Sébastien's first steps

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1. Connection between rank and symbolic complexity.
2. Application to spectral theory of substitutions.
3. Application to diophantine approximation.

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2. Which functions from $\mathbb{N}^*$ to $\mathbb{N}^*$ are complexity functions ?

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Then, he considers $\zeta_2 : 0 \to 0012,\ 1 \to 12,\ 2 \to 012$ a primitive substitution. The minimal system $X(\zeta_2)$ and the Chacon's system generated by $\zeta_1^\infty(0)$ (minimal too) are topologically conjugate and

$$p_2(n) = 2n + 1.$$

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Theorem (S3)

Minimal systems with sub-linear complexity are generated by a finite number of substitutions (or are S-adic systems). More precisely, there exist a finite set of substitutions $(\sigma_j,\ 1 \leq j \leq r)$, *S, on the alphabet* $D = \{0, \ldots, d-1\}$, *a map* $\pi : D \to A$ *and an infinite sequence* $(j_n), 1 \leq j_n \leq r$ *such that*

$$\inf_{0 \leq a \leq d-1} |\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_n}(a)| \to \infty$$

and every word of the language occurs in some $\pi\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_n}(0),\ n \geq 1$.

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It was known for sturmian sequences with $r = 2$, Arnoux-Rauzy sequences with $r = 3$ and an interpretation of (j_n).

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If $x = (x_1, x_2, \dots)$,

$$\tau(x) = (1, x_2, \dots) \textit{ if } x_1 = 0,$$
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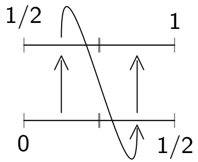
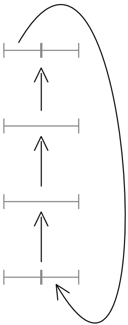


and then, by cutting and stacking,



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A Rokhlin tower is a collection $(T^j F)_{j=0}^{h-1}$ *of disjoint sets (*F *is the basis,* h *the height).*

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For an ergodic transformation T *preserving the finite measure* μ*, for every* $\varepsilon > 0$*, there exists a tower of total measure* $> \mu(X) - \varepsilon$*.*

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2.*The system (or T) has rank one if there exists a sequence of towers $(\{T^j F_n\}_{j=0}^{h_n-1})_n$ generating the σ-algebra :*

$$\forall A \in \mathcal{B}, \exists A_n \text{ union of levels of the n-th tower s.t. } \mu(A \Delta A_n) \to 0.$$

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So is every ergodic translation on a compact group (del junco 1976).

Connection between rank and complexity

Standard model of a rank one system :

- Stage one : F_1 is the basis ;
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Theorem (S2)

Consider a sequence taking its values in a finite alphabet and the associated system with complexity function p.
1. If it is a rank one system, then $\liminf_{n\to\infty} p(n)/n^2 \leq 1/2$ *(with possible sub-exponential peaks).*
2. If the system is minimal and $p(n) \leq an + b$ *for some* $a \geq 1$*, then the rank of the system is* $\leq 2[a]$.

Extensions and conjectures

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- The S-adic conjecture states that, for a minimal system,

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- A hudge amount of results are devoted to S-adic words and systems (Berthé, Delecroix, Leroy,...).
- Link between the complexity of a sequence (system) and its mixing properties. The Chacon system was the first example of weakly mixing not mixing system.

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$$\sigma_m = w^* - \lim_N \prod_{n \leq N} |P_n(e^{2i\pi t})|^2 \cdot \lambda,$$

where $P_n(z) = \frac{1}{\sqrt{p_n}} \sum_{j=0}^{p_n - 1} z^{-(jh_{n-1} + \sum_{k \leq j} s_{n,k})}$, *$h_n$ height of the n-th tower.*
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3. In any case, the spectrum is singular if $(1/p_n) \notin \ell^2$ (Klemes-Reinhold).
- Does there exist a Lebesgue rank one system ?
Banach's question : does there exist a Lebesgue simple spectrum ?

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- Weak mixing may occur but strong mixing never occurs.
- Substitution systems are finite rank systems (with sublinear complexity) :

Proposition

For every $n > 0$, we set

$$\mathcal{P}_n = \{T^k(\zeta^n[\alpha]), \quad \alpha \in A, \ 0 \leq k < |\zeta^n(\alpha)|\}.$$

1. $\mathcal{P}_n$ is a metric partition of X.
2. The σ-algebra generated by $\mathcal{P}_n$ increases to the σ-algebra on X.

