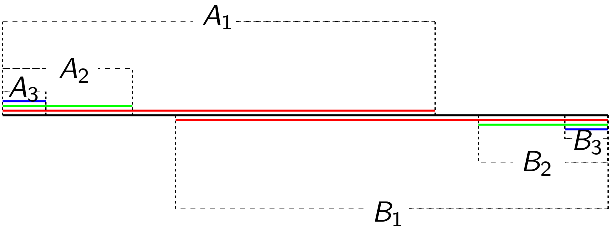
Almost every IET is uniquely ergodic (H. Masur - W. Veech, 1982).
One can associate with each IET a dynamical cocycle (Zorich cocycle).
Lyapunov spectrum of Zorich cocycle is simple (A. Avila - M. Viana, 2006).

Systems of isometries: a new hope

Systems of isometries were introduced by D. Gaboriau, G. Levitt and F. Paulin in 1994.

Definition

A *system of isometries* is a pair $S = (D, \{\phi_j\}_{j=1,\cdots,k})$ where D is a multi-interval and each $\phi_j : A_j \to B_j$ is an isometry between closed subintervals of D.



Systems of isometries of thin type

Definition

System of isometries S is called *balanced* if the following hold:

- all ends of D are covered by some subintervals A_i or B_i;
- $\Sigma_{i=1}^{n}|A_i| = |D|$, where $|A|$ means length of subinterval A;

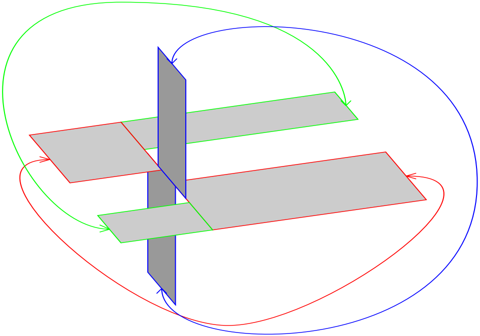
Definition

Balanced system of isometires is of *thin type* if all orbits are everywhere dense.

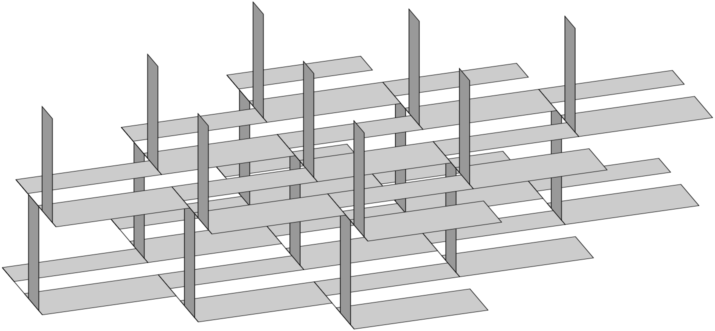
Thin case was discovered by G. Levitt in terms of $\mathbb{R}$-trees in 1993.
Main conjecture:
(S.P. Novikov - A. Maltsev, 2003; I. Dynnikov, 2008)
Thin systems of isometries form a zero measure set in parameter space.

2-dimensional foliated complex: suspension construction

There exists a kind of a suspension of systems of isometries that provides us with a foliated 2-complex (X, ω). Leaves of foliations correspond to orbits of systems.

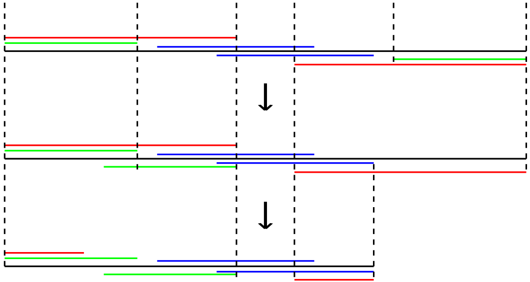


From systems of isometries to Novikov's problem



Rauzy induction: translation + reduction

We say that an system of isometry has a *hole* if there are some points in the support interval that are not covered by an interval from S. Rauzy induction stops when we obtain a hole.

