From dual substitutions to Rauzy fractals

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New Advances in Symbolic Dynamics

Previously on New Advances in SD

From Thierry Monteil's lecture

It allowed me to survive a lot of talks on Rauzy Fractals

In the 80's , G. Rauzy introduces the Rauzy fractal associated with the Tribonacci substitution

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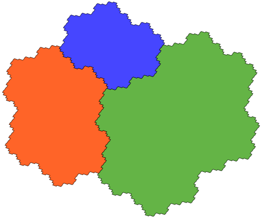

In 1991 S. Ito and M. Kimura write the seminal paper On Rauzy fractal on the generation of the boundary of the Rauzy fractal via free group morphisms by using Dekking's method

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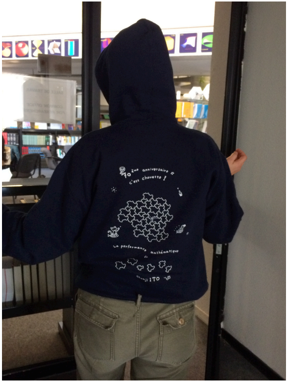
and then in 2000
P. Arnoux and S. Ito introduce in the formalism of

Dual Substitutions....

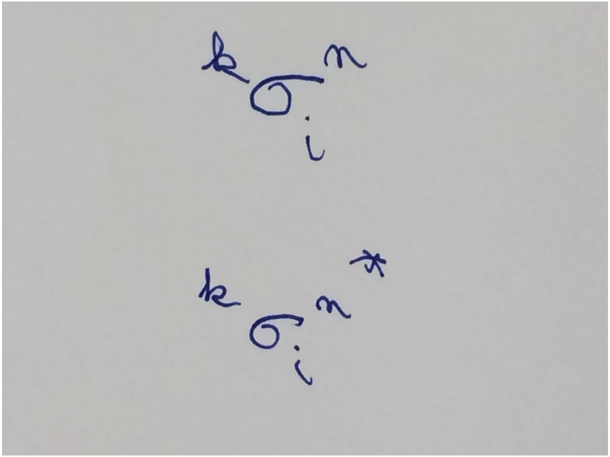
How to survive a talk on

Dual Substitutions?

DS are fun and fashion



DS have a nice name



OK... why not trying $\hat{E}_1$?


Layla Rose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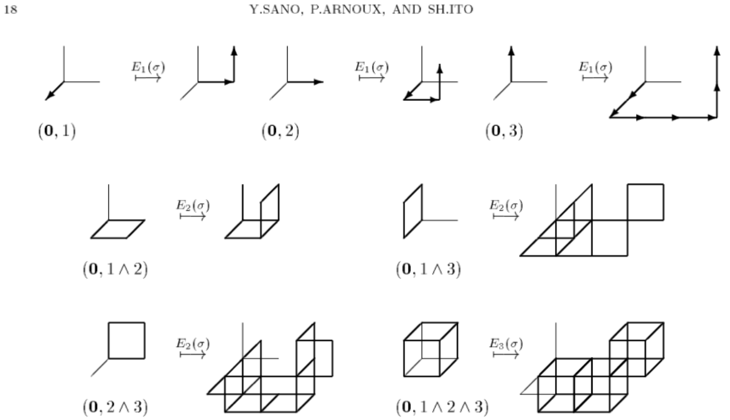
PUMA

Genesis of a star



DS like diagrams

The following commutative diagram holds:

$$
\begin{array}{ccccc}
 & \mathcal{G}_0^* & \xrightarrow{E_0^*(\sigma)} & \mathcal{G}_0^* & \\
\delta_1^* & \downarrow & & \downarrow & \delta_1^* \\
 & \mathcal{G}_1^* & \xrightarrow{E_1^*(\sigma)} & \mathcal{G}_1^* & \\
\delta_2^* & \downarrow & & \downarrow & \delta_2^* \\
 & \vdots & & \vdots & \\
\delta_{d-1}^* & \downarrow & & \downarrow & \delta_{d-1}^* \\
 & \mathcal{G}_{d-1}^* & \xrightarrow{E_{d-1}^*(\sigma)} & \mathcal{G}_{d-1}^* & \\
\delta_d^* & \downarrow & & \downarrow & \delta_d^* \\
 & \mathcal{G}_d^* & \xrightarrow{E_d^*(\sigma)} & \mathcal{G}_d^*. &
\end{array}
$$

