Guy David

Singular Sets of Minimizers for the Mumford-Shah Functional

INSTITVT D'ESTVDIS CATALANS MCMVII

Ferran Sunyer i Balaguer
Award winning monograph

$$J(u, K) = a \int_{\Omega} |u - g|^2 + bH^{n-1}(K) + c \int_{\Omega \setminus K} |\nabla u|^2, \tag{4}$$

where a, b, c are positive constants, and the competitors are pairs (u, K) for which

$$K \subset \Omega \text{ is closed in } \Omega, \tag{5}$$

$$H^{n-1}(K) < +\infty, \tag{6}$$

and

$$u \in W^{1,2}(\Omega \setminus K) \tag{7}$$

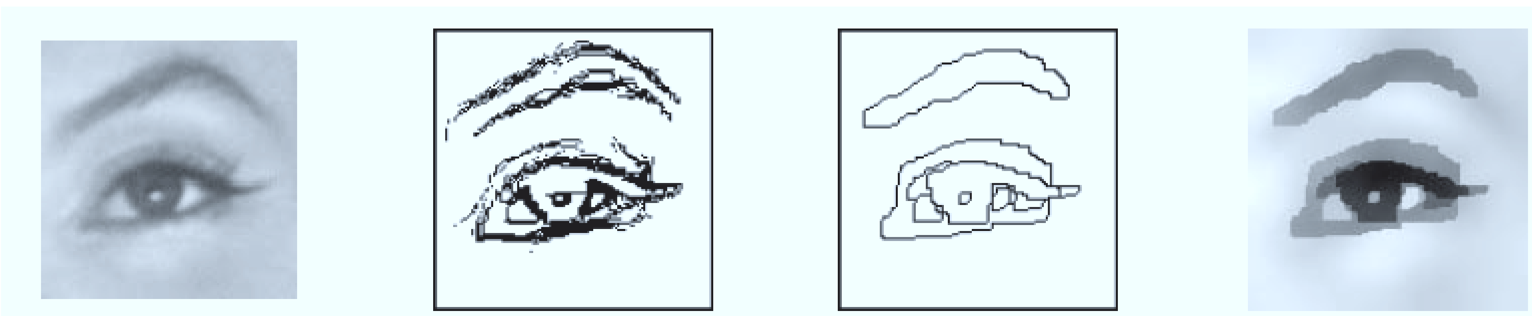


Image approximation with Mumford-Shah functional. (left) The image of an eye. (center-left) areas of high gradient in the original image. (center-right) boundaries in the Mumford-Shah model, (right) piecewise-smooth function approximating the image.

(From the Wikipedia article on the Mumford-Shah functional)

Extracts from the book's Introduction to 600 pages on the Mumford-Shah conjecture

« The project of this book started in a very usual way: the author taught a course in Orsay (fall, 1999), and then trapped himself into writing notes, that eventually reached a monstruous size. »

« The official goal of this book is to take the optimistic view that we first need to digest the previous progress, and then things will become easier »

« This book is too long (…). Even the author finds it hard to find a given lemma. »

« It should be stressed at the outset that there is a life outside of the Mumford-Shah conjecture »

Visual perception, geometry and the structure of images

Jean-Michel Morel,
CMLA, Ecole Normale Supérieure de Cachan, France

with contributions of
Miguel Colom, Axel Davy, Thibaud Ehret,
Gabriele Facciolo, Marc Lebrun, Nicola Pierazzo, Martin Rais, Yiqing Wang,

Marseille CIRM Luminy 3 octobre 2017

It has been long admitted that the structure of 2D functions is described by local characteristics, for example a local Fourier or wavelet expansion, or more trivially a Taylor expansion of some order. The regularity of the function would be encoded in the decay of a local expansion or in the boundedness of some norm measuring regularity (Sobolev, Besov, BV,…).

Image processing has strayed away from this model inherited from harmonic analysis and geometric measure theory. It looks directly at the "patch space".

Patches are 8 * 8 or 10 * 10 square images cropped from any image. Image characteristics seem to be better described in the patch space (of dimension 64, 100,…). This is a dimension reduction (from the space of images that would have a dimension of several millions), or on the contrary a dimension lifting from two dimensions (the image) to many more.

Can we explore the patch space and find some evidence about its regularity? This is an experimental question, because we can now analyze patches by billions. Still, this is a far too small number to sample a space in such a high dimension, unless it shows some regularity.

Even the sparse information that we have gathered on the patch space changes our view of image perception.

I'll illustrate it on two classic image processing problems: image denoising and anomaly detection.

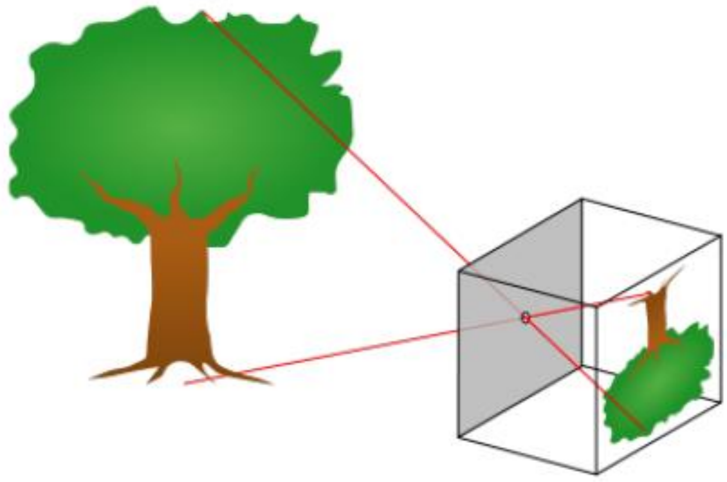
Image denoising is a complex mathematical operation performed routinely in billions of cameras. Every digital image and every video is systematically processed numerically.

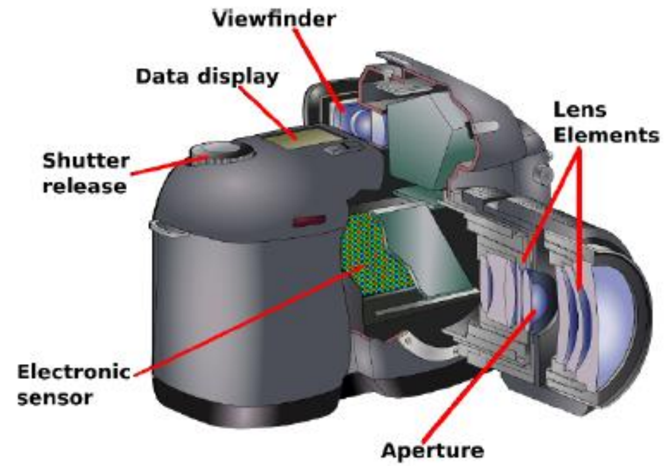
Simple integral formulas have been invented in the past ten years and account for the steady improvement of image quality.

Science and technology require verification: Most algorithms that I'll show can be tested on any image in the electronic journal

Image Processing on Line (IPOL)

http://www.ipol.im/





A « camera oscura » (this goes back at least to the 16th century) and a modern camera are in fact quite similar. The photons emitted by objects pass through the virtual pinhole (the lens focus) or through a real one, and hit a photosensitive surface. In modern cameras , this photosensitive surface is an electronic sensor matrix, so the image is directly sampled on a rectangular grid. It follows that digital image is nothing but a matrix of observed numbers.

Each pixel (for picture element) $\mathbf{x}$ receives a number of photons $u(\mathbf{x})$. This number is a random variable due to quantum effects of light emission and other random perturbation. Its expectation is the "ideal image" $\mathbb{E}u(\mathbf{x})$, and the difference $u(\mathbf{x}) - \mathbb{E}u(\mathbf{x})$ between observed and ideal is the noise. Our goal is to remove the noise, namely to guess $\mathbb{E}u(\mathbf{x})$.

$u(\mathbf{x})$

$\mathbb{E}u(\mathbf{x})$ ideal image

$u(\mathbf{x}) - \mathbb{E}u(\mathbf{x})$ "image noise"

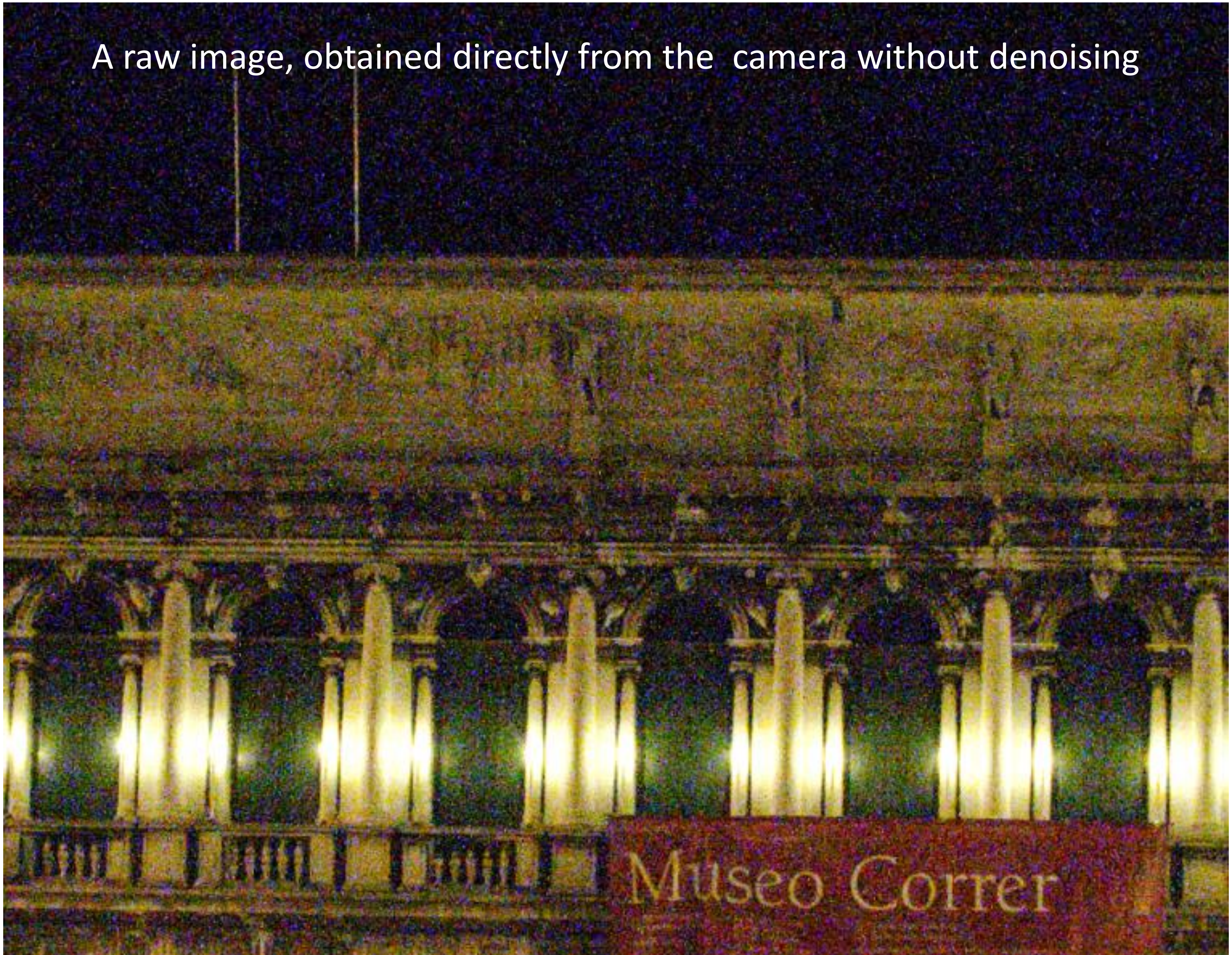
- Discrete image domain $\Omega = [[0, m]] \times [[0, n]]$

For a color image, $u(\mathbf{x})$ is actually a three component vector (red, green, blue), $u(\mathbf{x}) = (r(\mathbf{x}), g(\mathbf{x}), b(\mathbf{x})) \in \mathbb{R}^3$. Each component $r(\mathbf{x})$, $g(\mathbf{x})$, $b(\mathbf{x})$ is itself a Poisson random variable and denoising therefore means estimating the three true values (expectations) of these components at each pixel $\mathbf{x}$.

Without loss of generality, we assume that the noise is white : Gaussian, independent at each pixel and for each color channel, with uniform variance $\boldsymbol{\sigma}$.

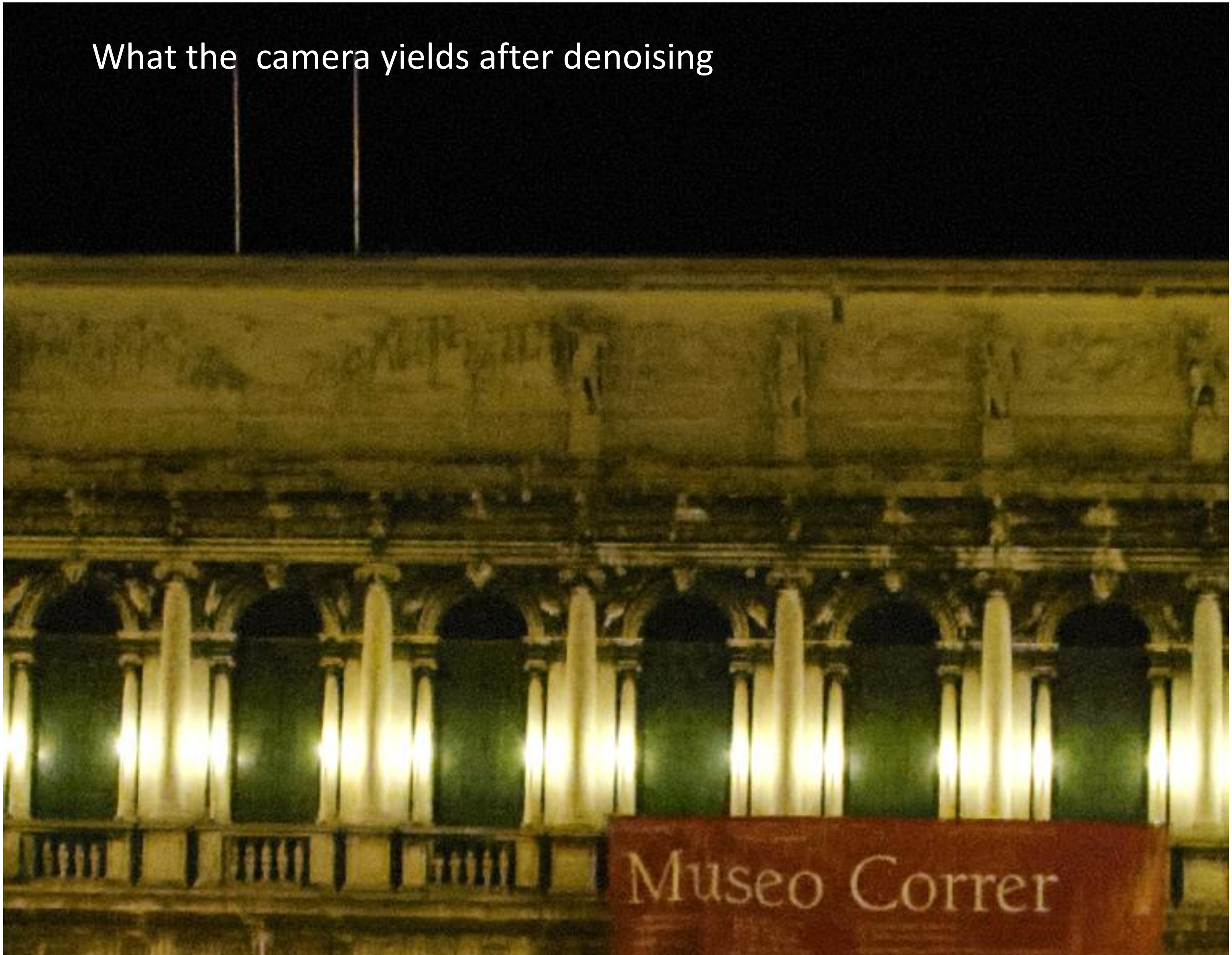
Each image pixel is a random vector, and we dispose of a single sample of each. Yet, we want to estimate the expectation of this random vector, from this single sample! This problem seems completely ill-posed but will be solved by grouping pixels into patches.