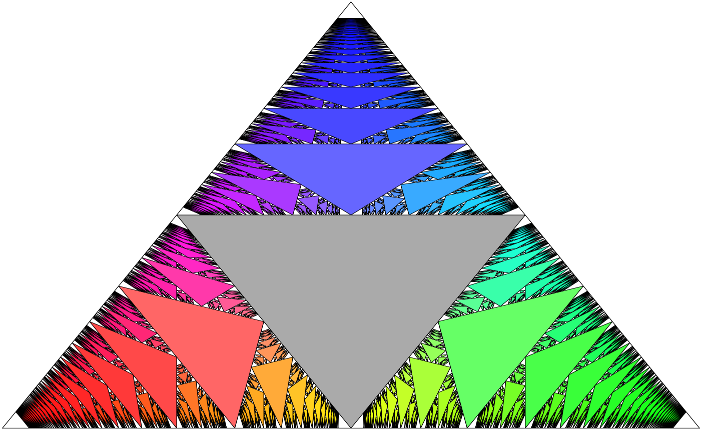


Baby Example: the Rauzy Gasket

Let us start from the easiest, two-dimensional case: $D = [0, 1]$, all A_i start in 0, all B_i end in 1.

- Triangle as a parameter space;
- Matrix of Rauzy induction has many in common with fully subtractive algorithm: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;
- Markov Partition (and therefore countable Markov shift);
- Contracting and Bounded Distortion Properties.

The Rauzy Gasket: Photo



The Rauzy Gasket: Bio

- **Date of Birth**: 2011 (2008, 1993)
- **Creators**: P. Arnoux and S. Starosta
- **Motivation**: episturmian words, multidimensional fraction algorithms
- **Alternative origin**:
  - I. Dynnikov and R. De Leo (3-dimensional topology: Novikov's problem);
  - G. Levitt (geometric group theory: pseudogroups of rotations);

The Rauzy Gasket: Metric Characteristics

Theorem

(G. Levitt and J.-C. Yoccoz, I. Dynnikov and R. De Leo, P. Arnoux and S. Starosta): The Rauzy Gasket has zero Lebesgue measure.

Open question (P. Arnoux): What about Hausdorff dimension?
Numerical estimations (I. Dynnikov and R. De Leo, 2008): between 1.7 and 1.8.

Theorem

(Artur Avila, Pascal Hubert and SS, 2013): The Hausdorff dimension of the Rauzy gasket is strictly less than 2.

The Cocycle and the Roof Function

- Accelerated matrix A:

$$\begin{pmatrix} n & 1 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

- Matrix of the cocycle B:

$$\begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ n & 0 & 1 \end{pmatrix}$$

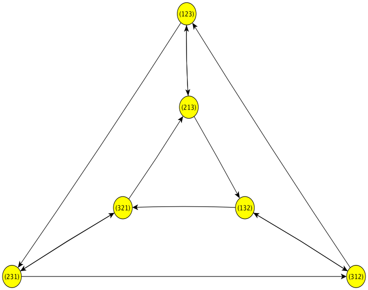
- the roof function: the first return time to the small simplex.

Theorem

The roof function r has exponential tails: there exists $\sigma > 0$ such that $\int_\Delta e^{\sigma r} dLeb < \infty$.

Combinatorics

The Rauzy graph illustrates that we work with a countable Markov shift:



Moreover, the shift is strongly recurrent (satisfies BIP property).

Key dynamical properties

Theorem (A. Avila - P. Hubert - SS, 2014)

There exists the measure of maximal entropy for the suspension flow of the Rauzy gasket, and this measure is unique.

The proof is based on thermodynamical formalism (O. Sarig' 2002).
Abramov's formula then gives us an invariant measure μ for the Rauzy gasket.

More subtle ergodic characteristics

Theorem (A. Avila - P. Hubert - SS, 2015)

Lyapunov spectrum of the cocycle described above is simple and Pisot.

Theorem (I. Dynnikov - P. Hubert - SS, 2016)

Almost all foliations associated with the Rauzy gasket are uniquely ergodic; however, non of them are weakly mixing.

So, systems of isometries are very similar to IET in some aspects and show completely opposite behavior - in some others.

What does it mean for Novikov's problem?

Theorem (Artur Avila, Pascal Hubert and SS, 2014)

The diffusion rate of the trajectories for almost all chaotic regimes (with respect to the measure μ) that correspond to the Rauzy gasket is strictly between $\frac{1}{2}$ and 1 :

$$\frac{1}{2} < \limsup_{t\to\infty} \frac{\log d(x, x_t)}{\log t} < 1,$$

where $d(x, y)$ is the standard distance between points x and y on the plane.

Thank you very much!