Wavelets on the hunt for gravitational waves

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credits: Eric Chassande-Mottin, Sergei Klimenko

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Harmonic Analysis and Geometric Measure Theory

CIRM October 2-6 2017

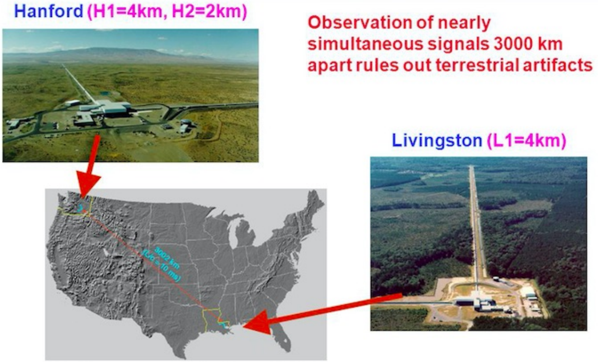
Le théorème $T(1)$

C'était au centre de mathématique de l'Ecole Polytechnique, je crois en septembre 1983. Guy me corrigera. Jean-Lin et Guy travaillaient dans un bureau contigu au mien. Et, soudain, Jean-Lin arrive en courant pour me dire que Guy et lui ont trouvé une condition nécessaire et suffisante de continuité L^2 pour tout opérateur d'intégrale singulière dont le noyau vérifie les estimations standard. Nous avons alors travaillé fiévreusement tous les trois pour améliorer l'énoncé jusqu'à obtenir la forme élégante du théorème $T(1)$. Je me souviendrai toujours de la fougue de Jean-Lin, de la lucidité ironique de Guy, de l'intense joie de la découverte et du bonheur communicatif que procurait l'amitié exceptionnelle qui liait Guy et Jean-Lin.

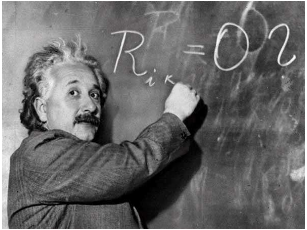
Yves Meyer

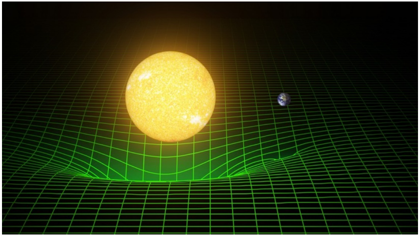
September 14th 2015 09 :50 :45 UTC

The LIGO (Laser Interferometer Gravitational-Wave Observatory) observatories in Hanford (state of Washington) and Livingston (Louisiana) performed the first detection of a gravitational wave



One minute of general relativity





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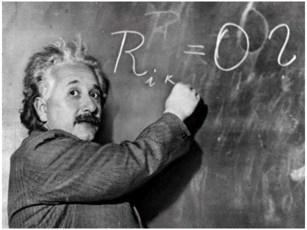

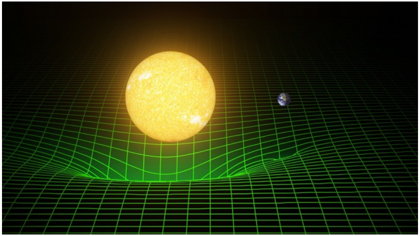

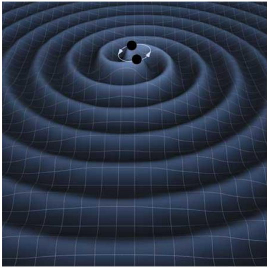

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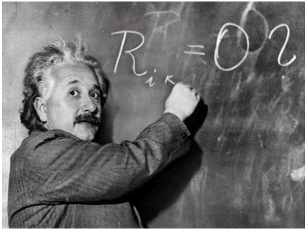

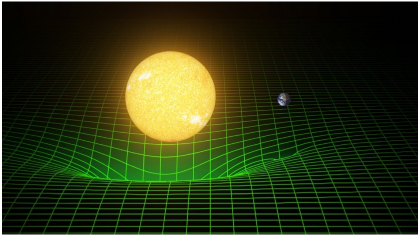

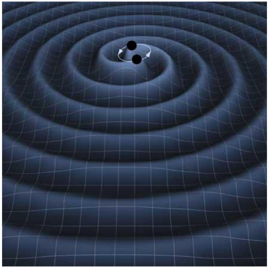

Einstein predicted the existence of gravitational waves (1916)