Image credit: DxO Labs



Experiment by DxO-Labs, a company specialized on image processing. One can compare the resulting image with-and-without denoising. By night the problem is more difficult because there are less photons and therefore more noise.

Image credit: DxO Labs



Denoising in five formulas from local to global

1-Transform thresholding: the paradigmatic example of DCT denoising

2-Neighborhood filters: an old and fantastic trick

3-A slight extension: nonlocal means

4-The Bayesian denoising paradigm from « non-local » to « global »

5-Anomaly detection and denoising

Where to test all algorithms: Image Processing on Line (IPOL)

http://www.ipol.im/

12

Linear frequency transform thresholding: the example of discrete cosine transform (DCT) denoising

Transform thresholding can be based on the Fourier transform (Shannon, Wiener), on wavelets (Meyer, Daubechies, Coifman, Donoho & Johnstone, Starck, Mallat,..), or on the discrete cosine transform (Yaroslavski).

Transform thresholding assumes that **the image is sparse on one of these bases** (Candès, Romberg, Tao). The noise frequency coefficients have instead all the same variance, and are therefore uniformly small.

Thus, by cancelling the small frequency coefficients of the noisy image (the threshold is typically lower than 3σ), the noise is reduced and the signal is mostly preserved.

In DCT denoising this operation is performed on each image 8x8 pixels « patch ».

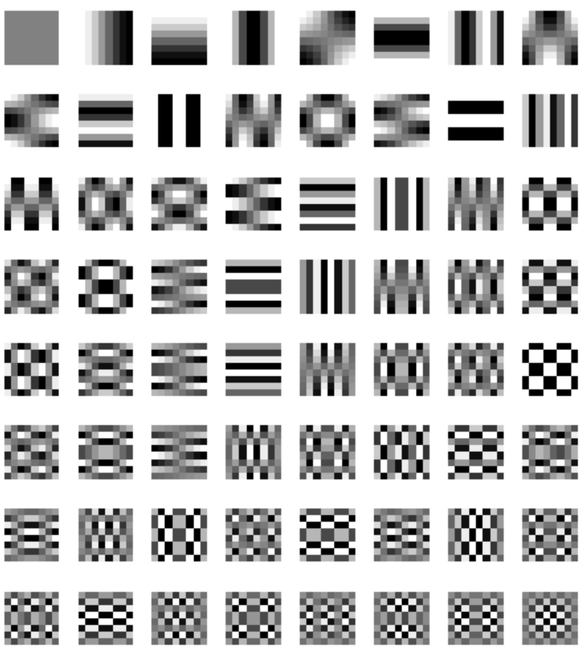
Wiener, Norbert (1949). ***Extrapolation, Interpolation, and Smoothing of Stationary Time Series*****. New York: Wiley.**

Linear frequency transform thresholding: the example of discrete cosine transform (DCT) denoising

"patch" $P_{\mathbf{x}}$: restriction of the image u to a small block around a pixel $\mathbf{x}$

σ: standard deviation of the noise

Each image patch is decomposed on the DCT basis and all of its small frequency coefficients are cancelled. The DCT basis is a local variant of the Fourier transform on 8x8 patches.



DCT basis for 8x8 patches:
cos(nx)cos(my), m,n =0,…,7

Input: noisy image, noise standard deviation σ:
For each 8x8 image patch **P_x** :

-Calculate 2D-DCT transform of the patch;

-Cancel all DCT coefficients with absolute value below 3σ.

-Calculate inverse 2D-DCT transform of the patch.

- Obtain the denoised color **u(x)** as the average of all those obtained for all 64 patches containing **x.**

DCT denoising algorithm

σ=15 noisy

DCT denoising is quite successful with moderate noise. **This amounts to cancel the smaller DCT coefficients of each patch**, as they mainly contain noise.

DCT transform threshold denoising: σ=15: G. Yu, G. Sapiro http://www.ipol.im/

σ=15
denoised

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DCT transform threshold denoising: σ=15: G. Yu, G. Sapiro http://www.ipol.im/

σ=15
original

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DCT transform threshold denoising: σ=15: G. Yu, G. Sapiro http://www.ipol.im/

DCT denoising and in general (wavelet) transform thresholding with a too large threshold creates **Gibbs effects** for large noise. This Gibbs effect or « ringing » appears everywhere

2-DCT transform threshold denoising: σ=40: G. Yu, G. Sapiro http://www.ipol.im/

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The trivial idea behind neighborhood filters: find the right samples in the image and average them. This is still a local smoothing but it is local in higher dimension (image + values)

A meaningful simple formula

Assume that $u(\mathbf{x}_1)$, $u(\mathbf{x}_2)$, ..., $u(\mathbf{x}_n)$ are observed noisy samples of the same underlying color $u(\mathbf{x})$. Then the mean $\frac{1}{n}\sum_{i=1}^{n} u(\mathbf{x}_i)$ is a much better estimate of $u(\mathbf{x})$ than each $u(\mathbf{x}_i)$. Indeed,

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} u(\mathbf{x}_i)\right) = \frac{1}{n}\mathrm{Var}(u(\mathbf{x}_1)).$$

Thus, it is enough to find, say, 16 samples for $u(\mathbf{x})$ to divide the noise by 4. Neighborhood filters propose a new way to select these samples.

Gaussian convolution as a denoiser

A first simple way to select the samples that will permit to estimate *u*(**x**) is to assume that <u>all pixels in a spatial (Gaussian weighted) neighborhood have the same underlying colour</u>. This amounts to convolve the image with a Gaussian, in other terms to replace *u*(**x**) by a weighted average of the values *u*(**y**) in a neighborhood of **x**.

$$G_\sigma^2 * u(\mathbf{x}) = \frac{\int_\Omega G_\sigma^2(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y}}{\int_\Omega G_\sigma^2(\mathbf{x}-\mathbf{y})d\mathbf{y}}$$

⟵ weighted average

⟵ normalization term

But a still much better way was invented with neighborhood filters: *u*(**y**) will contribute to the estimate of u(**x**) if and only **x** is close to **y** **but also** *u*(**y**) is close to *u*(**x**). The next slide gives the formula.

Reminder: $G_\sigma^N(\mathbf{x}) = \frac{1}{(2\pi\sigma)^{\frac{N}{2}}} e^{-\frac{||\mathbf{x}||^2}{2\sigma^2}}$

2-Neighborhood filters: an old and fantastic trick

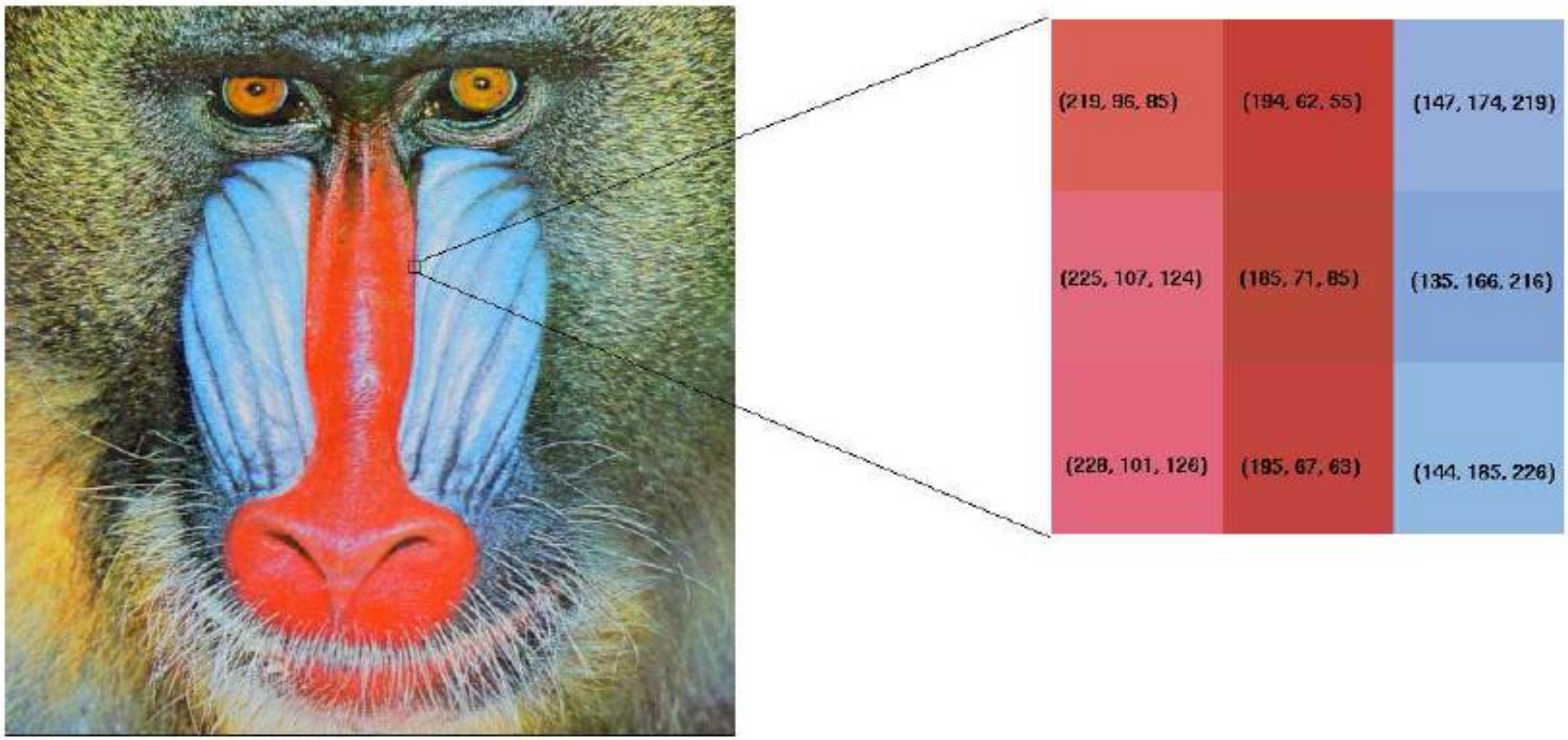


Figure 1: A. Buades, B. Coll, and J.M M "Neighborhood filters and PDE's", Numerische Mathematik, 105 (1), 2006.

3-Neighborhood filters: an old and fantastic trick

Neighborhood filter, Sigma filter, SUSAN, Bilateral filter

[Yaroslavski 80, Lee 83, Smith-Brady 97, Tomasi-Manduchi 98]

Now the Gaussian weights guarantee a proximity in space and color:

$\mathbf{x}, \mathbf{y}$: pixels; $u(\mathbf{x})$: image color at $\mathbf{x}$

$$G_\sigma^N(\mathbf{x}) = \frac{1}{(2\pi\sigma)^{\frac{N}{2}}} e^{-\frac{||\mathbf{x}||^2}{2\sigma^2}}$$

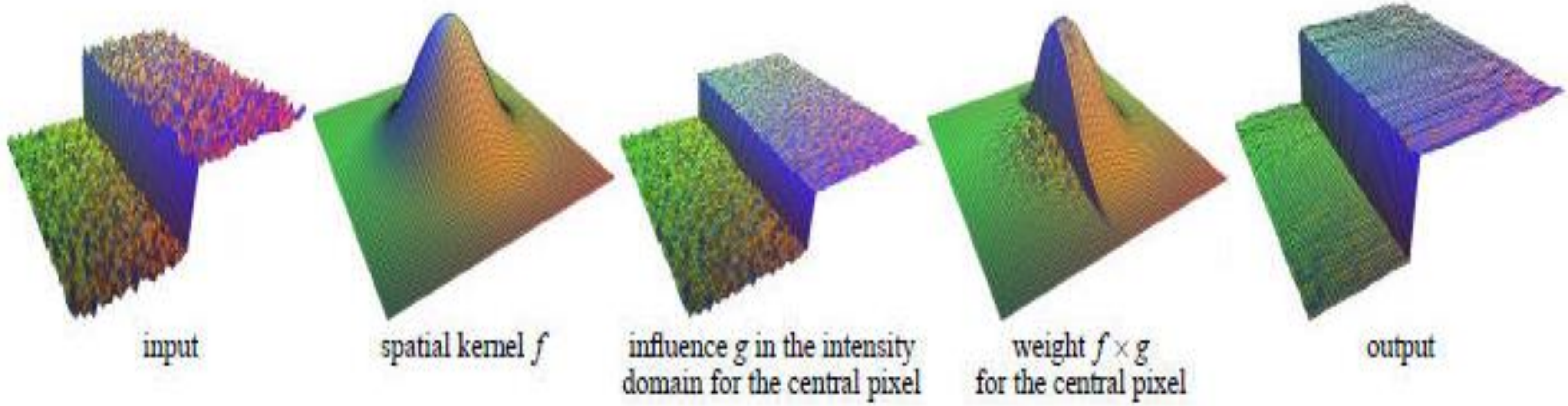
Spatial Filter f

Range Filter g

$$NFu(\mathbf{x}) = \frac{\int_\Omega G^2_{\sigma_s}(\mathbf{x}-\mathbf{y}) G^3_{\sigma_r}(u(\mathbf{x}) - u(\mathbf{y}))u(\mathbf{y})d\mathbf{y}}{\int_\Omega G^2_{\sigma_s}(\mathbf{x}-\mathbf{y}) G^3_{\sigma_r}(u(\mathbf{x}) - u(\mathbf{y}))d\mathbf{y}}$$

← weighted average

← normalization term



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Nonlocal-means denoising (2004)

Regularity in the patch space?

$$NL(P_{\mathbf{x}}) = \frac{\int_\Omega G^N_{\sigma_p}(P_{\mathbf{x}} - P_{\mathbf{y}}) P_{\mathbf{y}} d\mathbf{y}}{\int_\Omega G^N_{\sigma_p}(P_{\mathbf{x}} - P_{\mathbf{y}}) d\mathbf{y}}$$

$$G^N_\sigma(\mathbf{x}) = \frac{1}{(2\pi\sigma)^{\frac{N}{2}}} e^{-\frac{||\mathbf{x}||^2}{2\sigma^2}}$$

N : patch dimension (typically 8×8)

Here, the idea is to retain as valid samples for $u(\mathbf{x})$ only samples $u(\mathbf{y})$ whose surrounding patch $P_{\mathbf{y}}$ is similar to the patch $P_{\mathbf{x}}$ surrounding $\mathbf{x}$.