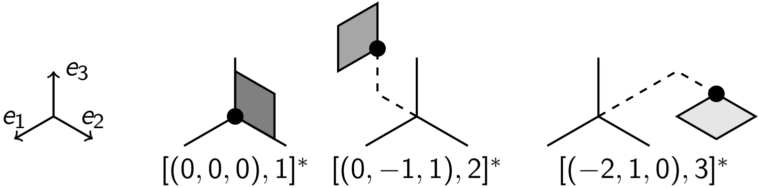
“There is certainly an underlying homological theory”

[Arnoux-Ito-Sano]

Dual substitutions act on unit faces

Unit faces $[\mathbf{x}, i]^*$

- position $\mathbf{x} \in \mathbb{Z}^3$
- type $i \in \{1, 2, 3\}$



Dual substitutions

Let σ be a substitution on the alphabet $\mathcal{A}$

Abelianisation

Let d be the cardinality of $\mathcal{A}$. Let $\vec{l} : \mathcal{A}^\star \to \mathbb{N}^d$ be the abelianisation map

$$\vec{l}(w) = (|w|_1, |w|_2, \cdots, |w|_d)$$

Dual substitutions [P. Arnoux-S. Ito]

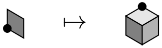
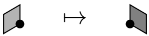
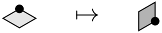
Let σ be a unimodular substitution

$$\mathbf{E}_1^*(\sigma)([\vec{x}, i^*]) = \sum_{j \in \mathcal{A}} \sum_{P,\, \sigma(j) = PiS} \left[\mathbf{M}_\sigma^{-1}\left(\vec{x} + \vec{l}(S)\right), j^*\right]$$

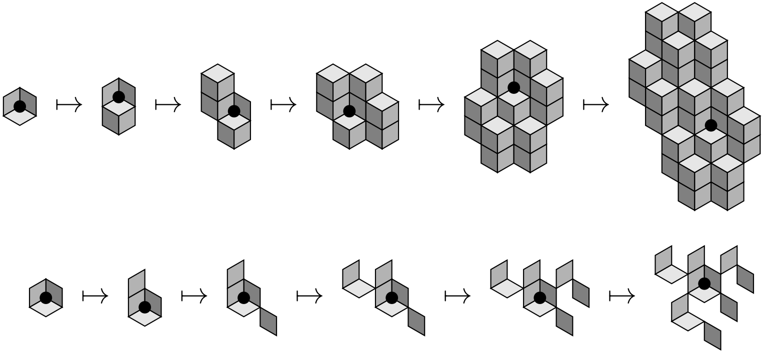
Example : $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

$$\mathbf{E}_1^*(\sigma)([\vec{x}, i^*]) = \sum_{j \in \mathcal{A}} \sum_{P,\, \sigma(j) = PiS} \left[\mathbf{M}_\sigma^{-1}\left(\vec{x} + \vec{l}(S)\right), j^*\right]$$

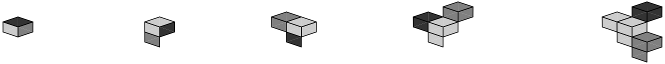
$$\begin{array}{rcl}
\mathbf{E}_1^*(\sigma)([\mathbf{x}, 1]^*) & = & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(1,0,-1),1]^* \cup [(0,1,-1),2]^* \cup [(0,0,0),3]^* \\
\mathbf{E}_1^*(\sigma)([\mathbf{x}, 2]^*) & = & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),1]^* \\
\mathbf{E}_1^*(\sigma)([\mathbf{x}, 3]^*) & = & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),2]^*
\end{array}$$

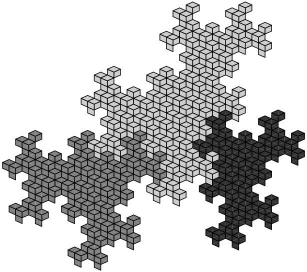




Examples



DS generate Rauzy fractals





Geometric representation vs. symbolic models

Looking for geometric representation

Theorem [Rauzy'82]

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Looking for symbolic models–Arithmetic dynamics

- We want to find symbolic realizations for a.e. Kronecker maps (toral translations)
- We want to reach nonalgebraic parameters by considering convergent products of matrices
- We consider not only one substitution but a sequence of substitutions S-adic formalism Non-stationary dynamics