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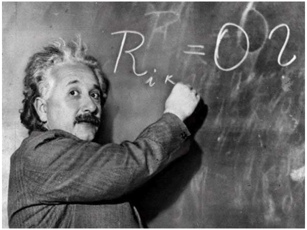

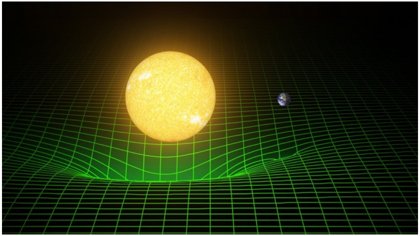

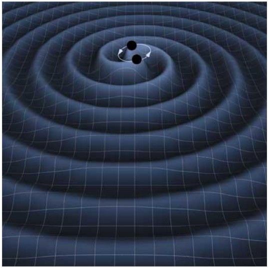

Einstein predicted the existence of gravitational waves (1916)

Consequences of the detection :

- Confirmation of general relativity in extreme conditions of mass and energy

- A new astronomy

Credit : http ://www.actusf.com/spip/L-actu-des-sciences-Avril-2014.html

A few orders of magnitude

Signal emitted 1,4 billion years ago

Coalescence of 2 black holes of 36 and 29 solar masses into 1 black hole of 62 solar masses

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Size of the recorded signal before denoising : $\sim 10^{-18}$ m
Size of the gravitational wave which crossed the earth : 10^{-21} m

Radius of the hydrogen atom : 10^{-11} m
Radius of the atomic nucleus : 10^{-15} m

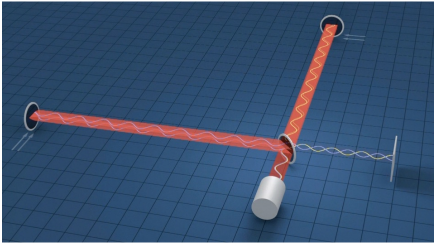
Scientific challenges

Instrumental and Physics :

Michelson Interferometer
with 2 arms of 4 km length

The laser beam is reflected
several hundreds of times

Rainer Weiss



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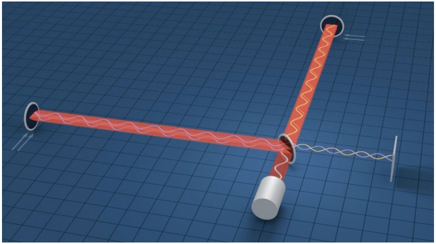
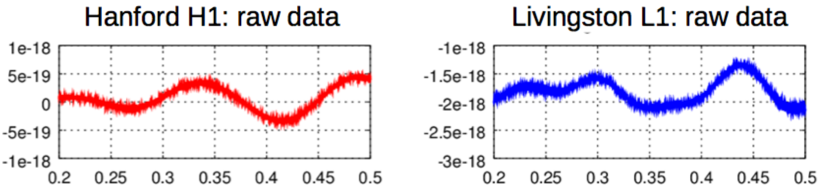


Mathematics and signal processing : Harmonic Analysis

The denoising algorithm : Frequency filtering



Gravitational wave GW150914 recorded by the LIGO detectors

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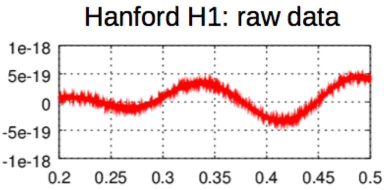