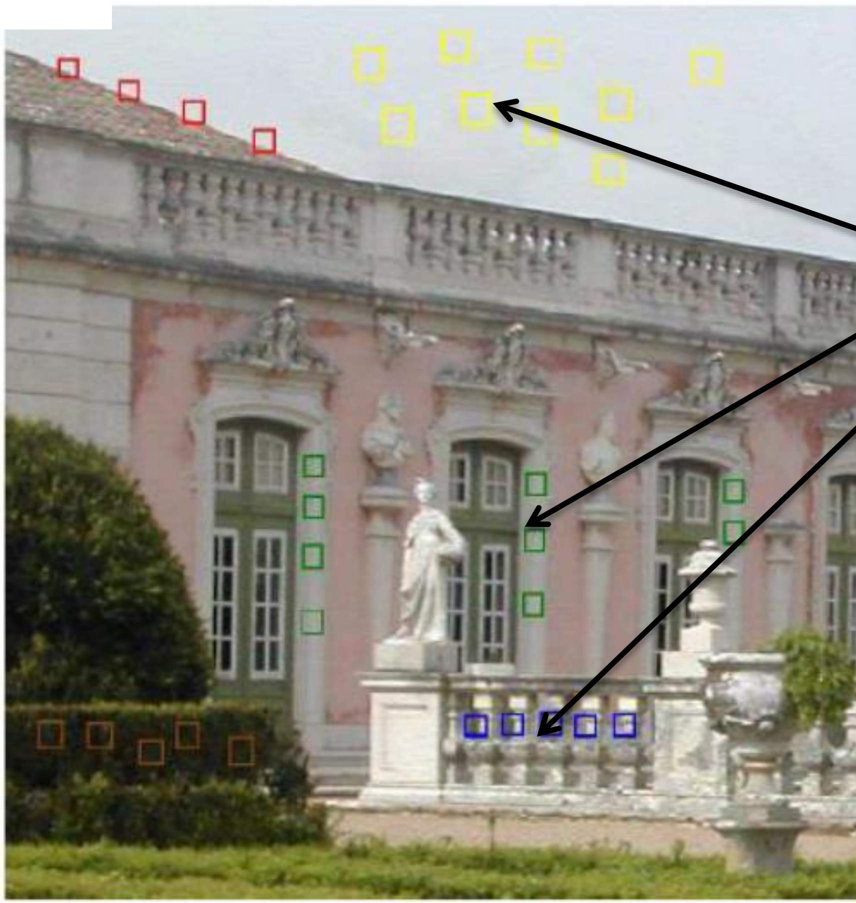


Image and its groups of « self-similar » patches. Nonlocal means computes an average of these groups to denoise them jointly.

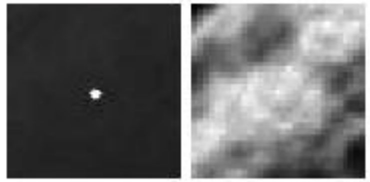
This idea goes back to Shannon (1948) who used it to simulate text.

Shannon's ideas extended by Efros and Leung (1999) to expand any texture from a small sample.

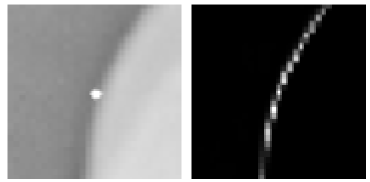
3-Nonlocal Means

Visualization of the patch « heat kernel » back-projected onto the image

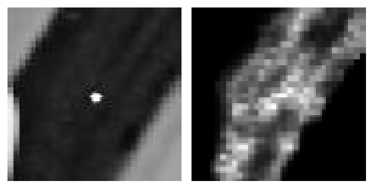
- Flat region. The large coefficients are spread out like a convolution.



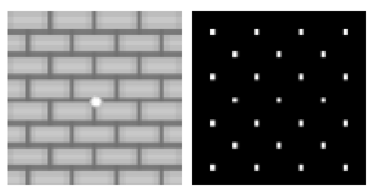
- Curved edge. The weights favor pixels belonging to the same contour.



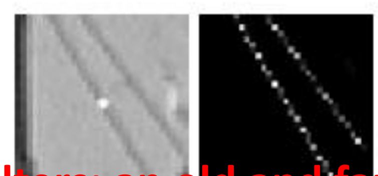
- Flat neighborhood. The average is made in the grey level neighborhood as the neighborhood filter.



- Periodic case. The large coefficients are distributed across the texture (non local).



- Repetitive structures. The weights favor similar configurations even they are far away (non local).



All these denoising algorithms boil down to three Gaussian convolutions with growing specificity: the first one is the classic Wiener spatial smoothing, the second is space+value nonlinear convolution, the third is a convolution with a Gaussian in the « patch space »

Gaussian convolution

$$G_\sigma^2 * u(\mathbf{x}) = \frac{\int_\Omega G_\sigma^2(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y}}{\int_\Omega G_\sigma^2(\mathbf{x}-\mathbf{y})d\mathbf{y}}$$

Neighborhood filter (Yaroslavski, Lee 1980)

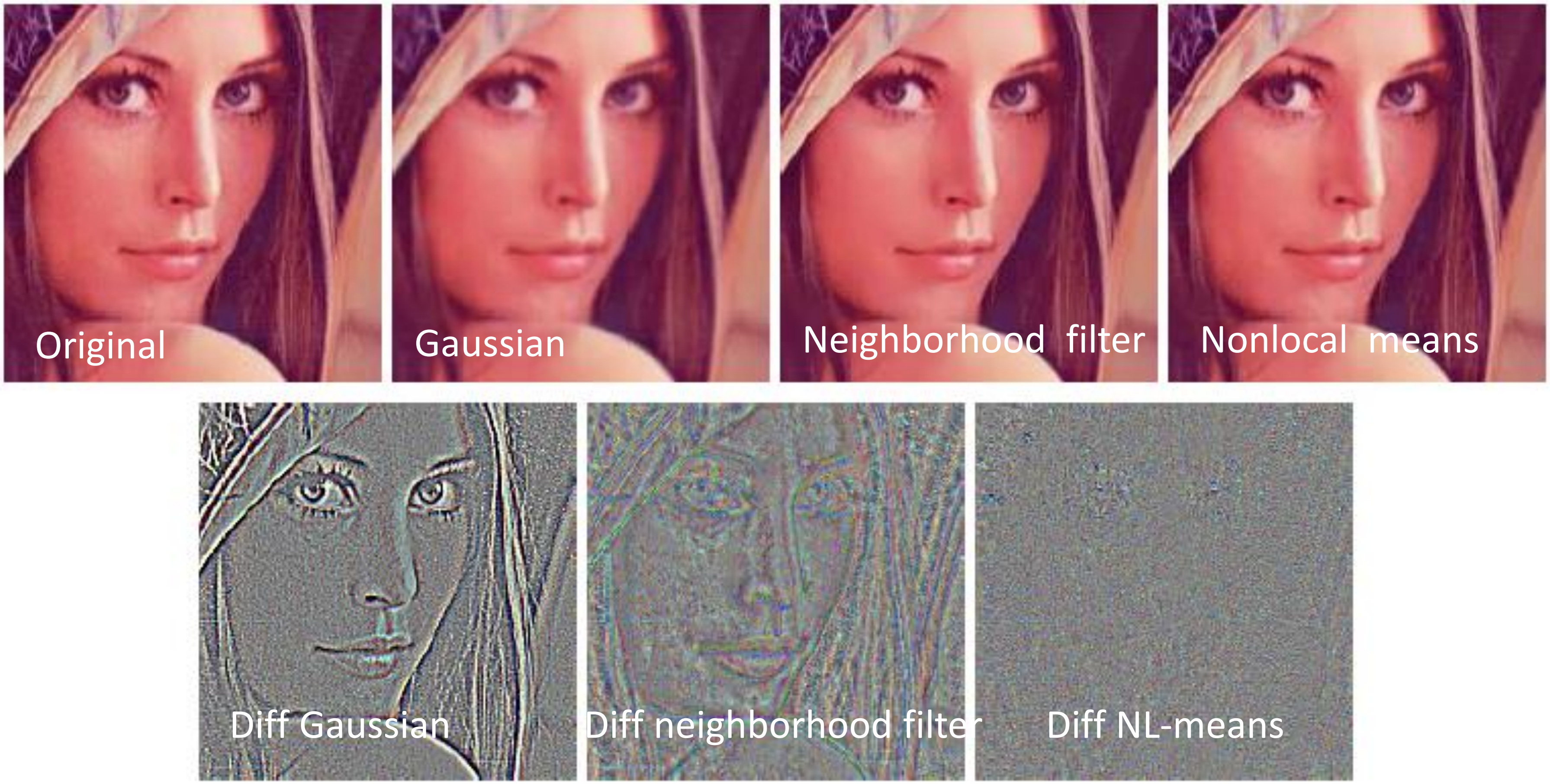
$$NFu(\mathbf{x}) = \frac{\int_\Omega G_{\sigma_s}^2(\mathbf{x}-\mathbf{y})G_{\sigma_r}^3(u(\mathbf{x})-u(\mathbf{y}))u(\mathbf{y})d\mathbf{y}}{\int_\Omega G_{\sigma_s}^2(\mathbf{x}-\mathbf{y})G_{\sigma_r}^3(u(\mathbf{x})-u(\mathbf{y}))d\mathbf{y}}$$

Nonlocal-means denoising (Buades & Coll & M. 2004)

$$NL(P_\mathbf{x}) = \frac{\int_\Omega G_{\sigma_p}^N(P_\mathbf{x}-P_\mathbf{y})P_\mathbf{y}d\mathbf{y}}{\int_\Omega G_{\sigma_p}^N(P_\mathbf{x}-P_\mathbf{y})d\mathbf{y}}$$

3-Nonlocal Means

Comparison of **Gaussian convolution**, **neighborhood filter**, **nonlocal means** on a real scanned image (Lena). The difference between the image and its filtered version should look like noise



Difference between the image and its filtered version. It should contain no visible structure. Hence NL-means is better than NF, which is better than the Gaussian convolution.

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31

Global Bayesian Denoising (Nadler & Levin 2011)

$\mathcal{P}$ is the set of all patches in the world

The probability of observing the noisy patch P given a perfect patch Q is

$$\mathbb{P}(P \mid Q) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{-\frac{||P-Q||^2}{2\sigma^2}} = G_\sigma^N(P-Q)$$

Thus, given a noisy patch P its optimal estimator for the mean square error is by Bayes' formula

$$GBD(P) = \mathbb{E}[Q \mid P] = \int \mathbb{P}(Q \mid P) Q dQ = \int \frac{\mathbb{P}(P \mid Q)}{\mathbb{P}(P)} \mathbb{P}(Q) Q dQ$$

Hence the Global Bayes Denoiser

$$GBD(P) = \frac{\int_{\mathcal{P}} G_\sigma^N(P-Q) Q dQ}{\int_{\mathcal{P}} G_\sigma^N(P-Q) dQ}$$

Gives the best estimate of the noisy patch P knowing all the patches $Q \in \mathcal{P}$ in the world! Tested on 2^{10} patches.

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N.B: the last integral is obtained by a change of variables. The preceding integral is w.r. to the Lebesgue measure. **The last integral is made on the « space of patches ».**

The Levin and Nadler optimal « global denoising algorithm » uses « all patches of the world »

- **Input**: Noisy image $\tilde{u}$, its patches $\tilde{P}$
- **Input**: Very large set of $M = 2^{10}$ patches P_i extracted from a large set of noiseless natural images (20000)
- **Output**: Denoised image $\hat{u}$.
- **for all patches $\tilde{P}$ extracted from $\tilde{u}$:** Compute the MMSE denoised estimate of $\tilde{P}$

$$G_\sigma^N(\mathbf{x}) = \frac{1}{(2\pi\sigma)^{\frac{N}{2}}} e^{-\frac{||\mathbf{x}||^2}{2\sigma^2}}$$

$$\hat{P} \simeq \frac{\sum_{i=1}^{M} \mathbb{P}(\tilde{P} \mid P_i) P_i}{\sum_{i=1}^{M} \mathbb{P}(\tilde{P} \mid P_i)}$$

  where $\mathbb{P}(\tilde{P} \mid P_i)$ is known from the Gaussian noise distribution.
- (Aggregation) : for each pixel $\mathbf{j}$ of u, compute the denoised version $\hat{u}_\mathbf{j}$ as the average of all values $\hat{P}(\mathbf{j})$ for all patches

A. Levin, B. Nadler. *CVPR* 2011. Natural image denoising: Optimality and inherent bounds
Zoran, D., & Weiss, Y. *ICCV* 2011. From learning models of natural image patches to whole image restoration.

All denoising integral formulas in one slide

Gaussian convolution

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$\mathcal{P}$ is the set of all patches in the world

Szlam, A. D., Maggioni, M., & Coifman, R. R. *JMLR* 2008. Regularization on graphs with function-adapted diffusion processes.

Comparison of six state of the art methods, which combine the above mentioned principles: transform thresholding, nonlocal means and the Bayesian global estimator. Some are used in real cameras.

Original, noisy, DCT sliding window, BLS-GSM. Experiments: IPOL

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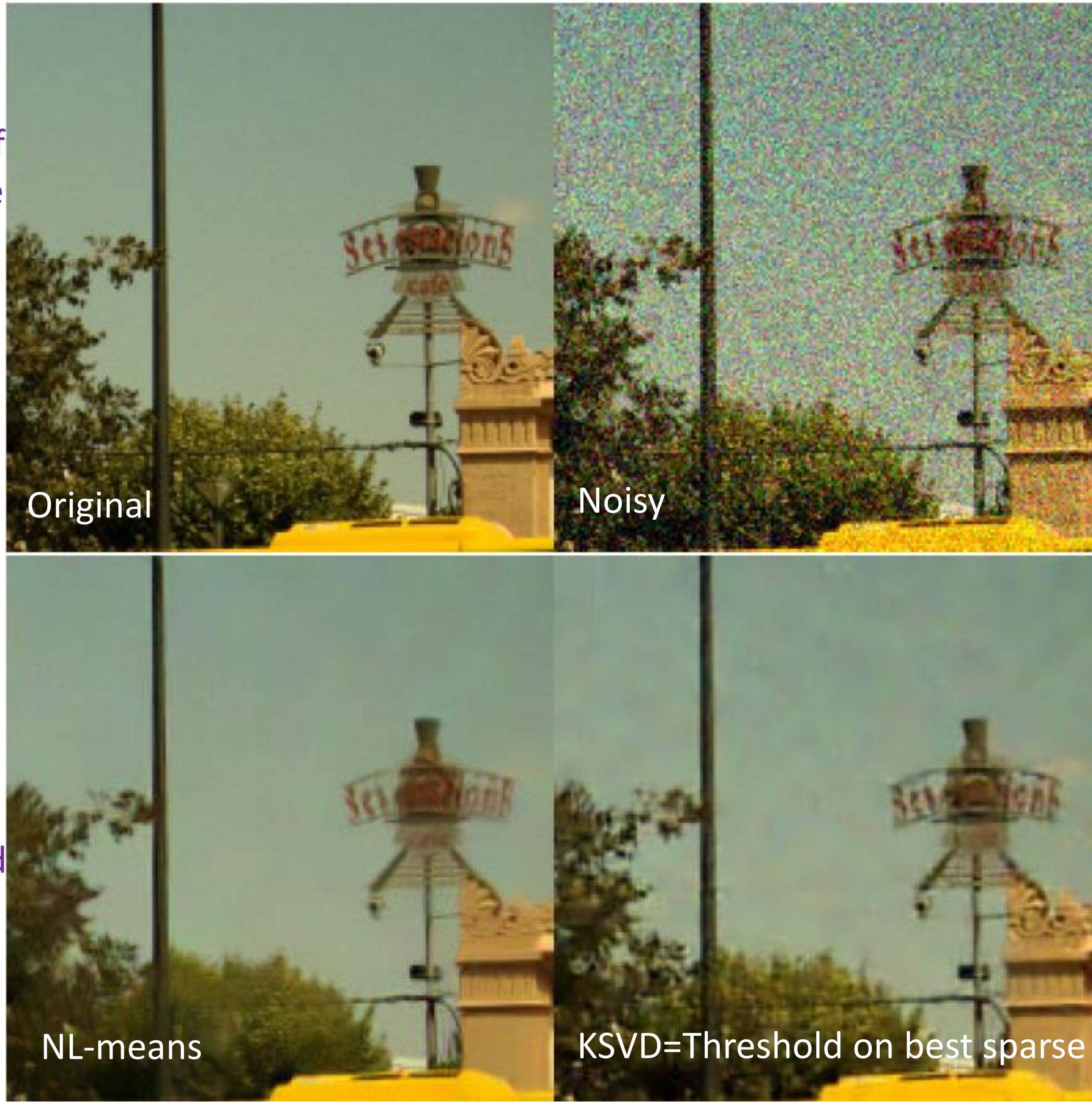