Continued fractions algorithm $\rightsquigarrow$ renormalization

Given a translation on $\mathbb{R}^2/\mathbb{Z}^2$

$$R_\alpha \colon x \mapsto x + \alpha \bmod 1, \quad \alpha = (\alpha_1, \alpha_2), \quad x = (x_1, x_2)$$

Find a suitable partition of $\mathbb{T}^2$ which provides a symbolic coding (X_α, S) of $(\mathbb{T}^2, R_\alpha)$ up to measure-theoretic isomorphism

$$\begin{array}{ccc} \mathbb{T}^2 & \xrightarrow{R_\alpha} & \mathbb{T}^2 \\ \downarrow & & \downarrow \\ X_\alpha & \xrightarrow[S]{} & X_\alpha \end{array}$$

- with factor complexity $2n + 1$
- such that the atoms corresponding to factors are Bounded Remainder Sets

$$|\mathrm{Card}\,\{0 \leq n \leq N ; R_\alpha(x) \in A\} - N\mu(A)| \leq C \quad \forall N \quad \text{a.e. } x$$

Balancedness on factors

Given a translation on $\mathbb{R}^2/\mathbb{Z}^2$

$$R_\alpha \colon x \mapsto x + \alpha \text{ mod } 1, \quad \alpha = (\alpha_1, \alpha_2), \quad x = (x_1, x_2)$$

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$$\begin{tikzcd}
\mathbb{T}^2 \arrow[r, "R_\alpha"] \arrow[d] & \mathbb{T}^2 \arrow[d] \\
X_\alpha \arrow[r, "S"'] & X_\alpha
\end{tikzcd}$$

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- In dimension 1, Sturmian words provide a coding with factor complexity $n + 1$
- (X_α, S) has pure discrete spectrum "Rauzy fractals tile"
- For a.e. every α, Brun algorithm does the job with a conjectured linear factor complexity [B.-Steiner-Thuswaldner]

DS are elegant

DS are algebraically robust and elegant

- They rely on duality
- They behave well with respect to S-adic formalism $\sigma_1 \cdots \sigma_n$

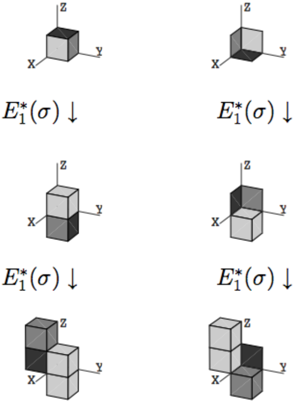
$$M_1 \cdots M_n \rightsquigarrow {}^t(M_1 \cdots M_n)$$

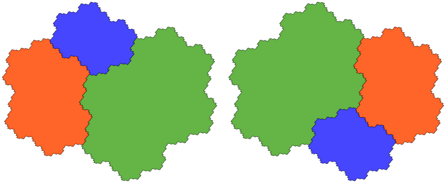
$$E_1^*(\tau \circ \sigma) = E_1^*(\sigma) \circ E_1^*(\tau)$$

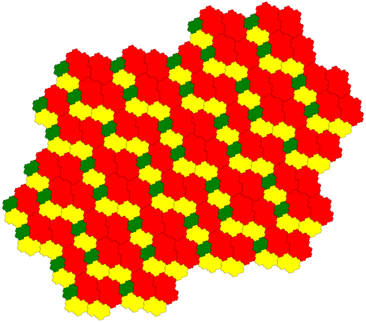
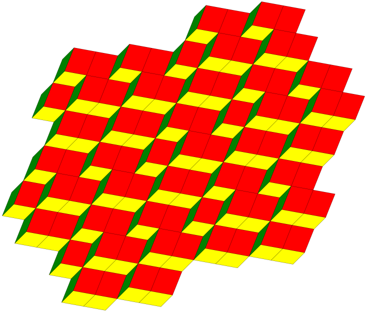
- Start from a dynamical system endowed with an induction/renormalization interval exchange transformation toral translation $\rightsquigarrow$ a symbolic S-adic coding
- Exduction $\rightsquigarrow$ build a fundamental domain for a toral translation R_α
- Characterization for pure discrete spectrum
- A geometric version of the IFS satisfied by Rauzy fractals (beyond Peron-Frobenius' arguments)