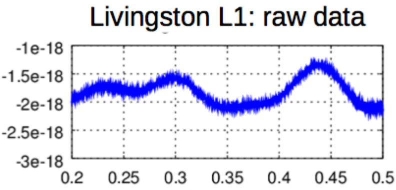

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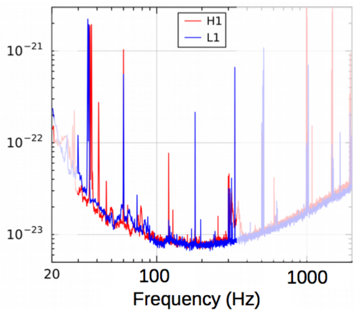
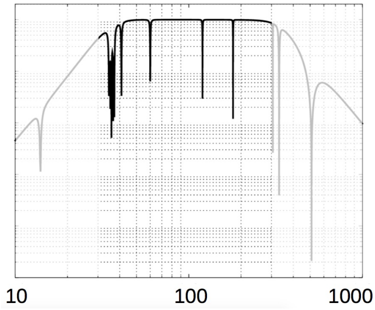


Frequencies in the data



Fourier transform of the filter

Denoising algorithm : "Glitches" resist !

"Glitches" are bursts of noise that remain in the data after filtering (several can be met in 1 second))

- Unknown origin
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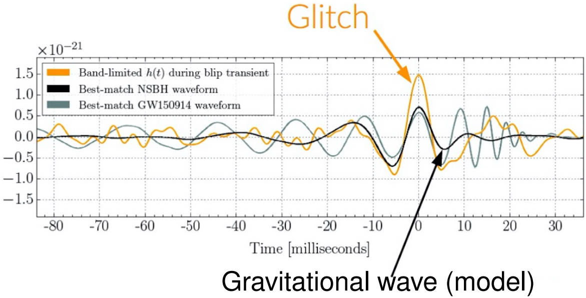

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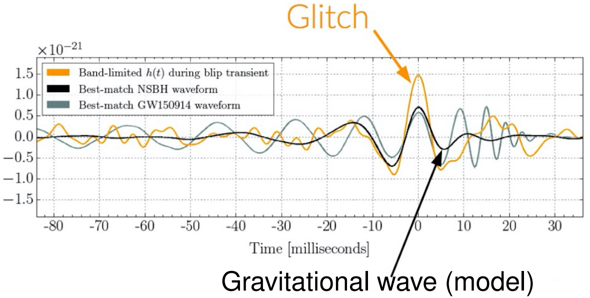
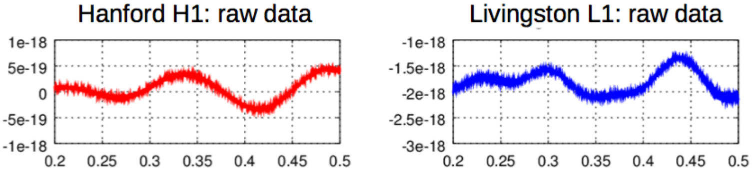
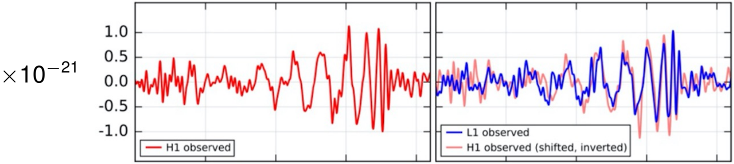


The final elimination is performed by comparing the shifted signals recorded in the two detectors (temporal shift < 10 ms)

The result of denoising





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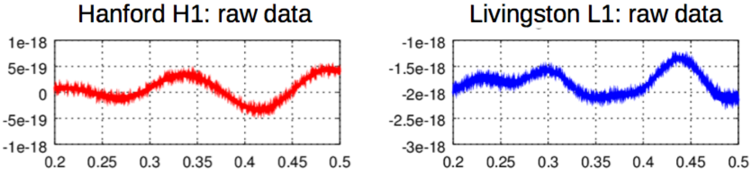

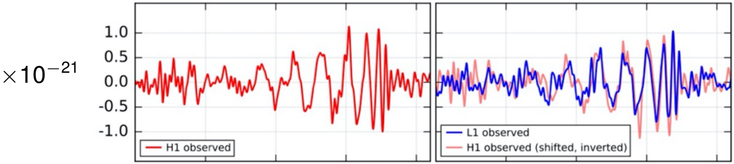

How can one detect these features in the signal ?

What does a gravitational wave look like ?