Original, noisy, NL-means, K-SVD Experiments: IPOL

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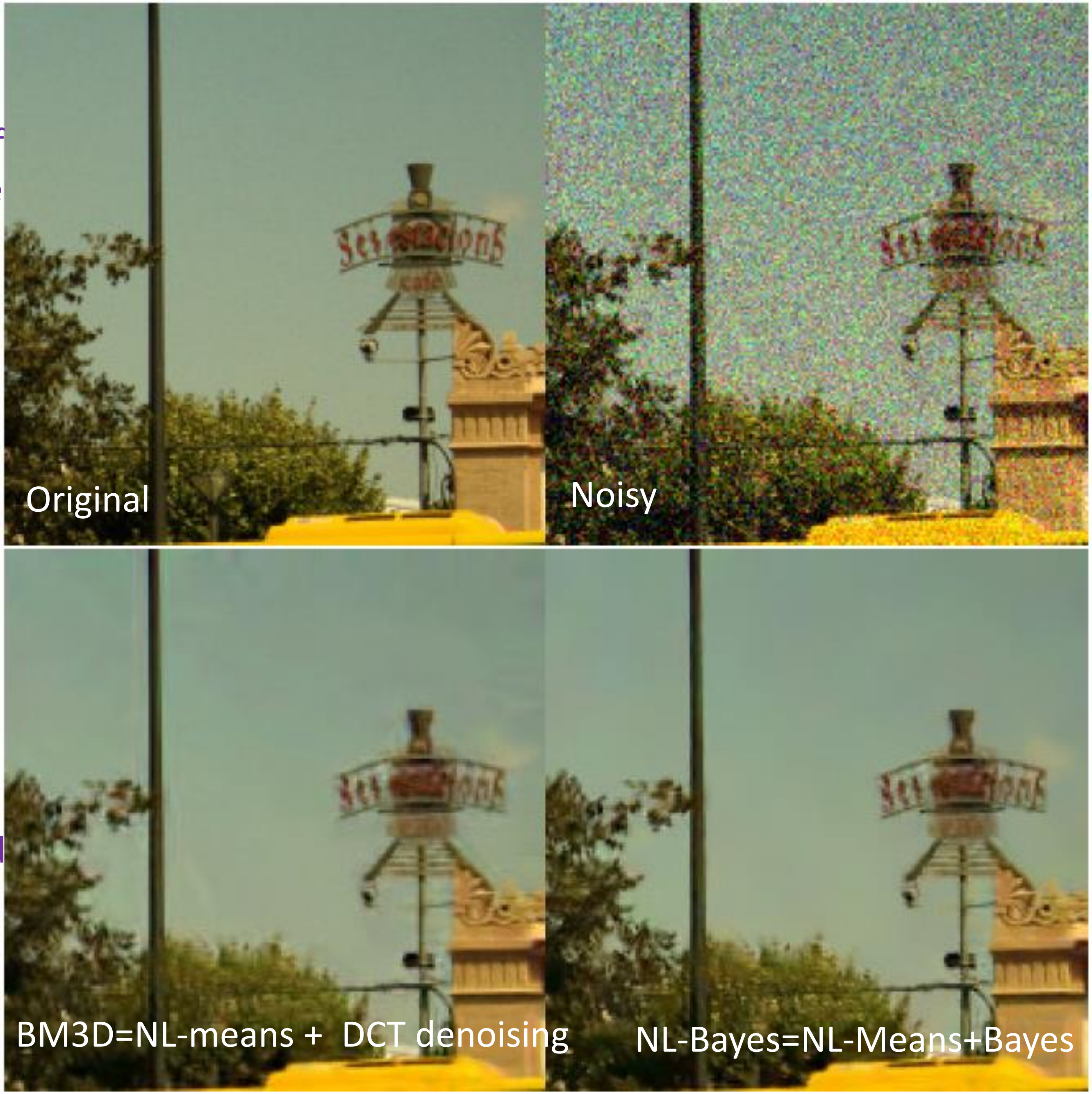

Original, noisy, BM3D and Non-local Bayes (selected by DxO) Experiments: IPOL

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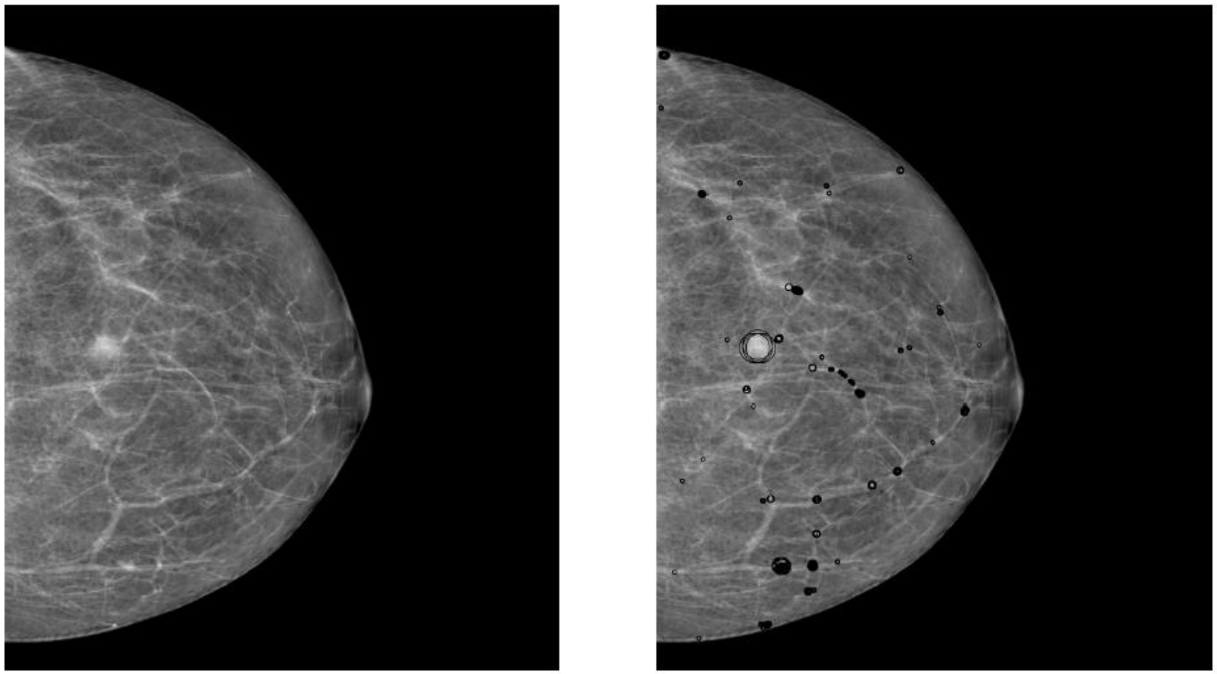
A-contrario Detectability of Spots in Textured Backgrounds (2009)

B. Grosjean and L. Moisan

Tumor detection in mammography (data: General Electric)

Problem: background too complex (and samples too sparse) for a simple stochastic model!



(a) Original image (b) $\varepsilon = 1$

“Figure 9: A-contrario detection of spots in a mammography image, for two values of the threshold “ applied to the detection metric NFA$_2$. The large opacity is well detected (its NFA$_2$ is equal to 0.15). Some clinically wrong detections also occur (small spots), mainly because the curvilinear breast structures are not taken into account by the texture model (these are false alarms clinically speaking, but are not with respect to the naive model used for the breast texture).”

A-contrario Detectability of Spots in Textured Backgrounds (**2009**) B. Grosjean and L. Moisan

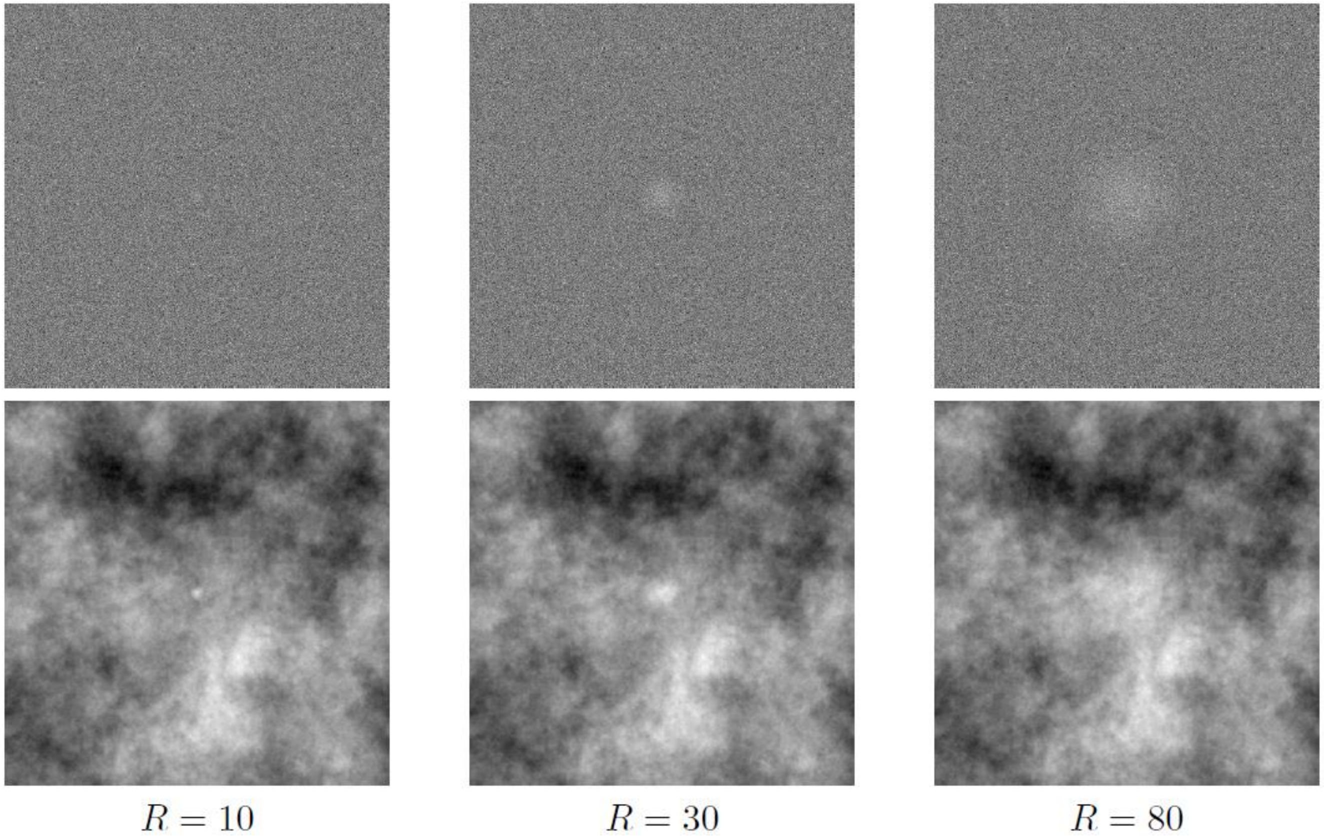
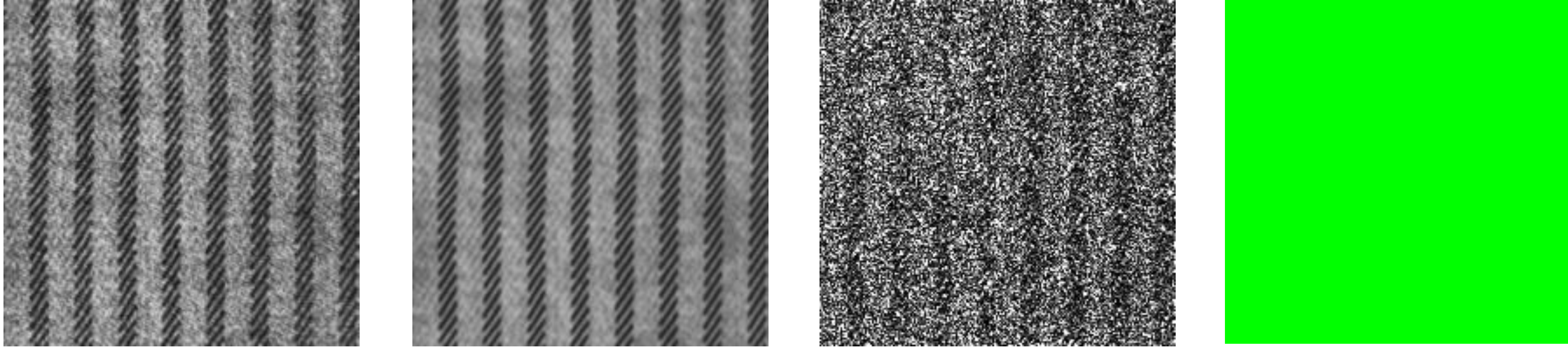


Figure 2: Examples of simulated spots with various sizes (R = 10, 30 and 80) but similar contrast, in a white noise texture (top row) and in a colored noise texture (bottom row). In the white noise texture, the saliency of the spot increases with its size. On the contrary, in the colored noise background, the unexpected reverse phenomenon occurs: the larger the spot, the less visible it is.

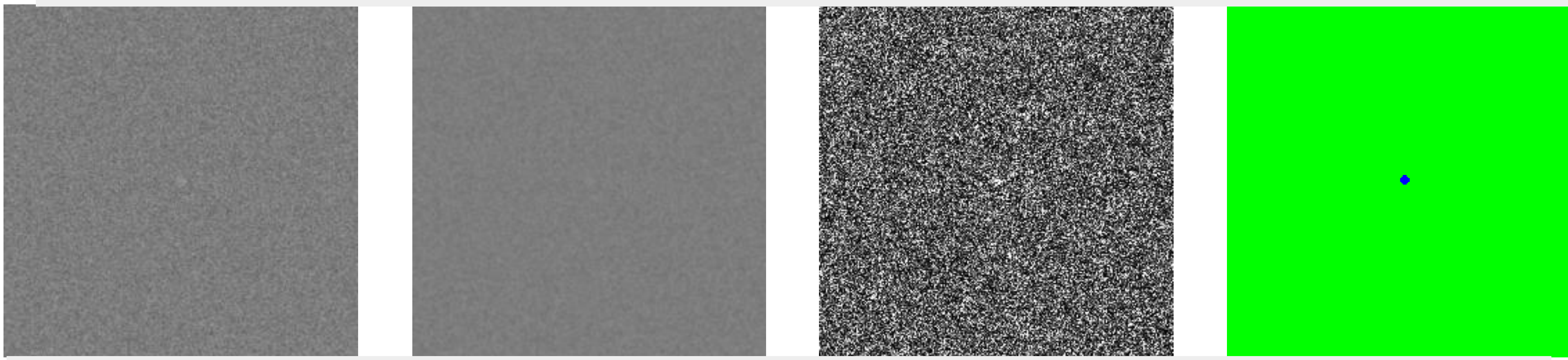
Type 2 Anomaly detection

- Take the difference image $N = \tilde{u} - u$ where $\tilde{u}$ is the estimated image model. N should be white noise.
- Compute the standard deviation σ of N
- Detect all **exceptional pixels** x, such that $\mathbb{P}(N(x) > s\sigma)$, (s=4)
- For each square window W with size n (e.g. $n = 16^2$); count the number k of exceptional pixels in W
- Compute the **Number of false alarms** of the **exceptional square window**, $NFA(k, s) := n' \begin{pmatrix} k \\ n \end{pmatrix} \mathbb{P}(N(x) > s\sigma)^k$. ($n'$ is the number of tested regions)

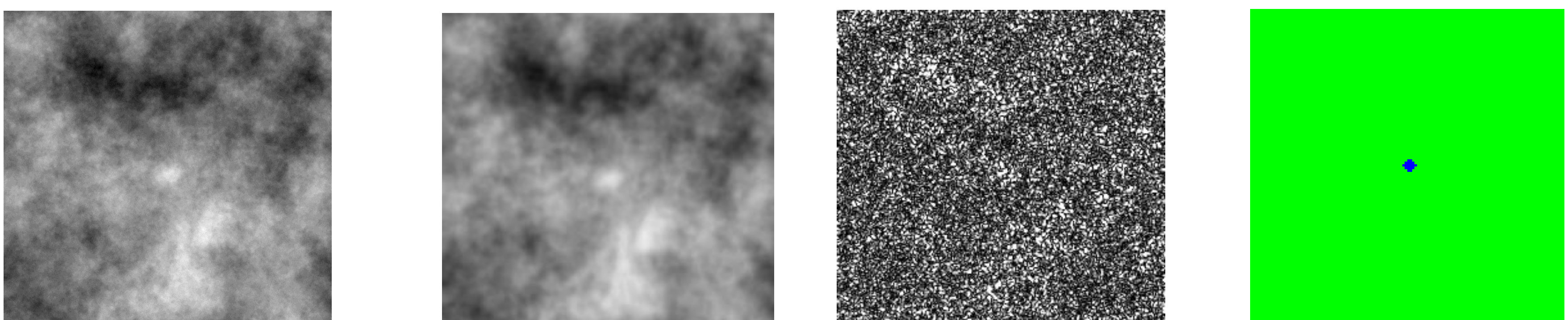
Desolneux, Agnes, Lionel Moisan, and JMM. *From gestalt theory to image analysis: a probabilistic approach*. Vol. 34. Springer Science & Business Media, 2007.