Exduction

P. Arnoux – S. Ito



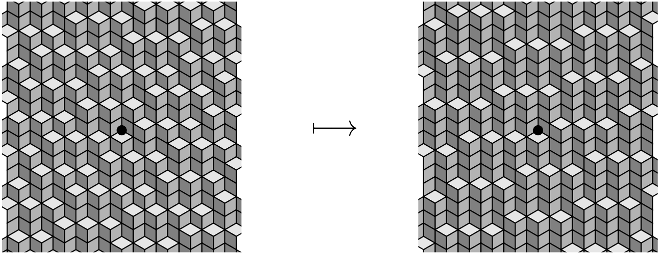


DS are powerful tools in discrete geometry

Theorem [Arnoux-Ito, Fernique]

Let σ be a unimodular substitution. The image of a discrete plane with normal vector v is a discrete plane with normal vector ${}^tM_\sigma v$



$\rightsquigarrow$ 2d Sturmian words, codings of $\mathbb{Z}^2$-actions by rotations on $\mathbb{T}$, Cut and project schemes

Generation of discrete planes

- Chose a vector v with positive entries
- Expand v with any continued fraction algorithm
$$v = M_1 \cdots M_n v_n$$
- Select substitutions $\sigma_1, \ldots, \sigma_n$ with matrices ${}^{t}M_1, \ldots, {}^{t}M_n$

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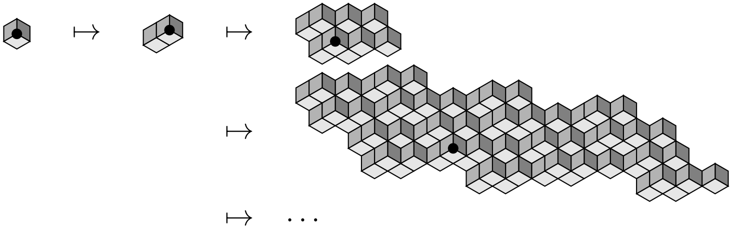
$$v = M_1 \cdots M_n v_n$$
- Select substitutions $\sigma_1, \ldots, \sigma_n$ with matrices ${}^t M_1, \ldots, {}^t M_n$
- Apply the substitutions starting from the unit cube $\mathcal{U} \subset \mathcal{P}_{\mathbf{v}}$
- Build increasing pieces of the discrete plane $\mathcal{P}_{\mathbf{v}}$

$$E_1^*(\sigma_1) \cdots E_1^*(\sigma_n)(\mathcal{U}) \subseteq \mathcal{P}_{\mathbf{v}}$$

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3d continued fraction algorithms Cheat Sheets by S. Labbé

Cassaigne algorithm

Definition

On $\Lambda = \mathbb{R}^3_+$, the map is [4]

$$F(x_1, x_2, x_3) = \begin{cases} (x_1 - x_3, x_3, x_2) & \text{if } x_1 > x_3 \\ (x_2, x_1, x_3 - x_1) & \text{if } x_1 < x_3. \end{cases}$$

Matrix Definition

The partition of the cone is $\Lambda = \cup_{\pi \in \mathcal{S}_3} \Lambda_\pi$ where

$$\Lambda_1 = \{(x_1, x_2, x_3) \in \Lambda \mid x_1 > x_3\},$$
$$\Lambda_2 = \{(x_1, x_2, x_3) \in \Lambda \mid x_1 < x_3\}.$$

The matrices are given by the rule

$$M(\mathbf{x}) = M_i \quad \text{if and only if} \quad \mathbf{x} \in \Lambda_i.$$

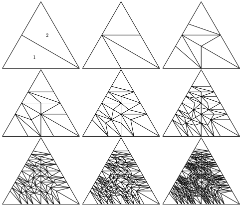
The map F on Λ and the projective map f on $\Delta = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\|_1 = 1\}$ are:

$$F(\mathbf{x}) = M(\mathbf{x})^{-1}\mathbf{x} \quad \text{and} \quad f(\mathbf{x}) = \frac{F(\mathbf{x})}{\|F(\mathbf{x})\|_1}.$$

Matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Cylinders

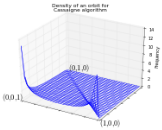


Density function

The density function of the invariant measure of $f : \Delta \to \Delta$ for the Cassaigne algorithm is [1]

$$\frac{1}{(1 - x_1)(1 - x_3)}.$$

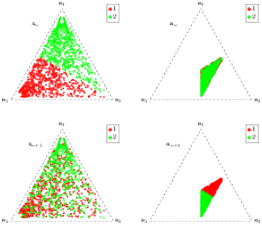
Invariant measure



Natural extension

Two sequences $(\mathbf{x}_{n+1})_{n\geq 0}$ and $(\mathbf{a}_{n+1})_{n\geq 0}$ defined such that

$$\mathbf{x}_{n+1} = M(\mathbf{x}_n)^{-1}\mathbf{x}_n \quad \text{and} \quad \mathbf{a}_{n+1} = M(\mathbf{x}_n)^\top \mathbf{a}_n.$$