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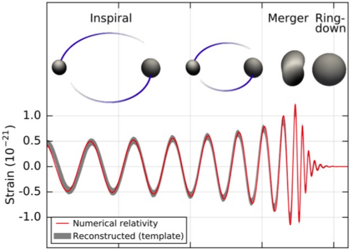
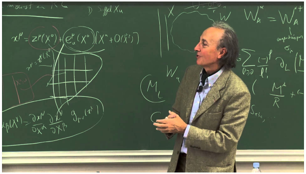
Circular binary black holes coalescence





Thibault Damour
(analytic computations)
+ numerical relativity

$\sim |t-t_0|^{-1/4} \cos(\omega |t-t_0|^{5/8} + \varphi) \Longrightarrow$ Instantaneous frequency $\sim |t-t_0|^{-3/8}$

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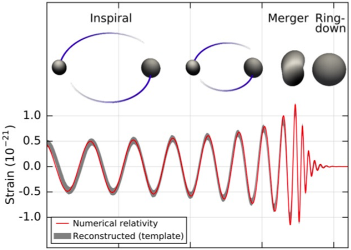

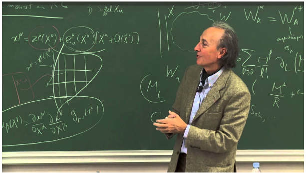

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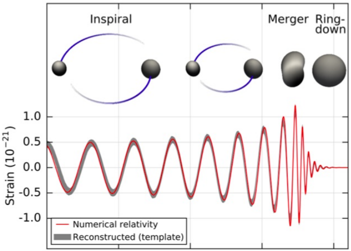

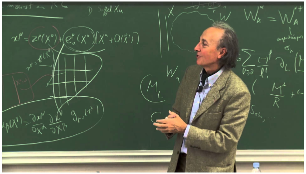

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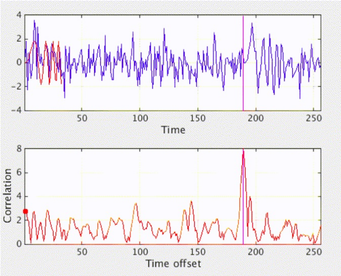
The shapes of most expected gravitational waves are either unknown or very partially known

The adapted filtering method (parametric)

The signal is correlated with all possible shapes of gravitational waves



The sampling of the parameters space leads to 250 000 templates of potential gravitational waves

The only events that are kept are those corresponding to the same filter and a physically compatible time shift ($\tau < 10$ ms)

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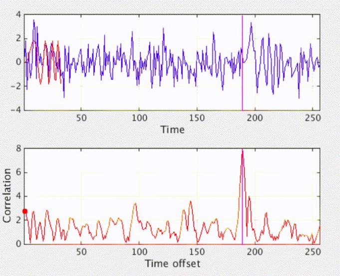

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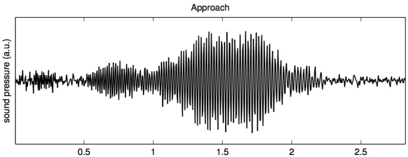
Drawbacks :

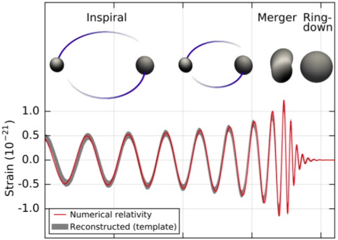
- One needs to know the exact shape of the gravitational wave
- Not feasible if the gravitational wave depends on too many parameters

Chirps everywhere



Credit : Patrick Flandrin

Ultrasound emitted by a bat



Gravitational wave

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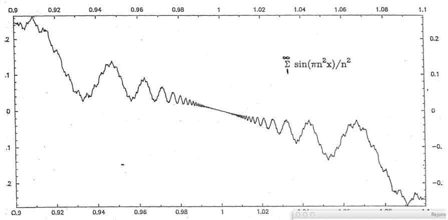

