Limiting curves for a class of self-similar adic transformations

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(based on a joint work with Andrei Lodkin)

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Let $(f_0, f_1, \ldots, f_n, \ldots)$ be a sequence with $f_i \in \mathbb{R}, i \in \mathbb{N}_0$.
Let F denote the partial sum, defined by

$$F(n) = \sum_{i=0}^{n-1} f_i, n \geq 0,$$

The function F is assumed to be linearly interpolated between consecutive integers.
Let the function $\varphi_n : [0,1] \to [0,1]$ be defined by

$$\varphi_n(t) = \frac{F(t \cdot n) - t \cdot F(n)}{R_n},$$

(The normalizing coefficient $R_n = \max_{t \in [0,1]} |F(t \cdot n) - t \cdot F(n)|$.)

Let (X, T) be a dynamical system, a function $f : X \to \mathbb{R}$ and a point $x \in X$.
É. Janvresse, T. de la Rue and Y. Velenik defined $f_i = f(T^i x), i \geq 0$, and considered cluster points in $C[0,1]$ of the set $\{\varphi_n\}_{n \geq 1}$. Any cluster point $\varphi = \varphi_{x,f}$ is called a *limiting function*.

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More formally: φ is a *limiting function* if there is a *stabilizing subsequence* $(l_n)_n$ such that $||\varphi - \varphi_{l_n}||_\infty \to 0$.

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Although, in general, limiting functions do not always exist and are never unique, they give an interesting insight to the fluctuations of ergodic sums (when an ergodic limit exists).

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Although, in general, limiting functions do not always exist and are never unique, they give an interesting insight to the fluctuations of ergodic sums (when an ergodic limit exists).
There are also interesting applications to combinatorics and number theory.

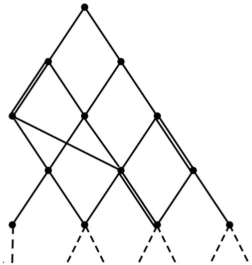


Figure: A graded graph (Bratteli diagram).

- The space X of infinite edge paths of some graded graph with some linear order on the incoming edges of each vertex. It is equipped with a partial order $\preceq$, which is lexicographical on the set of edge paths in X that belong to the same class of the tail partition.
- The adic transformation T is defined on $X \setminus \big(X_{\max} \cup X_{\min}\big)$ by sending $x \in X$ to its successor Tx, that is, the smallest y that satisfies $y \succ x$.

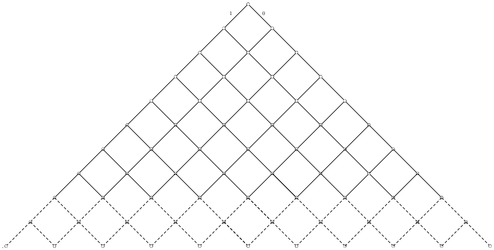
Let (X, T) be an adic transformation. This assumption is not restrictive due to the following theorem by A. M. Vershik:

Theorem

Any ergodic measure preserving transformation on a Lebesgue space is isomorphic to some adic transformation.

Let $\mathscr{F}_N$ denote the space of cylindric functions of rank N (i.e., functions that depend only on the first N coordinates of $x = (x_n)_0^\infty$).

The Pascal adic transformation



Let I be $\{0,1\}^\infty$ and μ_q be the dyadic Bernoulli measures $\prod_1^\infty (q, 1-q)$, $q \in (0,1)$.
We denote by P the Pascal adic transformation can be explicitly defined by:

$$x \mapsto Px; \quad P(0^{m-l}1^l\mathbf{10}\dots) = 1^l0^{m-l}\mathbf{01}\dots$$

(that is only the initial $m+2$ coordinates of x are being changed).

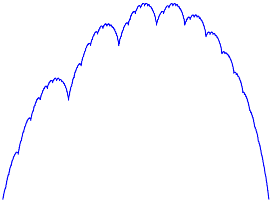
The measures $\mu_q, q \in (0,1)$, give the list of all invariant ergodic measures.

A function $g \in L^\infty(X,\mu)$ of the form $g = h \circ T - h + const$, $h \in L^\infty(X,\mu)$, is called *cohomologous to a constant*.