Lyapunov exponents

(using 30 orbits of 100000000 iterations each)

30 succesfull orbits	min	mean	max	std
θ_1	0.1824	0.1827	0.1829	0.00013
θ_2	-0.07083	-0.07072	-0.07060	0.000054
$1 - \theta_2/\theta_1$	1.38698	1.38712	1.38725	0.000070

Substitutions

$$\sigma_1 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 13 \\ 3 \mapsto 2 \end{cases} \qquad \sigma_2 = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 13 \\ 3 \mapsto 3 \end{cases}$$

S-adic word example

Using vector $v = (1, e, \pi)$:

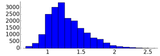
$$\begin{aligned} w &= \sigma_2\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1 \cdots (1) \\ &= 232321323232313232321323232132323231323232\ldots \end{aligned}$$

Factor Complexity of w is $(p_w(n))_{0\leq n\leq 20} =$

$$(1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41)$$

Discrepancy

Discrepancy [10] for all 19701 S-adic words with directions $v \in \mathbb{N}^3_{>0}$ such that $v_1 + v_2 + v_3 = 200$:



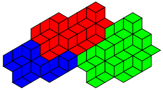
Dual substitutions

$$\sigma_1^* = \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 3 \\ 3 \mapsto 2 \end{cases} \qquad \sigma_2^* = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 23 \end{cases}$$

E one star

Using vector $v = (1, e, \pi)$, the 13-th iteration on the unit cube is:

$$E_1^*(\sigma_2^*)E_1^*(\sigma_1^*)E_1^*(\sigma_2^*)E_1^*(\sigma_1^*)E_1^*(\sigma_1^*) \cdots (\text{cube}) =$$



Cassaigne/Selmer algoritm

Selmer algorithm Subtract the smallest entry to the largest entry
An absorbing set

Cassaigne algorithm

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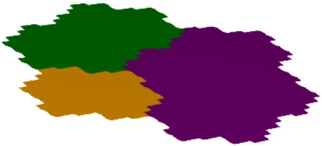
TODO list

- Use E_1^* to prove pure discrete spectrum
- Balancedness on factors?


Vittel

Markov partition

- One takes the Rauzy fractal in the contracting plane
- One lifts each piece along the expanding direction
- Periodic tiling $\rightsquigarrow$ a fundamental domain for $\mathbb{T}^3$



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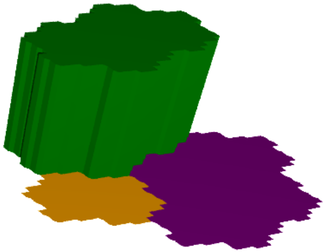

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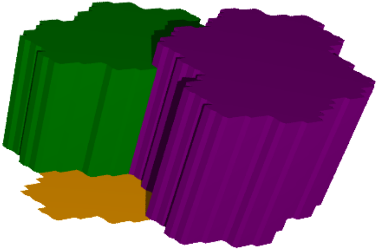

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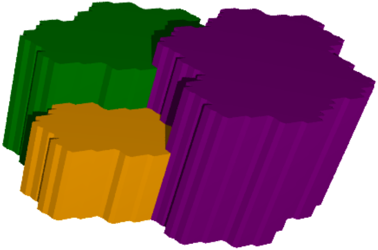

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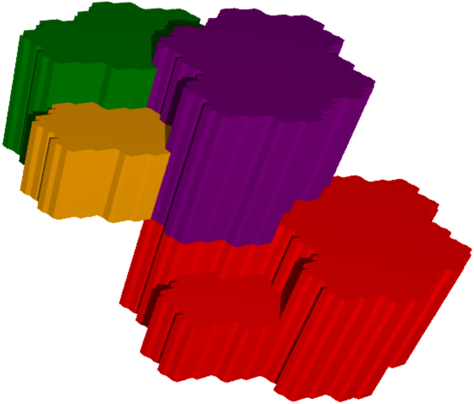