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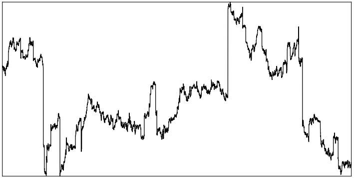
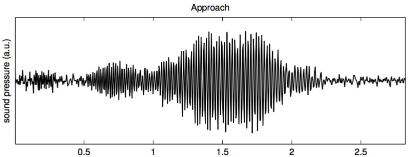


Lévy process

Chirps everywhere



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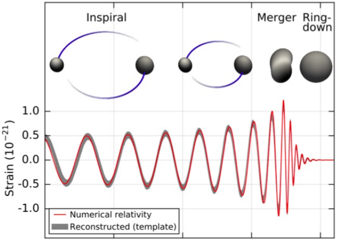

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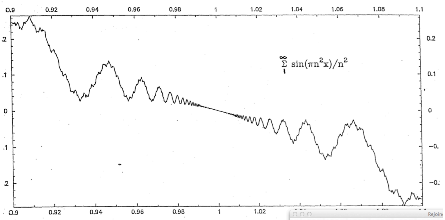

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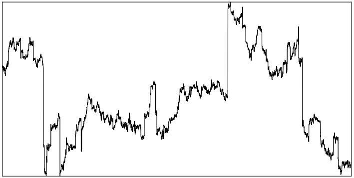
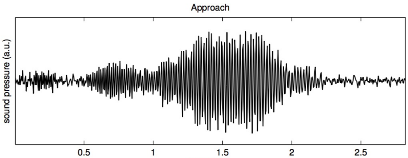


Lévy process

The instantaneous frequency evolves with time

Chirps everywhere



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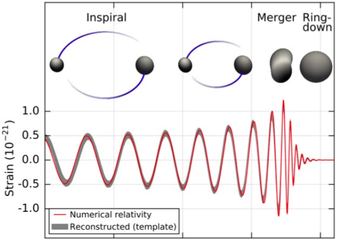

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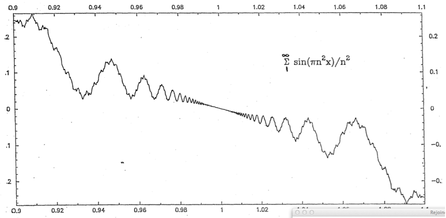

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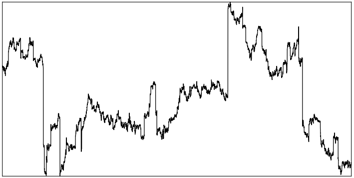

Lévy process

The instantaneous frequency evolves with time

$$f(t) = Re\left(a(t)e^{i\varphi(t)}\right) \quad \text{where} \quad \left|\frac{a'(t)}{a(t)}\right| << \varphi'(t)$$

I . Time-Frequence analysis :

The short-time Fourier transform (STFT)

Let φ be a smooth well localised “window” (e.g. a Gaussian function) the short-time Fourier transform of a function f defined on $\mathbb{R}$ is

$$G_f(x,\xi) = \int_{\mathbb{R}} f(x)\varphi(t-x)\; e^{-2i\pi t\xi} dt$$

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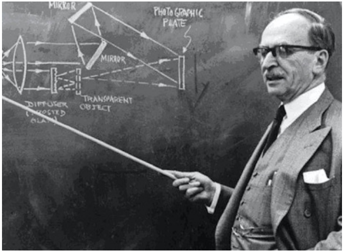

D. Gabor

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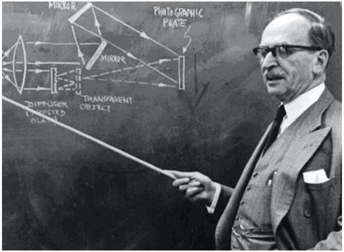
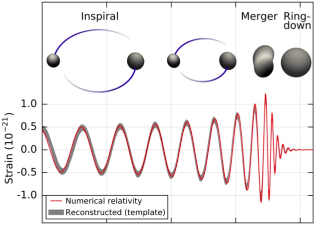


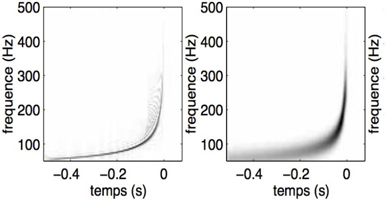
D. Gabor

Wigner-Ville transform :

$$W(t,\xi) = \int_{-\infty}^{\infty} f\left(t+\frac{\tau}{2}\right) f\left(t-\frac{\tau}{2}\right) e^{-i\xi\tau} d\tau$$

Time-Frequency analysis of gravitational waves





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Wigner-Ville vs. STFT

Variants of time-frequency analysis have been tested on chirps, see e.g.