Theorem

(*É. Janvresse et.al., Theorem* 2.4.) *Let P be the Pascal adic transformation defined on the Lebesgue probability space $(X, \mathcal{B}, \mu_q), q \in (0,1)$, and g be a cylindric function from $\mathscr{F}_N$. Then for μ_q-a.e. x the limiting curve $\varphi_x^g \in C[0,1]$ exists if and only if g is not cohomologous to a constant.*

Example of the limiting curve, $q = 0.4$:



Main results

Limiting curves and cohomologous to a constant functions (necessary condition)

Self-Similar dynamical systems

Limiting curves for the Pascal adic transformation

- Illustration
- Transition regimes

Smooth limits of limiting curves

Limiting curves and cohomologous to a constant functions (necessary condition)

Theorem (É. Janvresse et al (2005), A.M.(2016))

- *If a continuous limiting curve* $\varphi_x^g = \lim_n \varphi_{x,l_n}^g$ *exists for* μ*-a.e.* x*, then the normalizing coefficients* R_{x,l_n}^g *are unbounded in* n*.*
- *The normalizing sequence* R_{x,l_n}^g *is bounded if and only if the function* g *is cohomologous to some constant.*

Self-Similar dynamical systems

Let $p(x)$ be a positive integer polynomial, for example, $p(x) = 1 + x + 3x^2$

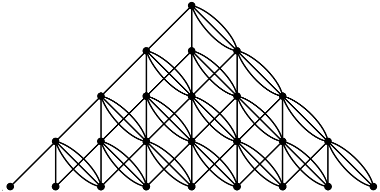


Figure: The graded graph associated to $p(x) = 1 + x + 3x^2$.

Self-Similar dynamical systems

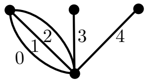


Figure: Canonical ordering for $p(x) = 1 + x + 3x^2$.

As for the Pascal adic, the set of all invariant ergodic measures for a polynomial system is a certain one-parameter family μ_q, $q \in (0, \frac{1}{a_0})$, of Bernoulli measures.

Denote by t_q the unique solution in $(0,1)$ of the equation

$$a_0 q^d + a_1 q^{d-1} t + \cdots + a_d t^d - q^{d-1} = 0.$$

X. Mela and S. Bailey showed that these measures are as follows:

$$\mu_q = \prod_0^\infty \left(\underbrace{q, \ldots, q}_{a_0}, \underbrace{t_q, \ldots, t_q}_{a_1}, \underbrace{\frac{t_q^2}{q}, \ldots, \frac{t_q^2}{q}}_{a_2}, \ldots, \underbrace{\frac{t_q^d}{q^{d-1}}, \ldots, \frac{t_q^d}{q^{d-1}}}_{a_d} \right).$$

Self-Similar dynamical systems

Theorem

Let (X, T, μ_q) be a polynomial system and g be a cylindric function from $\mathscr{F}_N$. Then for μ_q-a.e. x a limiting curve $\varphi_x^g \in C[0,1]$ exists if and only if the function g is not cohomologous to a constant.

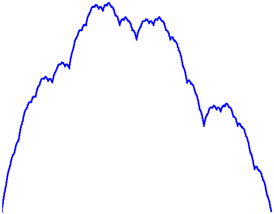


Figure: An example of a limiting curve

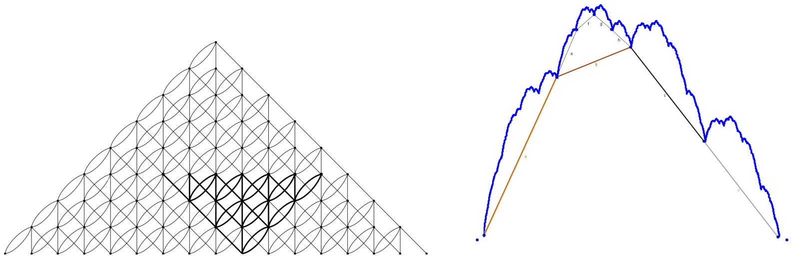


Figure: The Bratteli diagram and the polygonal approximation $p(x) = 2 + x + x^2$.

The distribution function of the measure μ_q

- We assume for simplicity that $p(x) = 1 + x$ (it is the Pascal adic case).
- The Bernoulli measure $\mu_q = \prod(q, 1-q)$ (if carried to $[0,1]$) can be defined by the distribution function $L_q(x) : [0,1] \to [0,1]$.

$$L_q : x = \sum_{k=1}^{\infty} \omega_k \frac{1}{2^k} \quad \mapsto \quad \sum_{k=1}^{\infty} \omega_k q^{k-s_{k-1}} (1-q)^{s_{k-1}},$$

where $s_k = \sum_{j=1}^{k} \omega_j, \ \omega_j \in \{0,1\}$.