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- Fabien Durand obtained a characterization of substitutive sequences : the set of "derivated" sequences must be finite.
- In a standard rank one system $(p_n, s_{n,j})$, the return times from F_n over F_{n-1} already appear in the Riesz product : $jh_{n-1} + \sum_{k \leq j} s_{n,k}$.

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Theorem (F-M-N)

Let ζ be a primitive aperiodic substitution ; the complex number λ of modulus 1 is an eigenvalue of ζ if and only if

$$\lambda^{r_n(C)} \to 1 \ \ \forall \ C \text{ return word.}$$

Eigenvalues of a substitution

Host (1986) and Livshits (1988–separately) gave a necessary and sufficient condition for a complex number to be an eigenvalue of a primitive aperiodic substitution, with computation in some classes of examples.
Sébastien, Christian Mauduit and Arnaldo Nogueira obtained a new formulation in terms of return words over some letter.

If $C = u_{[i,j-1]}$ is a return word over some letter, they define

$$r_n(C) = |\zeta^n(C)| = |\zeta^n(u_i)| + \cdots + |\zeta^n(u_{j-1})|.$$

Theorem (F-M-N)

Let ζ be a primitive aperiodic substitution ; the complex number λ of modulus 1 is an eigenvalue of ζ if and only if

$$\lambda^{r_n(C)} \to 1 \ \ \forall\ C \text{ return word.}$$

An algebraic criterion for such a substitution to be weak mixing is deduced.

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Then $\omega \in \mathbb{Q}(\theta)$ with θ the golden number; finally $\omega \in \mathbb{Z}\theta + \mathbb{Z}$.

Extensions and conjectures

1. Bufetov-Solomyak (2014) investigate the Hölder properties of spectral measures of substitutions in the general primitive and aperiodic case. The spectral study requires a matrix analogue of Riesz products (appeared in rank one systems) and return words.

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3. Weak mixing of S-adic systems and interval exchanges.

Diophantine approximation and transcendence

An infinite word w on the alphabet $A = \{0, 1, \ldots, q-1\}$ can be viewed as the q-adic expansion of some real number in $[0, 1)$

$$x_w = \sum_{k=1}^{\infty} w_k q^{-k} \text{ si } w = w_1 w_2 \cdots \in \{0, 1, \ldots, q-1\}^{\infty}.$$

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Roth's theorem in diophantine approximation gives the transcendence by truncation :

Theorem (Roth)

Let $\alpha \notin \mathbb{Q}$ and $\varepsilon > 0$ be such that

$$|\alpha - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}}$$

for infinitely many rational numbers p/q ; then α is a transcendental number.

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A *p*-adic version of Roth's theorem, due to Ridout, involves the arithmetic structure of the denominators in the rational approximants to the number.

Theorem (Ridout)

Let $p_1, p_2, \dots, p_k$ be k arbitrary prime numbers. If there exist $\varepsilon > 0$ and infinitely many rational numbers p/q such that

$$\Big(\prod_{i=1}^{k} |p|_{p_i} \prod_{i=1}^{k} |q|_{p_i}\Big)|\alpha - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}}$$

then α is a transcendental number.

Transcendence of low complexity expansions

Notation : If W is some word and $a \geq 1$ some positive integer ; W^a denotes the word $WW \cdots W$ with a repetitions ;
if $a \in]0, 1[$ is a rational number, W^a denotes the prefix of W of length $a|W|$.

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Theorem (F-M)

If the adic-expansion of the irrational number α begins, for every n, by $0.U_n V_n^s \cdots$ where $s > 2$, $|V_n| \to \infty$ and $|U_n|/|V_n|$ bounded, then α is a transcendental number.

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Consequences :

1. Sturmian numbers (on ℓ letters), A-R numbers, and some automatic numbers are transcendental numbers.
2. First estimate for the complexity of an algebraic number. If α is an algebraic irrational number, then, for any k, $\lim(p(n) - n) = +\infty$.

Extensions and conjectures

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3. (A-B 2011) Numbers with sub-linear complexity are either Liouville numbers, or S– or T–numbers in the Mahler classification.