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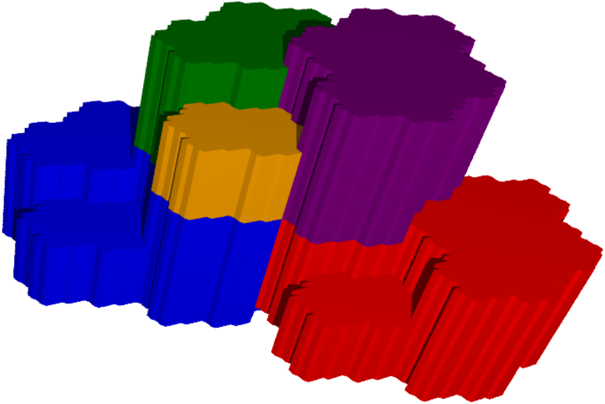

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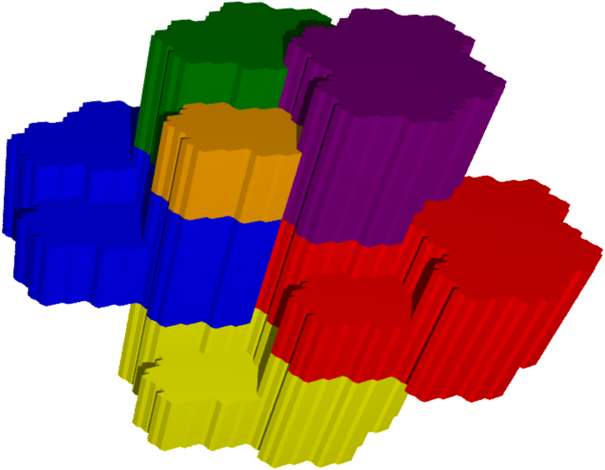

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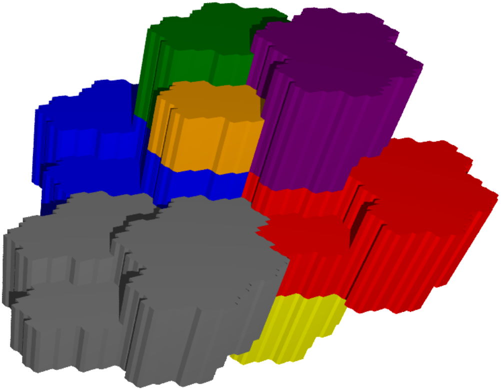

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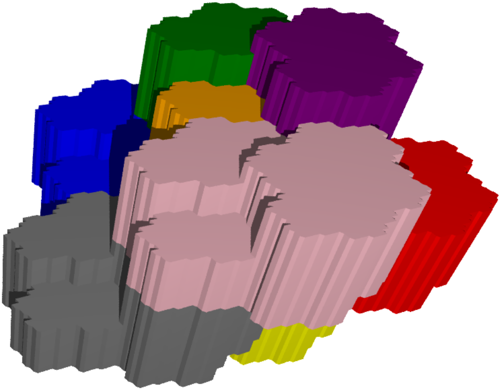

Markov partition

- One takes the Rauzy fractal in the contracting plane
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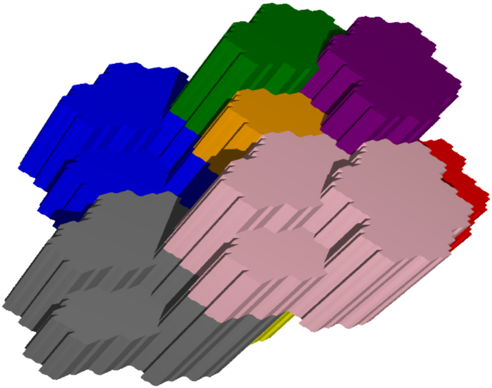

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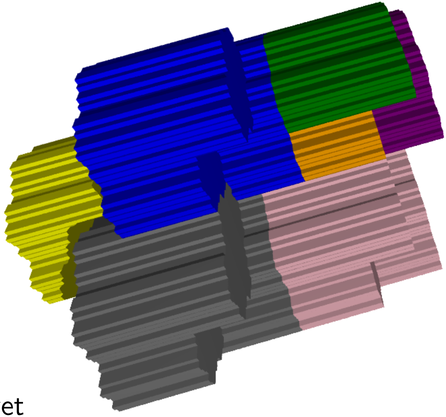