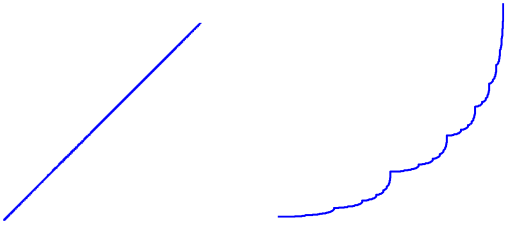


Figure: The graphs of $L_{0.5}$ (left) and $L_{0.3}$ (right)

A class of self-affine functions

Let q_1 and q_2 be distinct parameters from $(0,1)$. We consider the function $S_{q_1,q_2} : [0,1] \to [0,1]$ defined by $S_{q_1,q_2} = L_{q_2} \circ L_{q_1}^{-1}$. For $k \in \mathbb{N}$ we define the function $\mathcal{T}_q^k$ by the identity:

$$\mathcal{T}_q^k := \frac{\partial^k S_{q,a}}{\partial a^k}\bigg|_{a=q}, \ k \in \mathbb{N}.$$

The function $\frac{1}{2}\mathcal{T}_{1/2}^1$ is the famous Takagi function (M. Hata and M. Yamaguti).

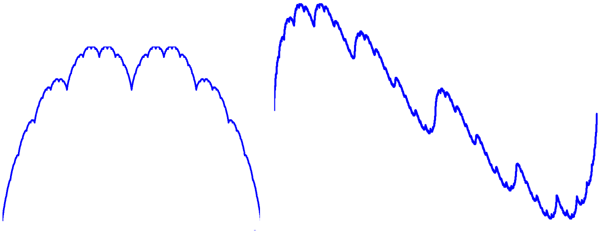


Figure: The graphs of $\mathcal{T}_{1/2}^1$ (left) and $\mathcal{T}_{1/2}^2$ (right)

A class of self-affine functions.

$$\mathcal{T}_q^k(q^i) = \frac{\partial^k}{\partial q^k} q^i = i(i-1)\dots(i-k)q^{i-k-1},$$

$i \in \mathbb{N}$, in particular,

$$\mathcal{T}_q^1(q^i) = iq^{i-1}, \qquad \mathcal{T}_q^2(q^i) = i(i-1)q^{i-2}.$$

Taking into account certain self-affinity relations, the functions $\mathcal{T}_q^k$ are uniquely defined by these values.

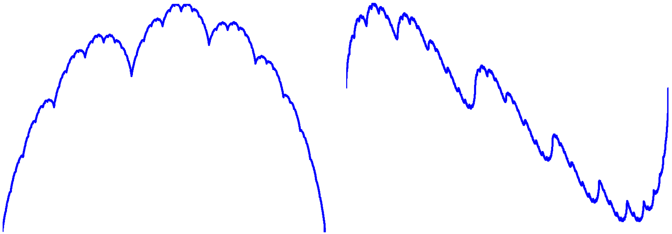


Figure: The graphs of $\mathcal{T}_{0.4}^1$ (left) and $\mathcal{T}_{0.4}^2$ (right)

A class of self-affine functions

Everything can be generalized for $p(x) = a_0 + a_1 x + \cdots + a_d x^d$:
As above, for $q_1,\ q_2 \in (0, 1/a_0)$, functions $S^p_{q_1,q_2} : [0,1] \to [0,1]$ can be defined.
Similarly,

$$\mathcal{T}^k_{p,q_1} := \left. \frac{\partial^k S^p_{q_1,q_2}}{\partial q_2^k} \right|_{q_2=q_1},\ \ k \in \mathbb{N}.$$

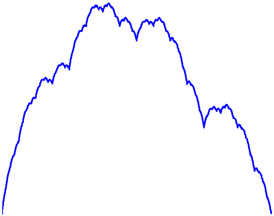
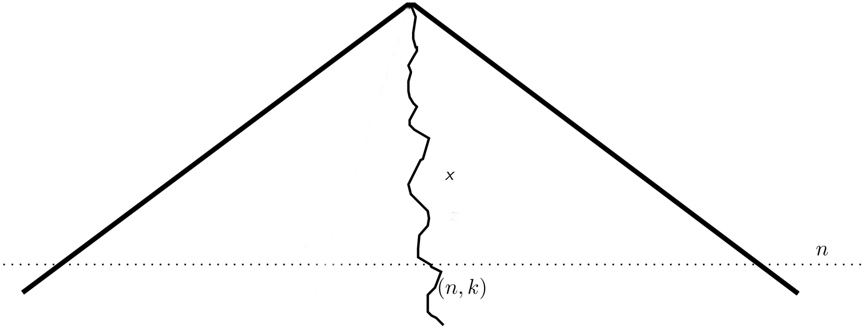


Figure: The graph of $\mathcal{T}^k_{p,q_1}$,$p(x) = 2 + x + x^2$.