Original denoised noise no detection



Original denoised noise detection: log NFA = -10.7

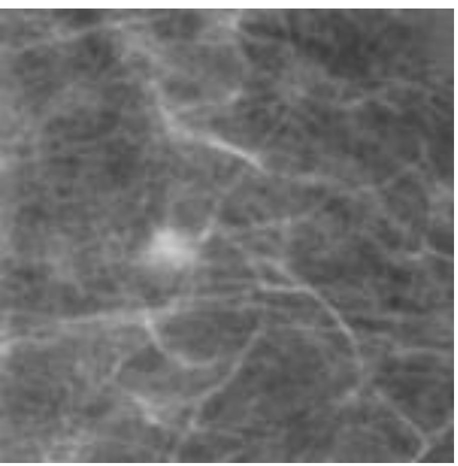


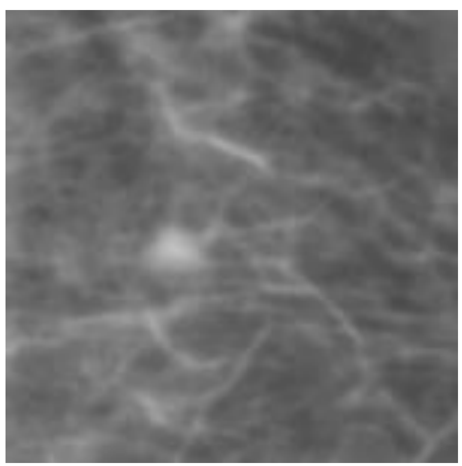
Original denoised noise detection: log NFA = -45.3

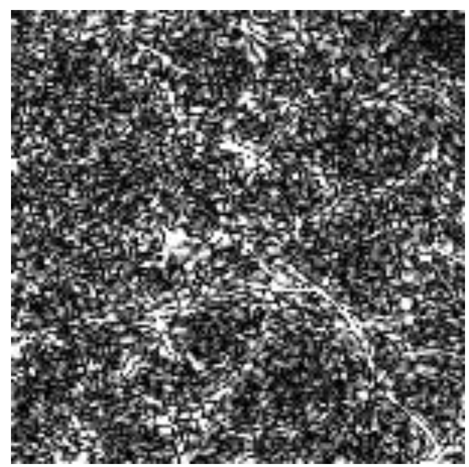
(a) Original image

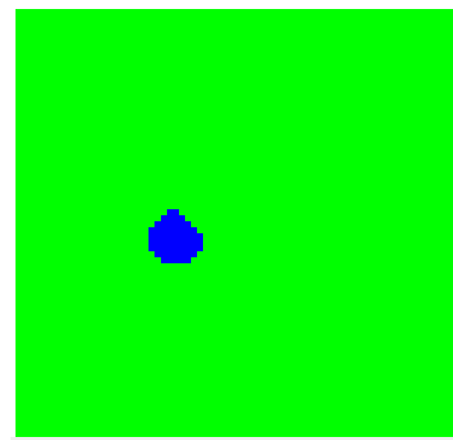
NFA = 0.15 for the tumor, many false detections

A-contrario Detectability of Spots in Textured Backgrounds (2009)
B. Grosjean and L. Moisan







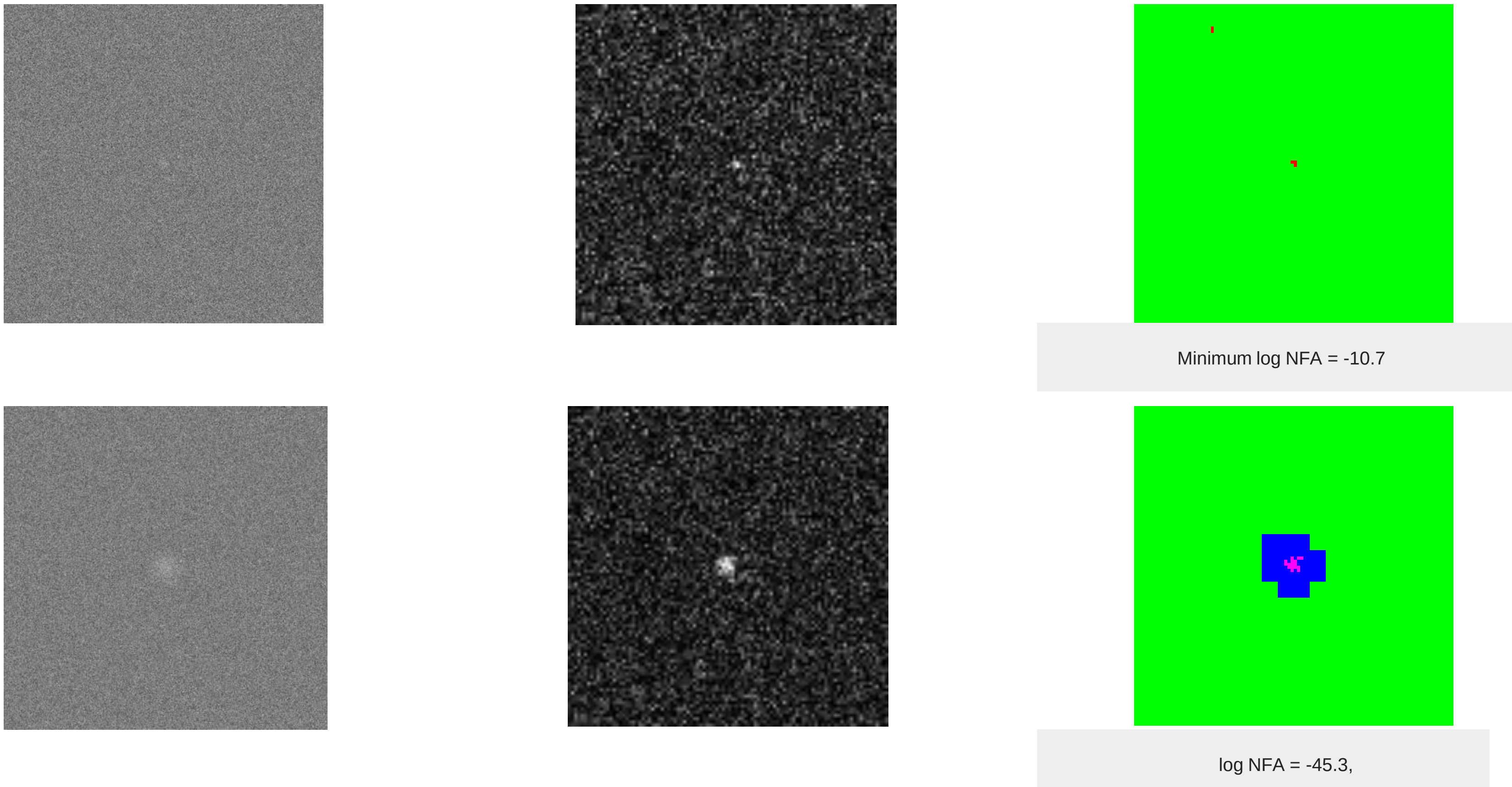


log NFA = -39.2857
By the very same method applied to the noise

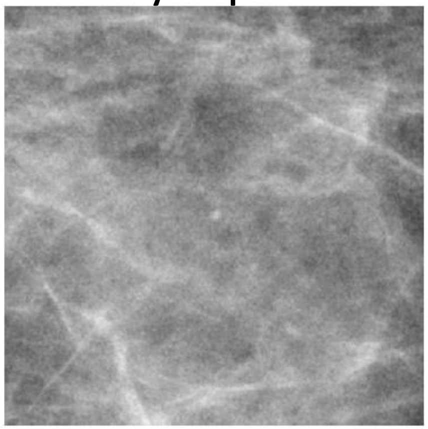
The oldest heliographic engraving known in the world, a reproduction of a 17th century Flemish engraving. Nicéphore Niépce in 1825, (Bibliothèque nationale de France).

Joseph-Nicéphore Niépce (1765-1833): first indoor photograph,
Denoised by the Noise Clinic, IPOL (Image Processing on Line www.ipol.im)

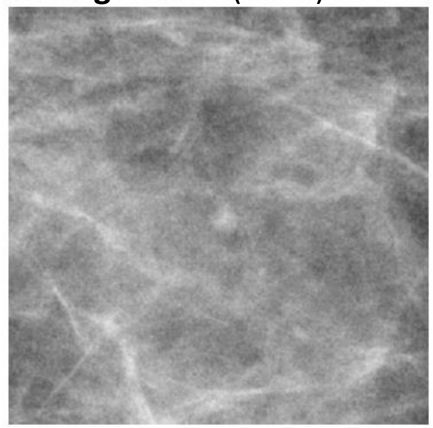
Joseph-Nicéphore Niépce (1765-1833): first indoor photograph,
Denoised by the Noise Clinic, IPOL (Image Processing on Line www.ipol.im)


Minimum log NFA = -10.7
log NFA = -45.3,

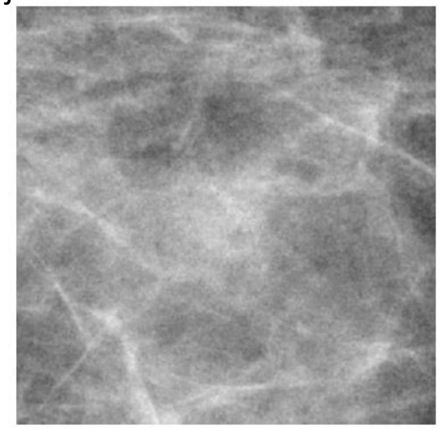
A-contrario Detectability of Spots in Textured Backgrounds (2009) B. Grosjean and L. Moisan



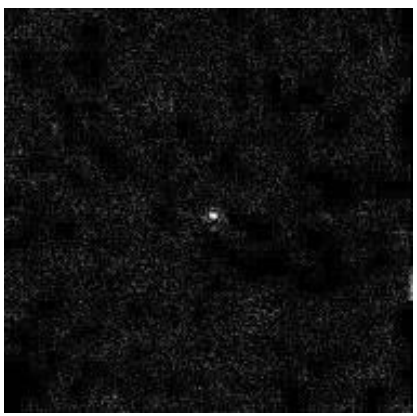
(a') R=5 (NFA_2=$2.6 \cdot 10^{-6}$)

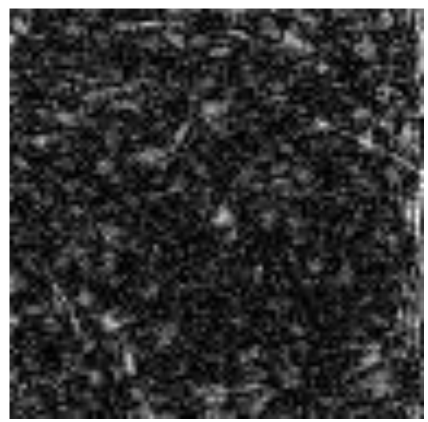


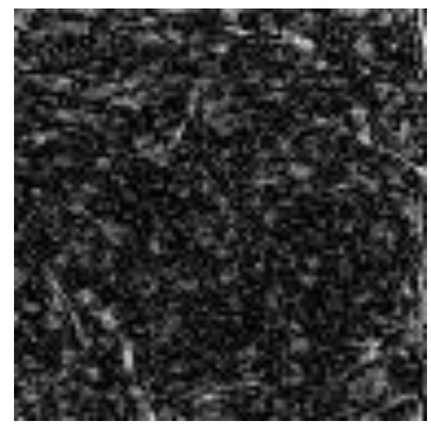
(b') R=10 (NFA_2=$4.6 \cdot 10^{-4}$)



(c') R=50 (NFA_2=0.2)





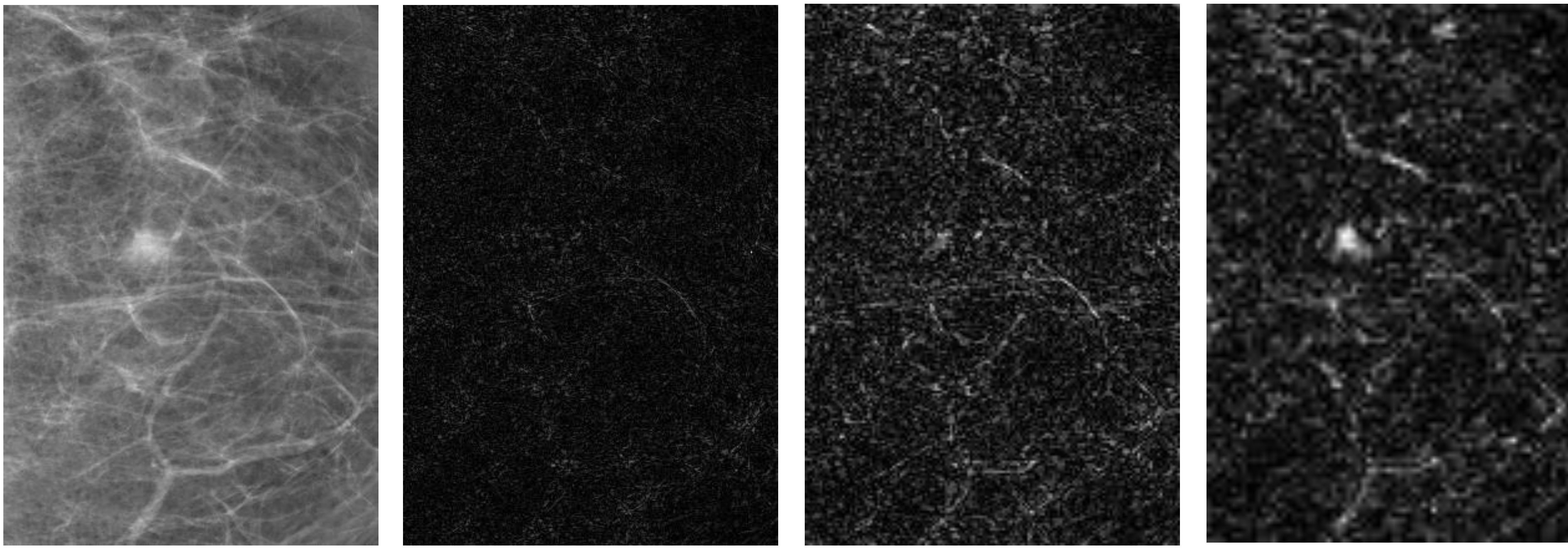


log NFA = -63.