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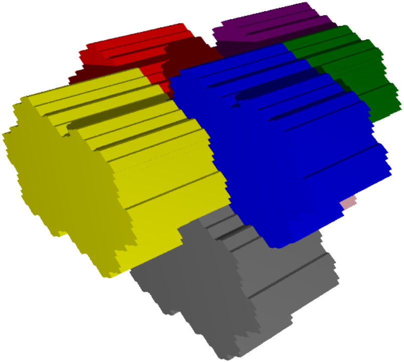

Dictionary

S-adic description of a minimal symbolic dynamical system $\rightleftharpoons$ multidimensional continued fraction algorithm that governs its letter frequency vector/ invariant measure

- S-adic expansion
- Unique ergodicity
- Linear recurrence
- Balance and Pisot properties
- Two-sided sequences of substitutions
- Shift on sequences of substitutions

- Continued fraction
- Convergence
- Bounded partial quotients
- Strong convergence
- Natural extension
- Continued fraction map

Continued fractions algorithm $\rightsquigarrow$ renormalization

S-adic Pisot dynamics

Theorem [B.-Steiner-Thuswaldner]

- For almost every $(\alpha, \beta) \in [0, 1]^2$, the translation by (α, β) on the torus $\mathbb{T}^2$ admits a natural symbolic coding provided by the S-adic system associated with Brun multidimensional continued fraction algorithm applied to (α, β)

- For almost every Arnoux-Rauzy word, the associated S-adic system has pure discrete spectrum

S-adic Pisot dynamics

Theorem [B.-Steiner-Thuswaldner]

- For almost every $(\alpha, \beta) \in [0, 1]^2$, the S-adic system provided by the Brun multidimensional continued fraction algorithm applied to (α, β) is measurably conjugate to the translation by (α, β) on the torus $\mathbb{T}^2$

Proof Based on Dual substitutions

- "adic IFS"
- Finite products of Brun substitutions have pure point spectrum [B.-Bourdon-Jolivet-Siegel]

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Proof Based on Dual substitutions

- "adic IFS"
- Theorem [Avila-Delecroix]
  - The Arnoux-Rauzy S-adic system is Pisot
- Theorem [Avila-Hubert-Skripchenko]
  - A measure of maximal entropy for the Rauzy gasket
- Finite products of Brun/Arnoux-Rauzy substitutions have pure point spectrum [B.-Bourdon-Jolivet-Siegel]

Mapping families [Arnoux-Fisher 2005]

Sequence of d-dimensional Riemann manifolds $(X_n)_{n\in\mathbb{Z}}$

$$X = \coprod_{n\in\mathbb{Z}} X_n$$

Let $f_n : X_n \to X_{n+1}$ be a C^1-diffeomorphism and define

$$f : X \to X \text{ with } f(x) = f_n(x) \text{ for } x \in X_n$$

(X, f) is a mapping family.

$$\cdots \xrightarrow{f_{-2}} X_{-1} \xrightarrow{f_{-1}} X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots$$

S-adic mapping families

$$\cdots \xrightarrow{f_{-2}} X_{-1} \xrightarrow{f_{-1}} X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots .$$

Let $\boldsymbol{\sigma}$ be a two-sided sequence of substitutions over an alphabet of cardinality d. For all i, let M_i be the incidence matrix of σ_i.

Consider the non-stationary composition of toral automorphisms

$$\cdots \xrightarrow{M_{-2}^{-1}} \mathbb{T}_{-1}^d \xrightarrow{M_{-1}^{-1}} \mathbb{T}_0^d \xrightarrow{M_0^{-1}} \mathbb{T}_1^d \xrightarrow{M_1^{-1}} \cdots$$

$$\mathbf{T} = \coprod_{n \in \mathbb{Z}} \mathbb{T}_n^d$$

$$f_{\boldsymbol{\sigma}} : \mathbf{T} \to \mathbf{T},\ f_{\boldsymbol{\sigma}}(x) = M_n^{-1}(x) \text{ for } x \in \mathbb{T}_n^d$$

$$(\mathbf{T}, f_{\boldsymbol{\sigma}}) = \text{mapping family}$$

- Nature of the mapping family, e.g. hyperbolicity?
- Tool: Oseledets' multiplicative ergodic theorem

Eventually Anosov mapping family

Let $\boldsymbol{\sigma}$ be a two-sided sequence of substitutions over an alphabet of cardinality d