Let's return to the Pascal adic

Let (I, P, μ_q) be the Pascal adic, $x \in I$ define the infinite path passing through the vertices $(n, k_n(x))$ of the Pascal graph, a stabilizing sequence can be chosen as $l_n = \binom{n}{k_n}$.



How to find the limiting curves explicitly?

Let $g \in \mathcal{F}_N$ be a cylindric function.
In order to find the limiting curve φ along the sequence (n, k_n) (we write simply (n, k)), we need

1. to represent the function g in the form $g = \sum_{j=0}^{2^N-1} c_j w_t^q$ for some convenient basis $\{w_t^q\}$,
2. to evaluate the partial sums $F_{n,k}$ at the points $x_{i,k,n} = \binom{n-i}{k-i}$, $i \in \mathbb{N}$, for each function w_t^q,
3. to find the leading term $R_n \varphi(q^i)$ of the asymptotic expansion

$$F_{n,k}(x_{i,k,n}) - \frac{x_{i,k,n}}{x_{0,k,n}} \cdot F_{n,k}(x_{0,k,n}) = R_n \cdot \varphi(q^i) + o(R_n)$$

(since $\lim_n \frac{x_{i,k,n}}{x_{0,k,n}} = q^i$ with $n \to \infty$, $\frac{k_n}{n} \to q$)

1. We take the Walsh-Paley functions $\{w_t^q\}$ (orthogonalized in $L^2_{\mu_q}$) for the basis in question.
2. Let $K_m(k, q, n)$ denote the *Krawtchouk polynomials* of a discrete variable k, defined by the identity

$$K_m(k, q, n) = {}_2F_1\left[\begin{matrix} -k, & -m \\ & -n \end{matrix}; \frac{1}{q}\right] \tag{1}$$

where ${}_2F_1$ is the Gauss hypergeometric function.
For the function $w_t^q, 0 < t < 2^N$, the partial sum $F_{n,k}(x_{i,k,n})$ is expressed as:

$$F_{n,k}(x_{i,k,n}) = (-2q)^m K_m(k - i, q, n - i) \cdot x_{i,k,n}, \tag{2}$$

with $m = \mathbf{s}_2(t)$.
We study the asymptotic behavior of $F_{n,k}$ for $n \to \infty, \frac{k}{n} \to q$.

Let $n \to \infty$, and $\xi = \frac{k-nq}{\sqrt{nq(1-q)}} = O(1)$.

- There is a classical asymptotic expansion of the Krawtchouk polynomial

$$K_m(k,q,n) = b_0 H_m(\xi) + O(n^{-(m+1)/2}),$$

where $H_m(x) \equiv (-1)^m e^{x^2} \frac{d^m}{dx^m} e^{-x^2}$ is the Hermite polynomial and $b_0 = (-1)^m \left(\frac{pq}{2n}\right)^{m/2}$.

- There are also expansions by N. Temme and J. Lopez (with asymptotic property)

$$K_m(k,q,n) = \sum_{j=0}^{m} b_j(\xi) H_{m-j}(\xi)$$

For instance, we have

$$K_m(k,q,n) = b_0 H_m(\xi) + b_3(\xi) H_{m-3}(\xi) \frac{1}{n} + o(n^{-(m+2)/2})$$

Limiting curves for the Pascal adic transformation

Theorem

Let P be the Pascal adic transformation of the Lebesgue probability space $(I, \mathcal{B}, \mu_q)$, $N \in \mathbb{N}$, and $g \in \mathscr{F}_N$ be a function that is not cohomologous to a constant. Then for μ_q-a.e. x there exists a stabilizing sequence $l_n(x)$ such that the limiting function is $\alpha_{g,x}\mathcal{T}_q^1$, where $\alpha_{g,x} \in \{-1, 1\}$.