log NFA = -18.

no detection

A-contrario Detectability of Spots in Textured Backgrounds (2009) B. Grosjean and L. Moisan



Minimum logNFA = -27.4337
Bayesian

Minimum logNFA = -27.05

Minimum logNFA = -39.2857

NL-Bayes : the only diference w.r. to global denoising is that a Gaussian model is locally estimated in the image patch space

- patch noise model $\mathbb{P}(\tilde{P}|P) = c \cdot e^{-\frac{\|\tilde{P}-P\|^2}{2\sigma^2}}$
- Bayes' rule $\mathbb{P}(P|\tilde{P}) = \frac{\mathbb{P}(\tilde{P}|P)\mathbb{P}(P)}{\mathbb{P}(\tilde{P})}$
- assume we got a patch Gaussian model $\mathbb{P}(Q) = c.e^{-\frac{(Q-\overline{P})^t \mathbf{C}_P^{-1}(Q-\overline{P})}{2}}$
- hence the variational problem

$$\begin{aligned}
\max_P \mathbb{P}(P|\tilde{P}) &\Leftrightarrow \max_P \mathbb{P}(\tilde{P}|P)\mathbb{P}(P) \\
&\Leftrightarrow \max_P e^{-\frac{\|P-\tilde{P}\|^2}{2\sigma^2}} e^{-\frac{(P-\overline{P})^t \mathbf{C}_P^{-1}(P-\overline{P})}{2}} \\
&\Leftrightarrow \min_P \frac{\|P-\tilde{P}\|^2}{\sigma^2} + (P-\overline{P})^t \mathbf{C}_P^{-1}(P-\overline{P}).
\end{aligned}$$

- An empirical covariance matrix $C_{\tilde{P}}$ can be obtained for the patches $\tilde{Q}$ similar to $\tilde{P}$. P and the noise n being independent,
  $\mathbf{C}_{\tilde{P}} = \mathbf{C}_P + \sigma^2 \mathbf{I}; \qquad E\tilde{Q} = \overline{P}$

A. Buades, M. Lebrun, J.M.M. : A Non-local Bayesian image denoising algorithm, SIIMS 2013
BLS-GSM: J. Portilla, V. Strela, M.J. Wainwright, E.P. Simoncelli, TIP 2003

Bayesian denoising : NL-Bayes

$\max_P \; \mathbb{P}(P|\tilde{P}) \Leftrightarrow \min_P \; \frac{\|P-\tilde{P}\|^2}{\sigma^2} + (P-\overline{\tilde{P}})^t(\mathbf{C}_{\tilde{P}} - \sigma^2\mathbf{I})^{-1}(P-\overline{\tilde{P}})$

one step estimation $\hat{P}_1 = \overline{\tilde{P}} + \left[\mathbf{C}_{\tilde{P}} - \sigma^2\mathbf{I}\right]\mathbf{C}_{\tilde{P}}^{-1}(\tilde{P} - \overline{\tilde{P}})$, where empirically:

$$\mathbf{C}_{\tilde{P}} \simeq \frac{1}{\#\mathcal{P}(\tilde{P}) - 1} \sum_{\tilde{Q}\in\mathcal{P}(\tilde{P})} \left(\tilde{Q} - \overline{\tilde{P}}\right)\left(\tilde{Q} - \overline{\tilde{P}}\right)^t, \qquad \overline{\tilde{P}} \simeq \frac{1}{\#\mathcal{P}(\tilde{P})} \sum_{\tilde{Q}\in\mathcal{P}(\tilde{P})} \tilde{Q}.$$

Iteration ("oracle estimation"): $\hat{P}_2 = \overline{\tilde{P}}^1 + \mathbf{C}_{\hat{P}_1}\left[\mathbf{C}_{\hat{P}_1} + \sigma^2\mathbf{I}\right]^{-1}(\tilde{P} - \overline{\tilde{P}}^1)$

where

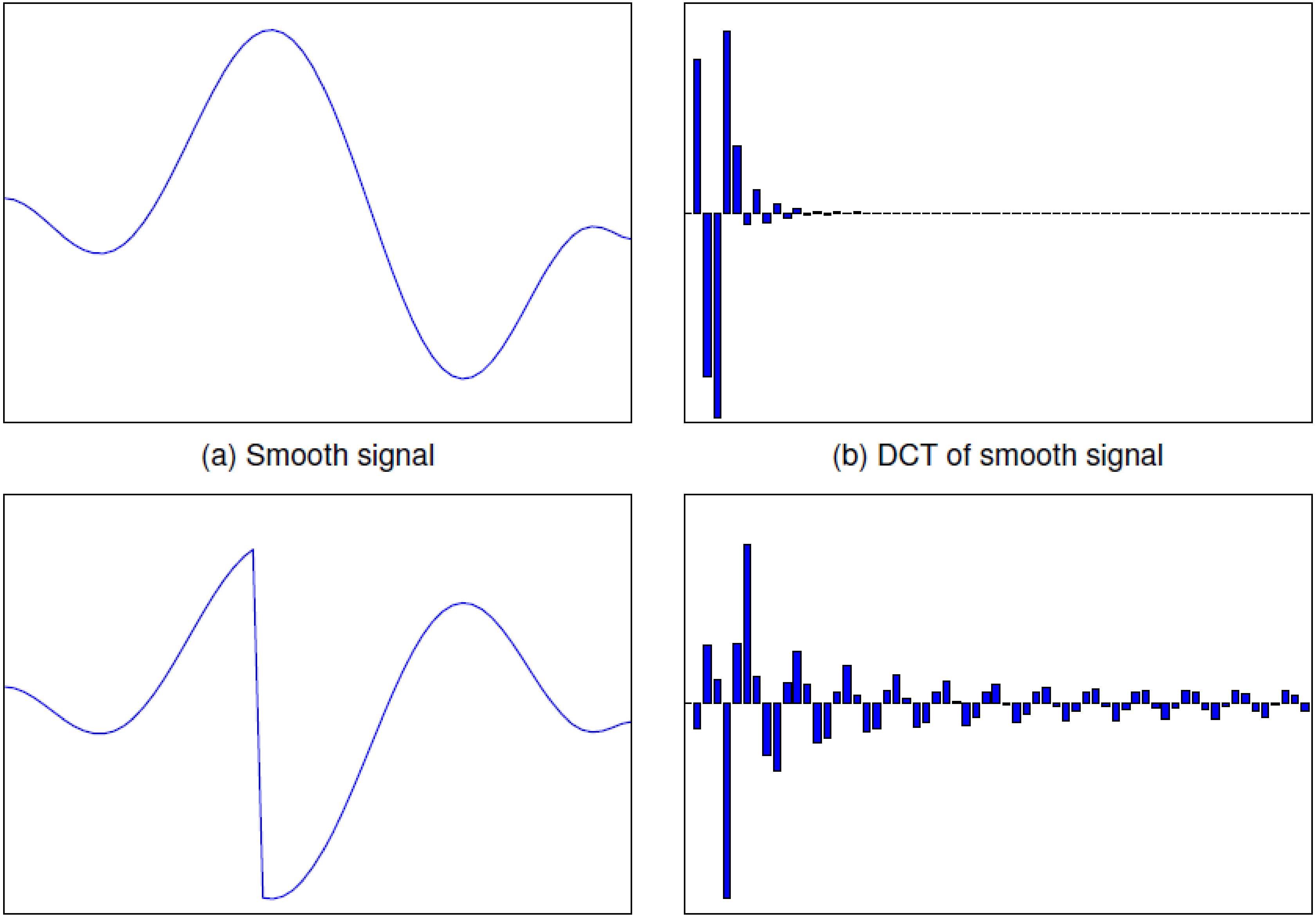
$$\mathbf{C}_{\hat{P}_1} \simeq \frac{1}{\#\mathcal{P}(\hat{P}_1) - 1} \sum_{\hat{Q}_1\in\mathcal{P}(\hat{P}_1)} \left(\hat{Q}_1 - \overline{\tilde{P}}^1\right)\left(\hat{Q}_1 - \overline{\tilde{P}}^1\right)^t, \quad \overline{\tilde{P}}^1 \simeq \frac{1}{\#\mathcal{P}(\hat{P}_1)} \sum_{\hat{Q}_1\in\mathcal{P}(\hat{P}_1)} \tilde{Q}.$$

A. Buades, M. Lebrun, J.M.M.: A Non-local Bayesian image denoising algorithm, SIIMS 2013

```
function x = DDID(y, sigma2)
    x = step(y, y, sigma2, 15, 7, 100, 4.0);
    x = step(x, y, sigma2, 15, 7, 8.7, 0.4);
    x = step(x, y, sigma2, 15, 7, 0.7, 0.8);
end

function xt = step(x, y, sigma2, r, sigma_s, gamma_r, gamma_f)

    [dx dy] = meshgrid(-r:r);
    h = exp(- (dx.^2 + dy.^2) / (2 * sigma_s^2));
    xp = padarray(x, [r r], 'symmetric');
    yp = padarray(y, [r r], 'symmetric');
    xt = zeros(size(x));

    parfor p = 1:numel(x), [i j] = ind2sub(size(x), p);

        % Spatial Domain: Bilateral Filter
        g  = xp(i:i+2*r, j:j+2*r);
        y  = yp(i:i+2*r, j:j+2*r);
        d  = g - g(1+r, 1+r);
        k  = exp(- d.^2 ./ (gamma_r * sigma2)) .* h;     % Eq. 4
        gt = sum(sum(g .* k)) / sum(k(:));               % Eq. 2
        st = sum(sum(y .* k)) / sum(k(:));               % Eq. 3

        % Fourier Domain: Wavelet Shrinkage
        V  = sigma2 .* sum(k(:).^2);                     % Eq. 5
        G  = fft2(ifftshift((g - gt) .* k));             % Eq. 6
        S  = fft2(ifftshift((y - st) .* k));             % Eq. 7
        K  = exp(- gamma_f * V ./ (G .* conj(G)));       % Eq. 9
        St = sum(sum(S .* K)) / numel(K);                % Eq. 8

        xt(p) = st + real(St);                           % Eq. 1
    end
end
```

Algorithm 1: *MATLAB code of Dual-Domain Image Denoising. This code reproduces all grayscale images in this paper.*

Dual domain denoising Claude Knaus, Matthias Zwicker, ICIP 2013



(a) Smooth signal

(b) DCT of smooth signal

(c) Signal with discontinuity (edge)

(d) DCT of signal with discontinuity

Figure 2.6 – Behaviour of DCT coefficients. Signals containing discontinuities have their energy less concentrated in the DCT domain. This makes the DCT basis less effective for denoising purposes in presence of edges.

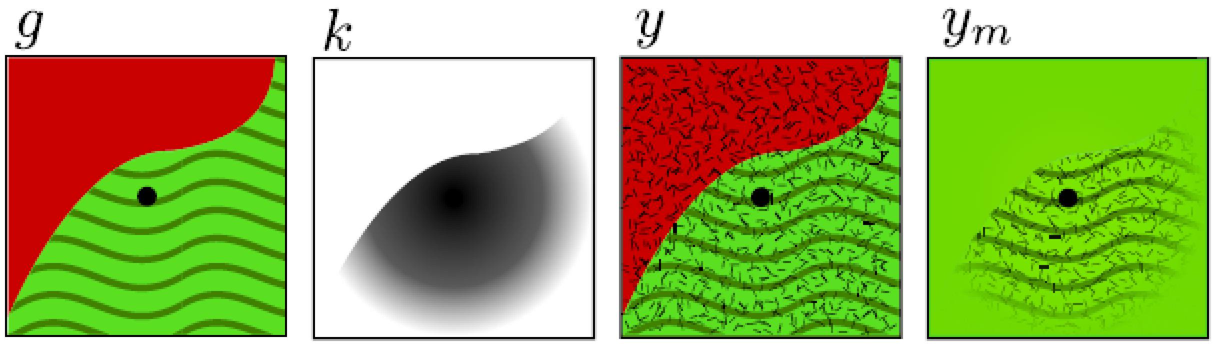
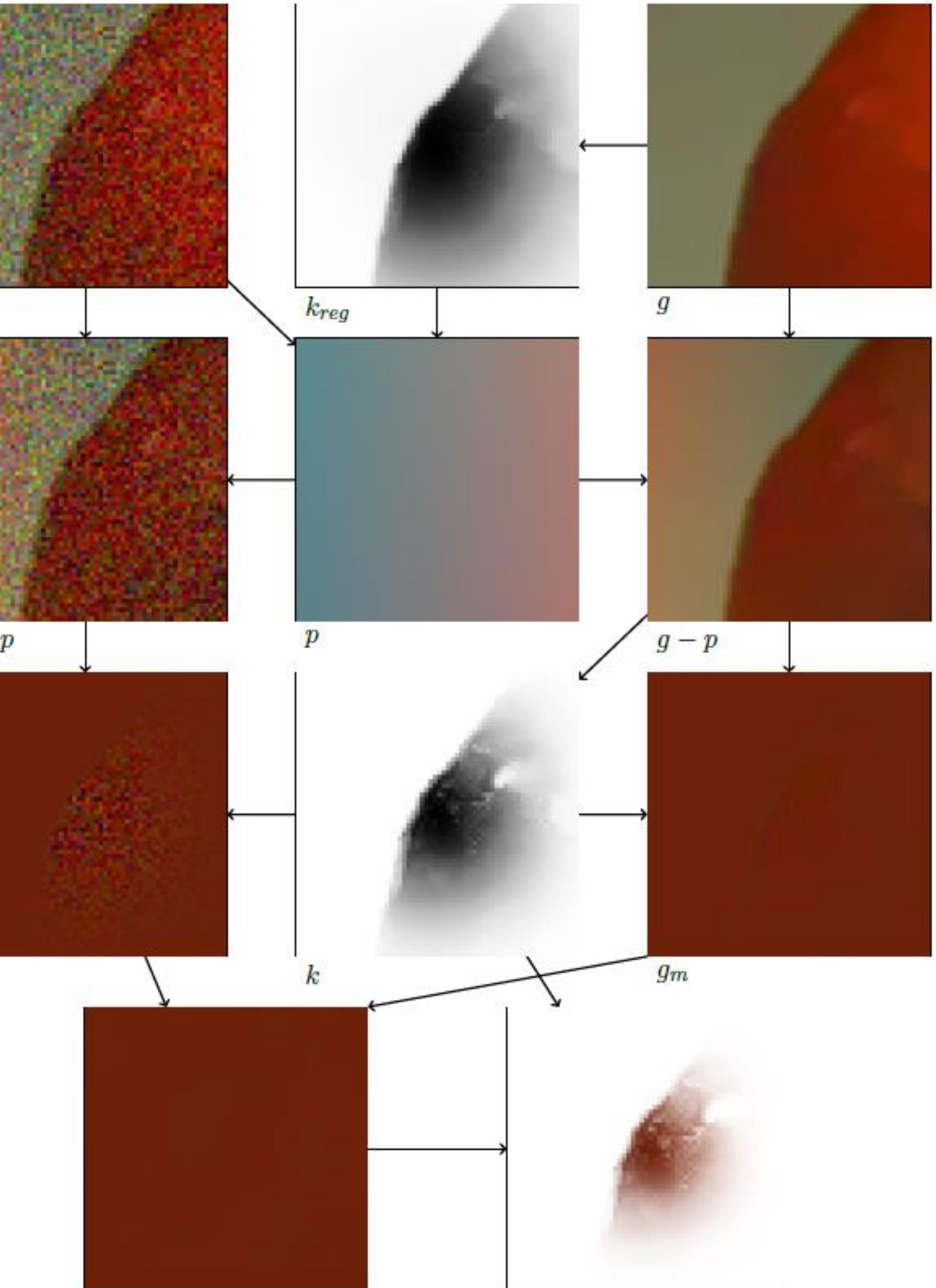


Fig. 2. Illustration of DDID's preprocessing of a patch. The kernel k is computed using the guide g. In the modified patch y_m all object discontinuities have been removed, leaving only the texture information corresponding to the object selected by the kernel k. The removed pixels are replaced by $\tilde{s}$: the average of the *meaningful* portion of the patch.