P. Flandrin : Explorations in Time-Frequency Analysis (Cambridge U. P. 2018)

B. Torresani 

 P. Flandrin

Orthonormal bases ?

Gabor “logons” : Expand any signal on the

$$\varphi(x-k)\, e^{2i\pi nx} \qquad k, n \in \mathbb{Z} \qquad \varphi \text{ is a Gaussian}$$

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The Balian-Low theorem (1981) : If

$$\int (1+t^2)|g(t)|^2 dt < \infty \quad \text{et} \quad \int (1+\xi^2)|\hat{g}(\xi)|^2 d\xi < \infty$$

then any system of the form

$$g(x-a\,k)\ e^{i\,b\,n\,x} \qquad k,\ n \in \mathbb{Z}$$

is either incomplete or over-complete

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$$\varphi(x-k)\; e^{2i\pi nx} \qquad k, n \in \mathbb{Z} \qquad\qquad \varphi \text{ is a Gaussian}$$

The Balian-Low theorem (1981) : If

$$\int (1+t^2)|g(t)|^2 dt < \infty \quad \text{et} \quad \int (1+\xi^2)|\hat{g}(\xi)|^2 d\xi < \infty$$

then any system of the form

$$g(x-ak)\; e^{ibnx} \qquad k,\; n \in \mathbb{Z}$$

is either incomplete or over-complete

T. Steger's Theorem : A Riesz basis of $L^2(\mathbb{R})$ cannot satisfy

$$\exists a_n,\, b_n : \int (1+|t-a_n|^2)|g_n(t)|^2 dt < \infty \;\; \text{et} \;\; \int (1+|\xi-b_n|^2)|\hat{g}_n(\xi)|^2 d\xi < \infty$$

(strong uncertainty principle for bases)

How to beat Balian-Low ?

Example of orthonormal basis compatible with Balian-Low :

$$1_{[k,k+1)}(x)\; e^{2i\pi nx} \qquad k,\; n \in \mathbb{Z}$$

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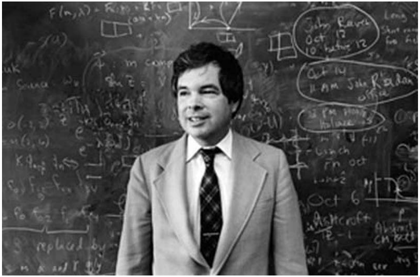

Come back to the definition of a chirp

$$f(t) = Re\left(a(t)e^{i\varphi(t)}\right) \quad \text{where} \quad \left|\frac{a'(t)}{a(t)}\right| << \varphi'(t)$$

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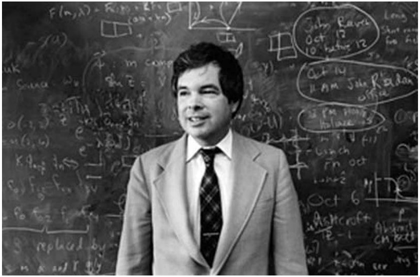

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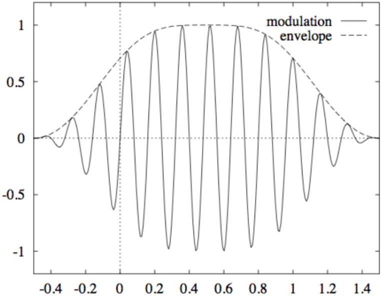
Allow a double Fourier localization around two frequencies of same amplitude and opposite signs