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Technically, for almost every sequence $(n, k_n(x))$ (in the sense of the μ_q measure) using the classical expansion of the Krawtchouk polynomials, we show that for the functions w_t^q with $m = \mathbf{s}_2(t)$

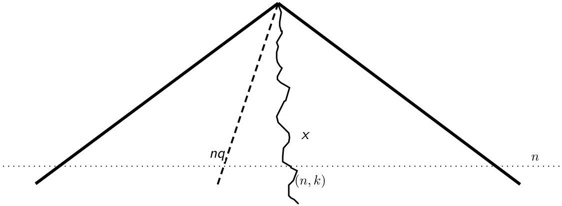
$$F_{n,k}(x_{i,k,n}) - \tfrac{x_{i,k,n}}{x_{0,k,n}} \cdot F_{n,k}(x_{0,k,n}) = iq^{i-1}R_{n,k} + o(R_{n,k})$$

$$R_{n,k} = \beta H_{m-1}(\xi),$$

where $\beta = \beta(m, n, k, q)$, provided ξ is not the root of H_{m-1}.

Limiting curves for the Pascal adic transformation

Illustration



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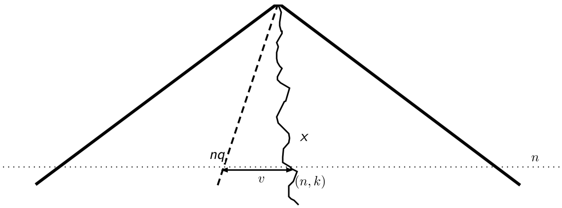

$$v = k - nq,$$

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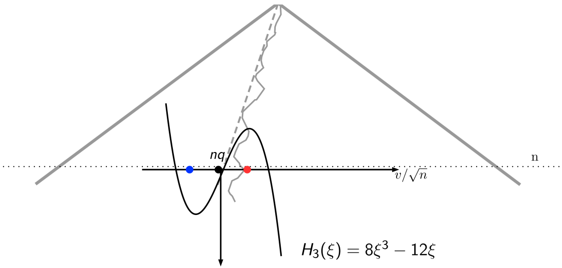

$$v = k - nq,\ \xi = \frac{k-nq}{\sqrt{2q(1-q)n}} = \frac{v}{\sqrt{2q(1-q)n}},\ \ m = 4.$$

Limiting curves for the Pascal adic transformation

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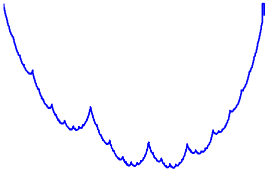
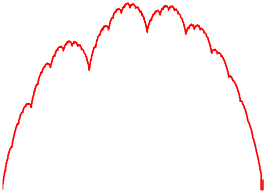


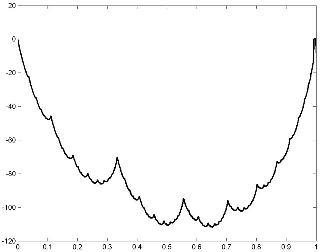


Limiting curves for the Pascal adic transformation

Transition regimes

What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$



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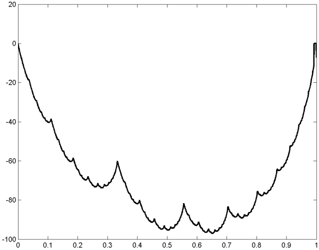

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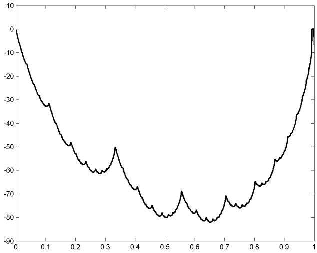

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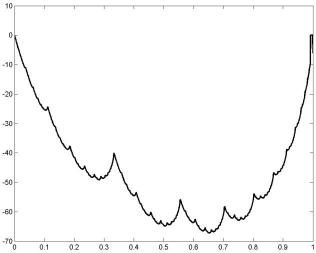

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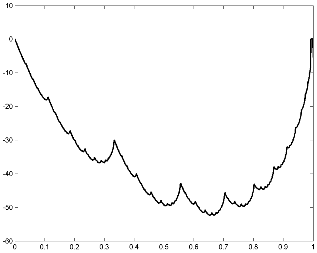

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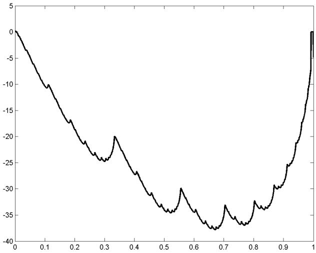

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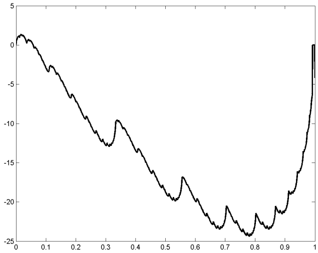