This explanation of Dual denoising comes from:

Non-local dual image denoising

N. Pierazzo, M. Lebrun, M. Rais, and G. Facciolo, ICIP 2014



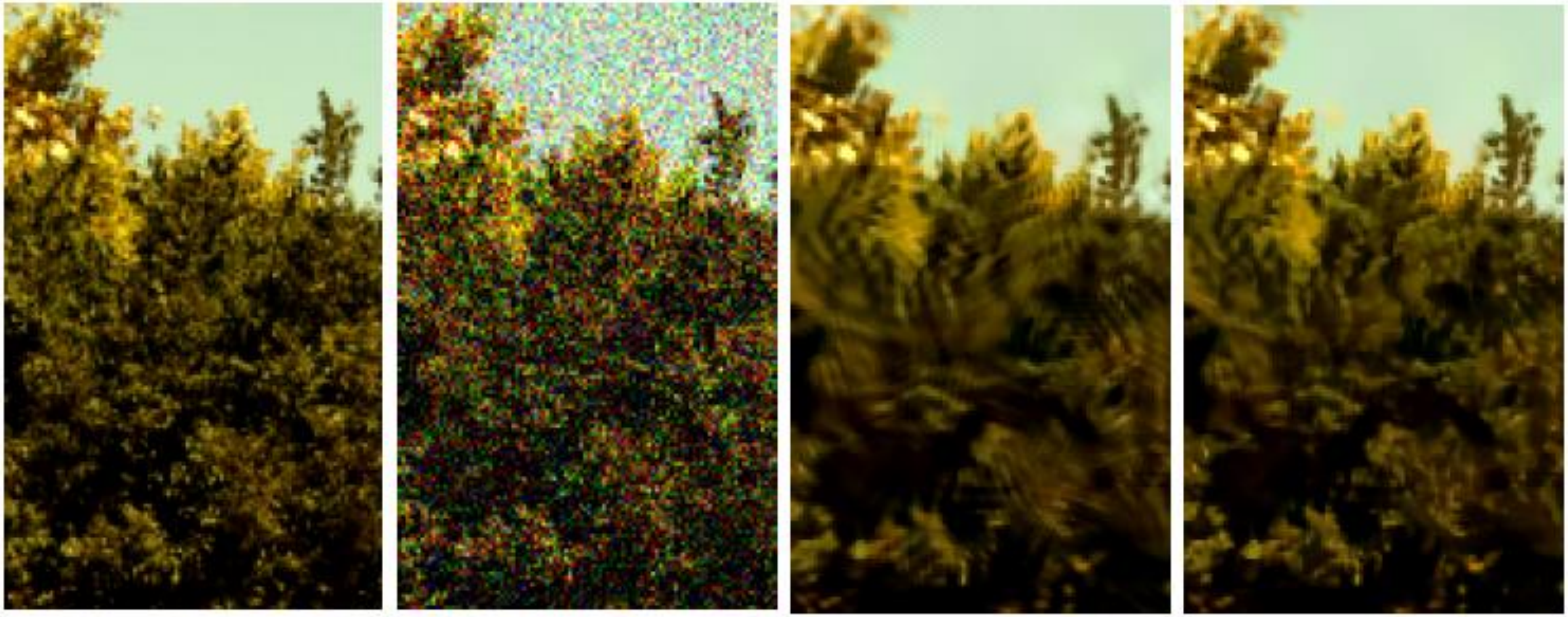
Steps DA3D
(Data adaptive dual domain denoising)

This figure shows what happens to a noisy patch taken in a natural image, containing an edge. The arrows indicate the elements needed to compute every step of the algorithm. Notice that, thanks to the weight function, the useful part of the patch is kept, while the discontinuities are completely removed.

Taken from

N. Pierazzo, PhD thesis, 2016

Fig. 3. Artifacts in DDID. From left to right: the noisy image (with $\sigma = 30$), the result of the first, second, and last iteration of the algorithm.



(a) Original (b) Noisy (c) DDID (d) NLDD

Fig. 1. A detail of the artifacts produced by DDID and the corresponding result of NLDD. In this example $\sigma = 30$.

Pierazzo, N., Lebrun, M., Rais, M. E. & Facciolo, G. (2014, October). Non-local dual image denoising. ICIP 2014

N. Pierazzo, PhD thesis, 2016

On line demo: http://dev.ipol.im/~pierazzo/ipol_demo/ddmd/

6-The noise clinic

The noise clinic at IPOL: estimating and denoising « any » image. This requires to estimate the noise before denoising. For image that have been manipulated, noise can be :

-signal dependent

-frequency dependent

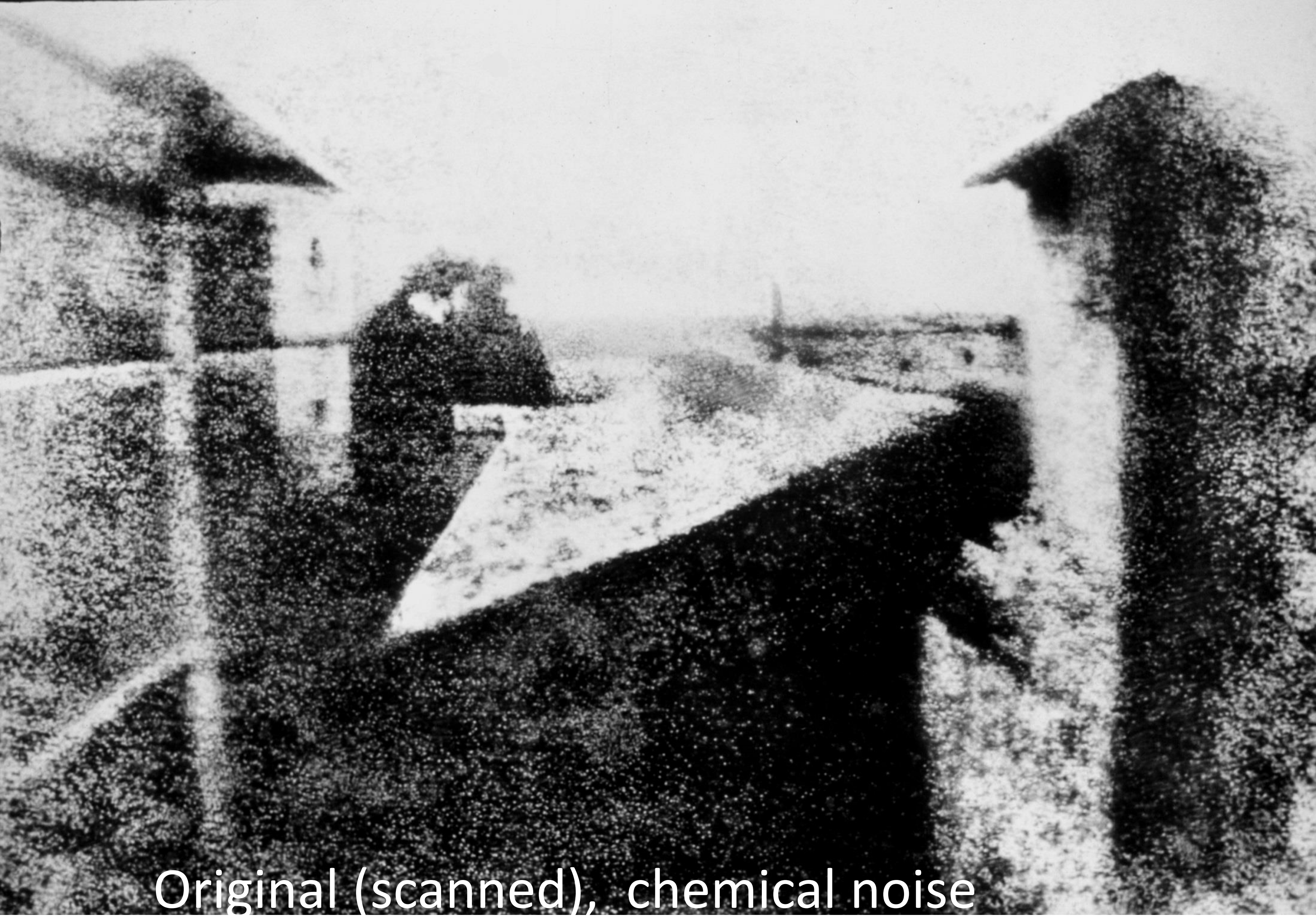
-scale dependent

Thus « noise curves » are established for each color level, each dyadic scale and each DCT frequency

Based on this a Bayesian algorithm can be applied (NL-Bayes)

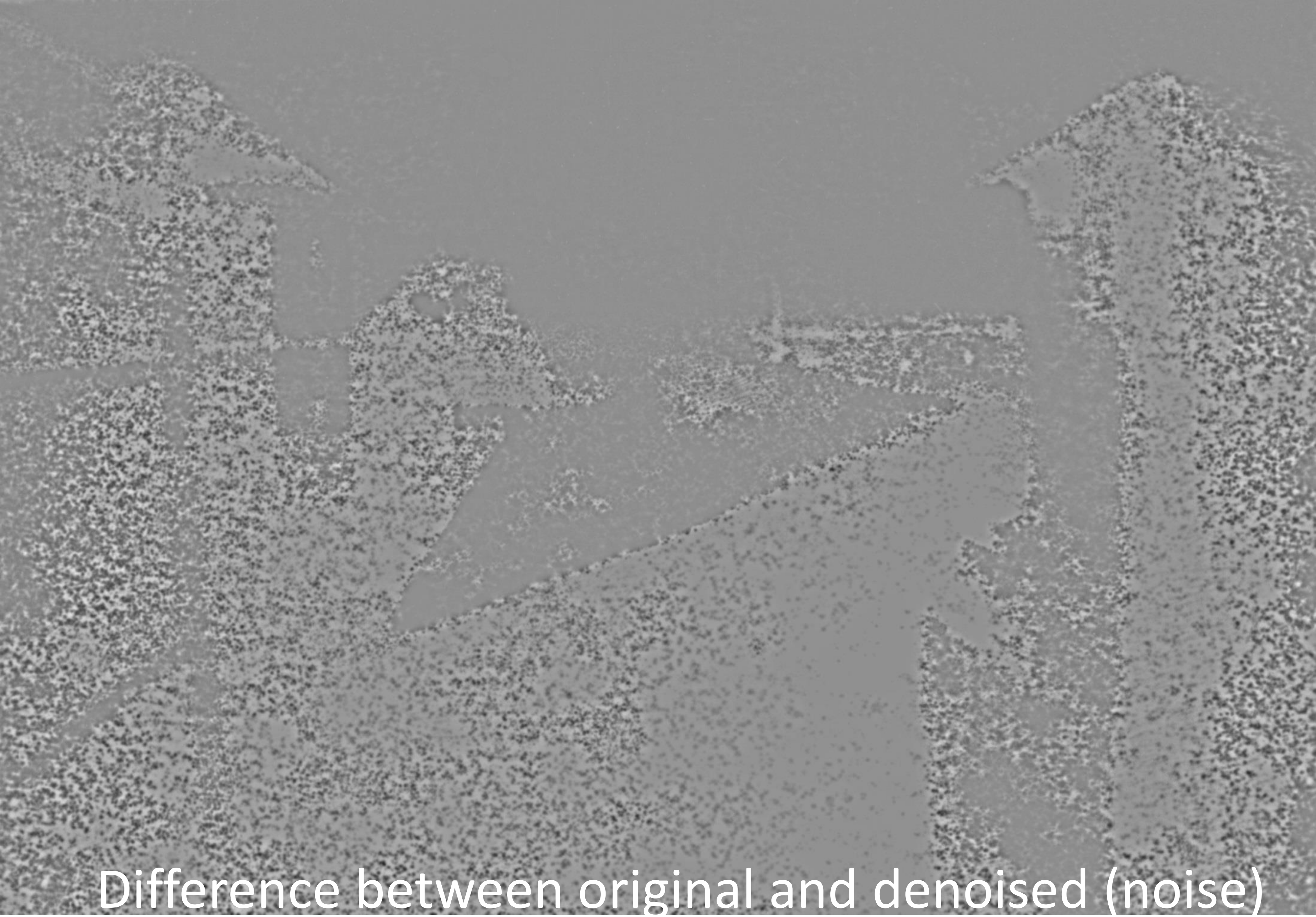
Where to test all algorithms: **Image Processing on Line (IPOL)** **http://www.ipol.im/**

Lebrun, Marc, Miguel Colom, and JMM. "The Noise Clinic: a blind image denoising algorithm." ***Image Processing On Line*** **5 (2015): 1-54.**



View from the Window at Le Gras (1826), Joseph Nicéphore Niépce

Denoised by the Noise Clinic, IPOL (Image Processing on Line www.ipol.im)



Difference between original and denoised (noise)

View from the Window at Le Gras (1826), Joseph Nicéphore Niépce

The oldest heliographic engraving known in the world, a reproduction of a 17th century Flemish engraving. Nicéphore Niépce in 1825, (Bibliothèque nationale de France).