The mapping family associated with $\boldsymbol{\sigma}$ is eventually Anosov if there exist splittings $E_s^{(n)} \oplus E_u^{(n)}$ of $\mathbb{R}^d$ so that the following properties hold

- f-invariance For all n, $f_n(E_s^{(n)}) = E_s^{(n+1)}$, $f_n(E_u^{(n)}) = E_u^{(n+1)}$
- Hyperbolicity For some (and hence for all) $k \in \mathbb{Z}$

$$\lim_{n\to+\infty} \inf\{\|M_{[k,n)}^{-1}\mathbf{x}\|/\|\mathbf{x}\| : \mathbf{x} \in E_u^{(k)} \setminus \{\mathbf{0}\}\} = +\infty, \quad n > k,$$

$$\lim_{n\to+\infty} \sup\{\|M_{[k,n)}^{-1}\mathbf{x}\|/\|\mathbf{x}\| : \mathbf{x} \in E_s^{(k)} \setminus \{\mathbf{0}\}\} = 0, \quad n > k,$$

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Our result $d = 3$

Theorem [Arnoux-B.-Minervino-Steiner-Thuswaldner] For a.e. pair of vectors, the associated two-sided orbit of Brun algorithm yields a mapping family which is eventually Anosov and whose stable and unstable spaces are provided by this pair of vectors, and the partition made of the suspensions of the Rauzy fractals form a Markov partition.

Remark The pieces of the associated Markov partition are connected [B.-Bourdon-Jolivet-Siegel] Dual substitutions!

Dynamically

- One has the shift acting on zero entropy systems (codings of toral translations) $\rightsquigarrow$ Basis of the Markov partition
- One has the shift acting on positive entropy systems (sequences of substitutions produced by Brun algorithm) $\rightsquigarrow$ Lifting of the Markov partition
- One has a renormalization cocycle given by the incidence matrices of the substitutions (inverse of the matrices of the Brun algorithm) Positive entropy, random shift of finite type
- We apply Oseledets theorem to get a splitting of the spaces to define stable and unstable spaces
- Dual substitutions formalise the cut and stack action

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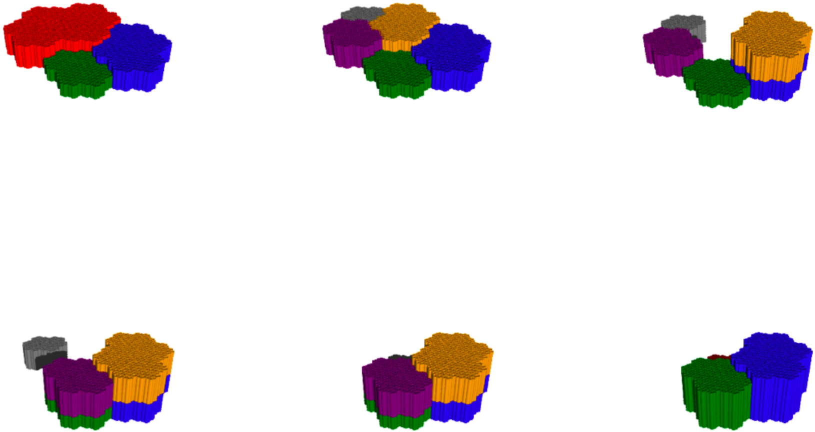
Toward a flow for Brun algorithm?

DS restack



Un message de la part d'Anne Siegel

Pierre m'a fait découvrir les systèmes dynamiques et a encadré mes premiers pas dans la recherche ; il a gardé un oeil sur moi depuis ma thèse, mêmeme si les pistes que j'ai explorées (biologie, topologie ou théorie des nombres) ne sont pas ses thèmes préférés. En plus des règles du métier, il m'a appris à prendre le temps d'identifier l'essentiel derrière une problématique. Il est pour moi un modèle d'intégrité, et je reste admirative de sa capacité à décrire l'essentiel d'un article obscur de 30 pages en moins de 10 lignes limpides.

Pour l'anecdote, les quelques fois où Pierre m'a fait "quelques" suggestions de corrections sur un article, l'article était raturé INTEGRALEMENT au feutre rouge toujours présent dans la poche droite.... La déprime à la vue de tout ce qu'il fallait faire... Et 20 ans plus tard je réalise que je fais pareil avec mes doctorant.e.s...