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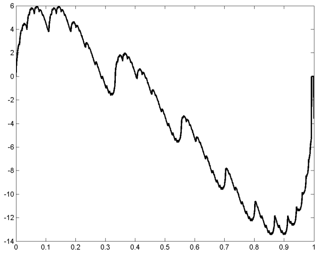

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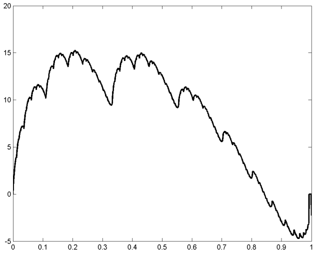

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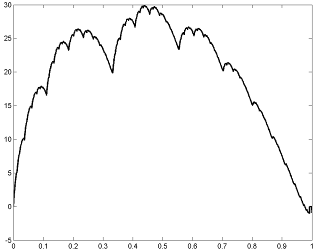

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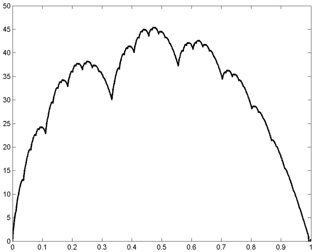

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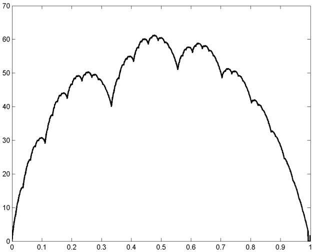
What if $v = O(1)$?

$$\alpha_v \mathcal{T}_q^1 + \beta_v \mathcal{T}_q^2$$



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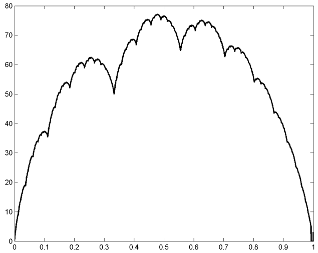

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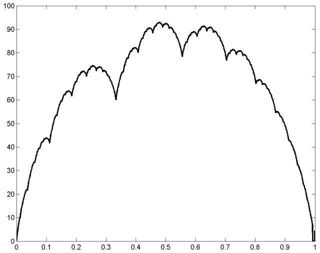

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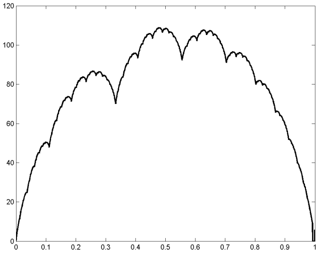

Smooth limits of limiting curves

We answer the question by É. Janvresse et.al: is there a smooth curve in the limit?

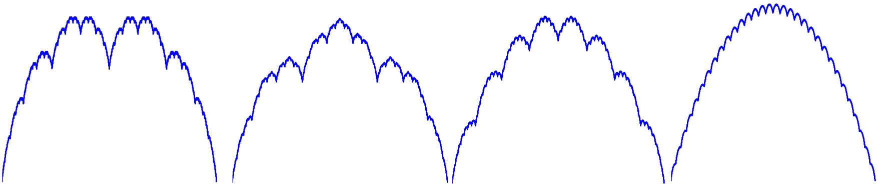

Figure: Limiting curves observed for the polynomial adic transformations associated with polynomial $p(x) = 1 + x + x^2 + \cdots + x^d$ for (from left to right): $d + 1 = 2, 3, 8, 32$ and symmetric measure.

Smooth limits of limiting curves

Statement:

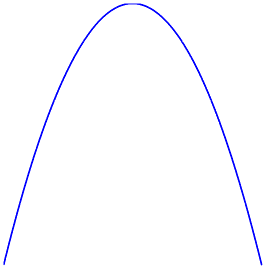

Figure: Limiting curve is parabola.

Question:
Is there a dynamical counterpart (a certain generalization of the self-similar adic transformations with infinite degree of each vertex)?

É. Janvresse, T. de la Rue, and Y. Velenik, *Self-similar corrections to the ergodic theorem for the Pascal-adic transformation*, Stoch. Dyn., 5:1 (2005), 1–25.

A. A. Lodkin,A. R. Minabutdinov, *Limiting Curves for the Pascal Adic Transformation*, Transl: J. Math. Sci.(N.Y.), 216:1 (2016), 94–119.

A. R. Minabutdinov, *Limiting curves for polynomial adic systems*, arXiv:1701.07617