Joseph-Nicéphore Niépce (1765-1833): first indoor photograph,
Denoised by the Noise Clinic, IPOL (Image Processing on Line www.ipol.im)

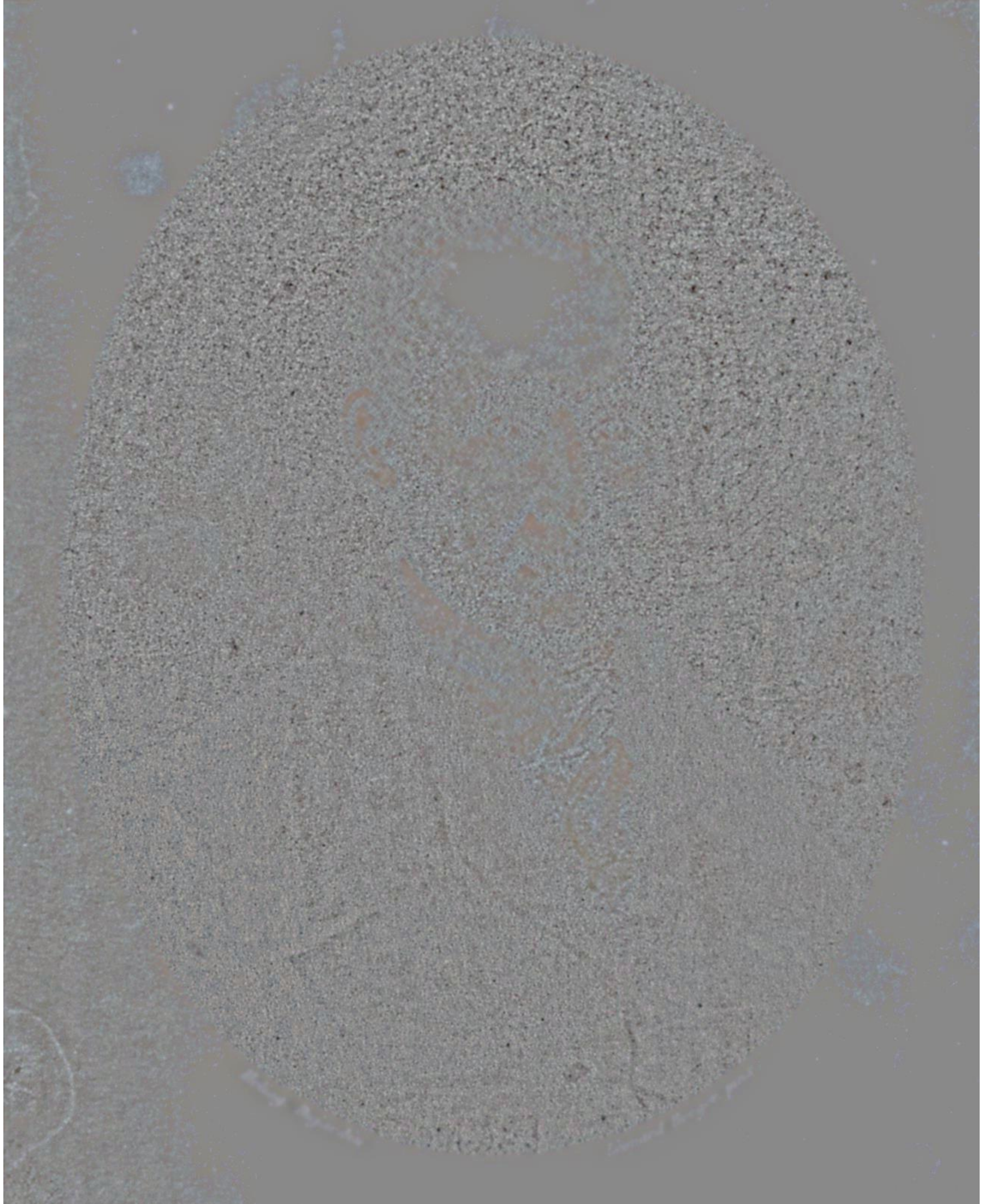
Joseph-Nicéphore Niépce (1765-1833): first indoor photograph,
Denoised by the Noise Clinic, IPOL (Image Processing on Line www.ipol.im)

Joseph-Nicéphore Niépce (1765-1833)

Photograph by Dujardin of a portrait of N. Niépce by L.F. Berger. Repr. by Günter Josef Radig, Wikipedia

We can as well denoise the scanned version of a 19th century photograph of this portrait…

Joseph-Nicéphore Niépce (1765-1833)
Denoised by the Noise Clinic,
IPOL (Image Processing on Line www.ipol.im)



Joseph-Nicéphore Niépce (1765-1833)

Difference between portrait and its denoised version by the Noise Clinic, IPOL (Image Processing on Line www.ipol.im)

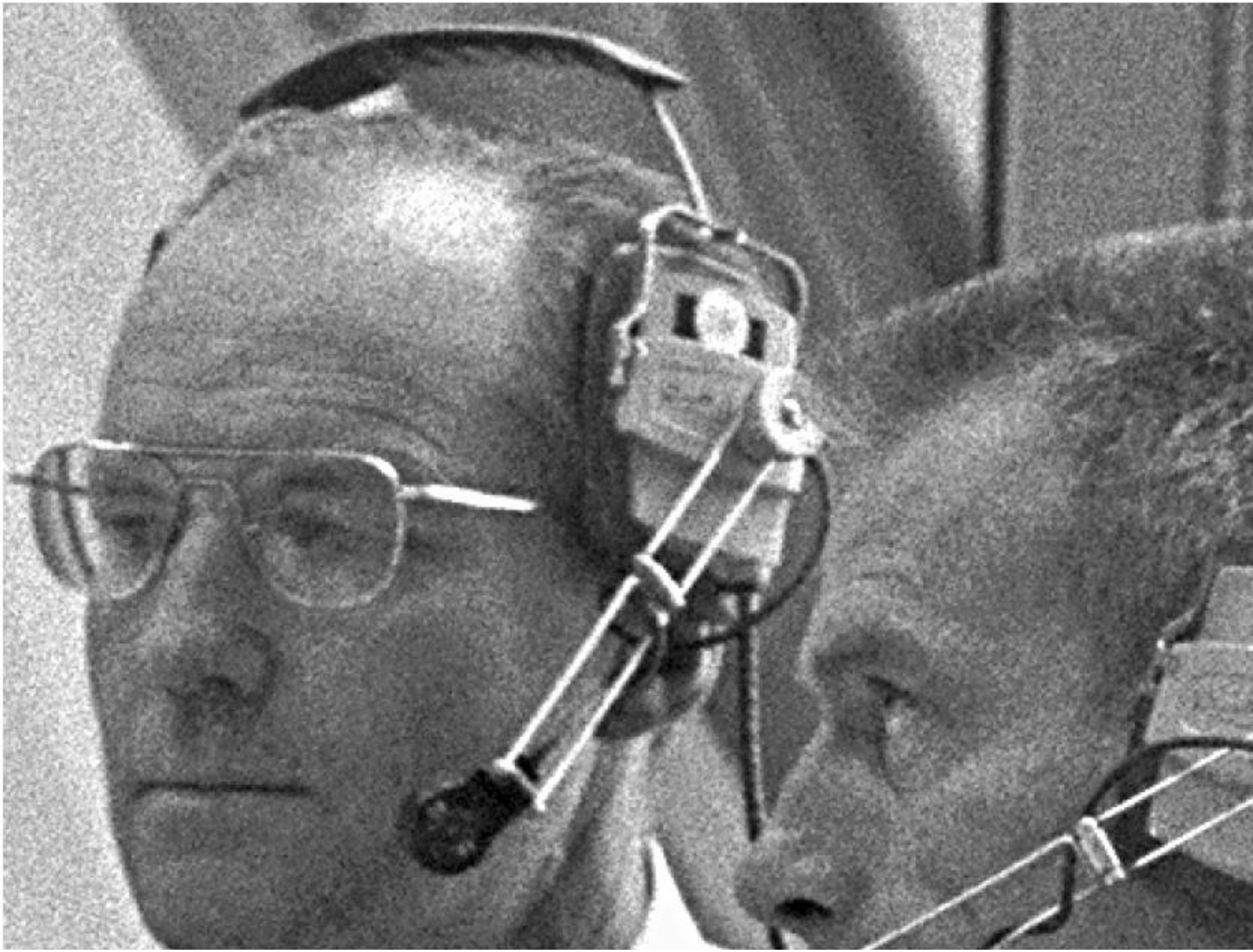
Making the difference between original and denoised permits to check if some detail has been removed at the same time as the noise. It is the case here.

Joseph-Nicéphore Niépce (1765-1833)

He made in 1826 the first outdoor successful photograph — an image of his courtyard, seen from his house — by putting a pewter plate coated with bitumen (a light-sensitive material) in the back of a camera obscura, a black box with a pinhole.

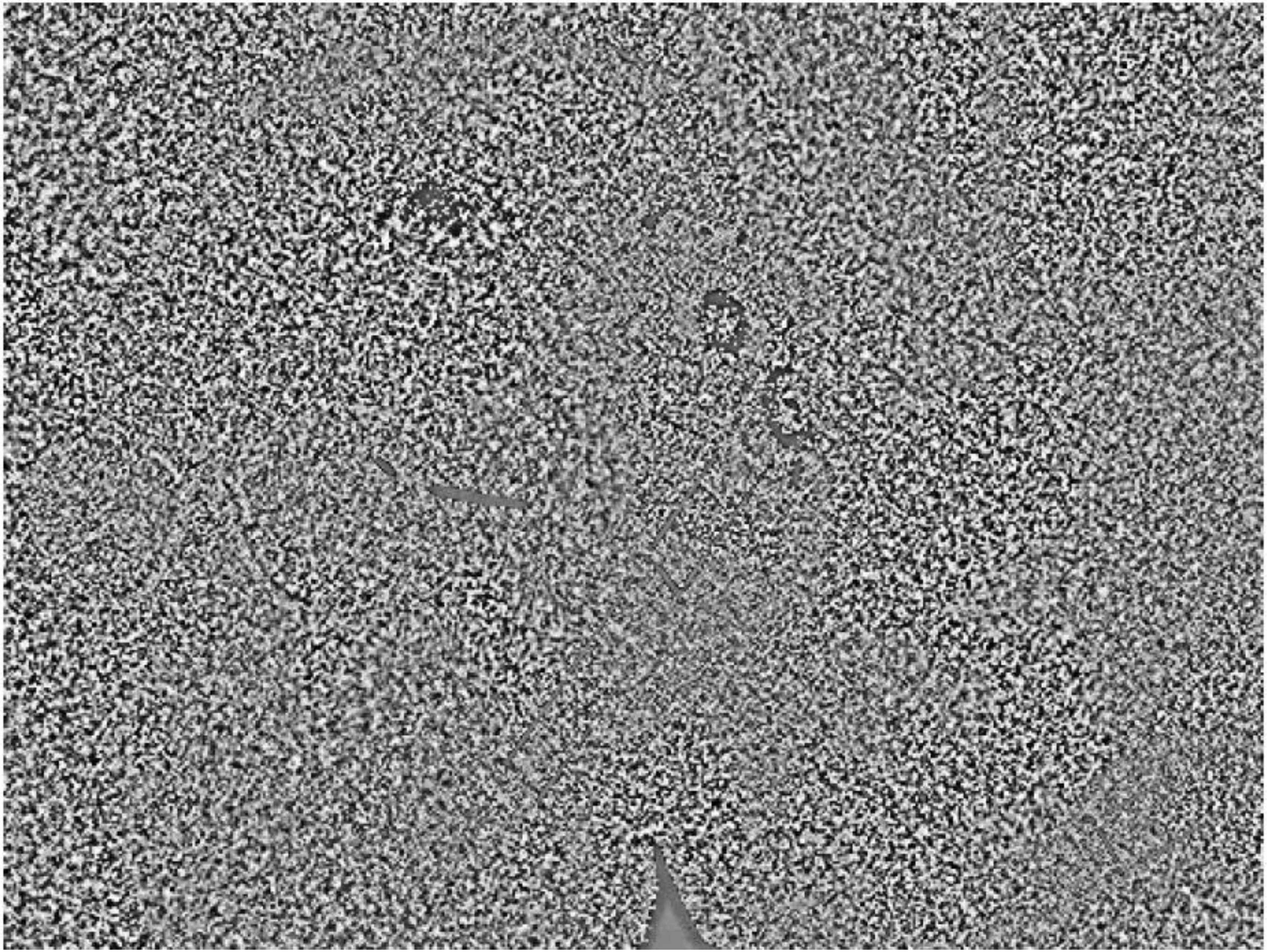
He also made the first known indoor heliographic engraving in 1825.

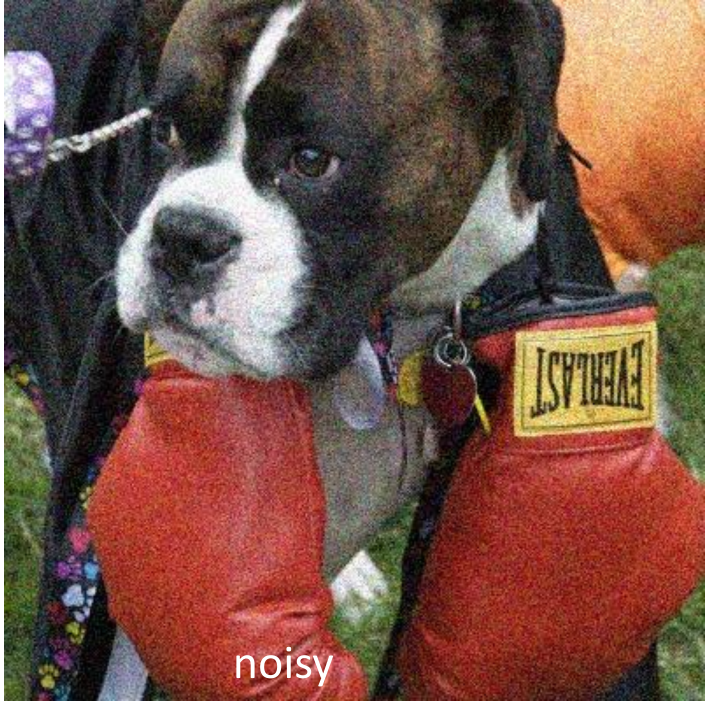
The Noise Clinic



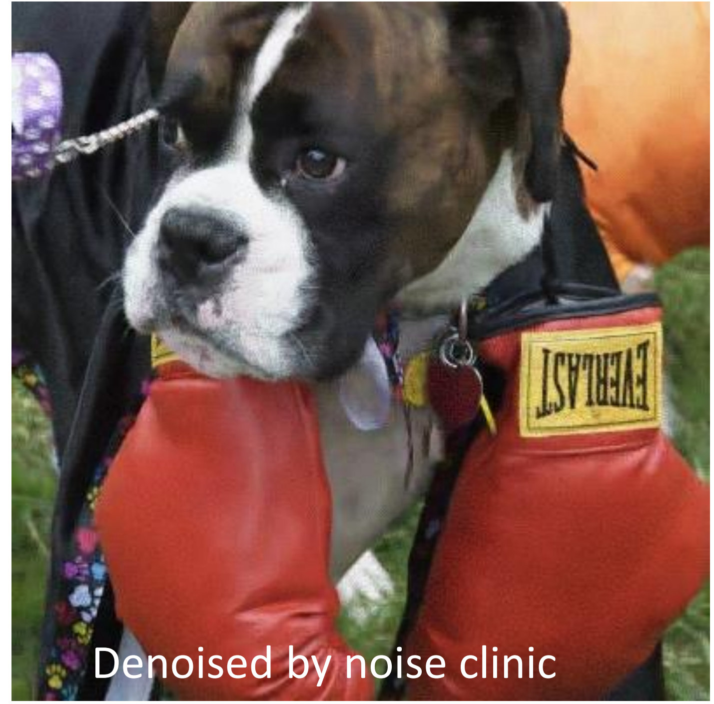
The Noise Clinic

The Noise Clinic



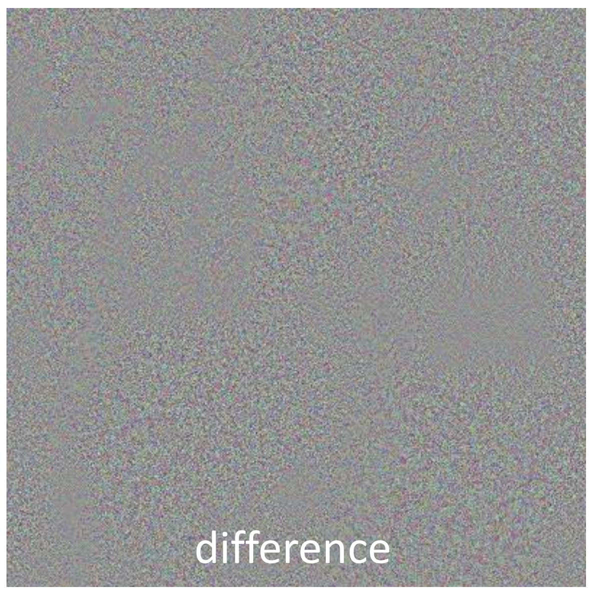


noisy



Denoised by noise clinic

Than you:



difference

questions?

Where to test all algorithms: Image Processing on Line (IPOL) http://www.ipol.im/