Wilson bases

Wilson bases (I. Daubechies, S. J., J.-L. Journé, 1991) are orthonormal bases of the form :

$$\begin{cases} \varphi(t-n), \ n \in \mathbb{Z} \\ \sqrt{2}\varphi\left(t-\frac{n}{2}\right)\cos(2\pi l t), \ l+n \in 2\mathbb{Z} \\ \sqrt{2}\varphi\left(t-\frac{n}{2}\right)\sin(2\pi l t), l+n \in 2\mathbb{Z}+1 \end{cases}$$



I. Daubechies

J.-L. Journé († April 2016)

Advantages of Wilson bases

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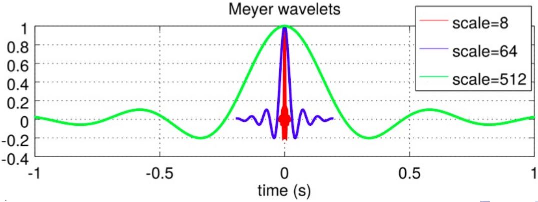
- Simple characterization of modulation spaces (H. Feichtinger and K. Gröchenig)
- gravitational waves are sparse in Wilson bases

Coherent Wave Burst

Algorithm due to S. Klimenko
and his collaborators in order to detect
gravitational waves generated by
the coalescence of two black holes

The window φ can be
Meyer scaling function
($\widehat{\varphi}$ is compactly supported)

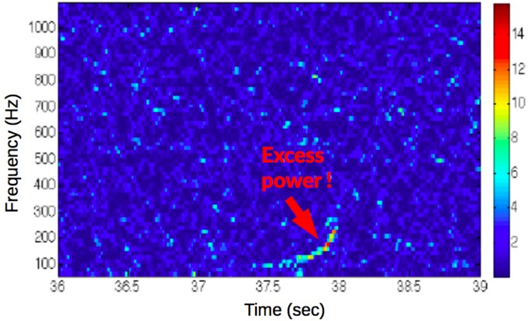
The signal processing is performed on 7 Wilson bases (and their quadrature bases) each obtained by a dilation of factor 2 of the window $\Longrightarrow$ overcomplete system of 14 orthonormal bases



Coherent Wave Burst

- Inspiral requires good frequency resolution

- Merger requires good time resolution



Optimal sparsity of the signal requires the use of 7 dilated Wilson bases

Compromise between time frequency and time scale analysis

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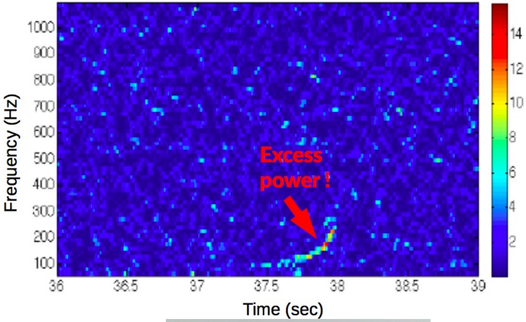

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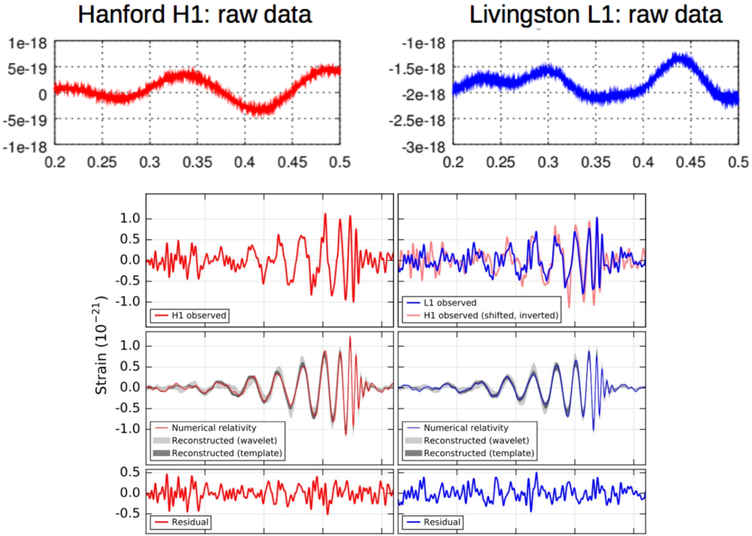
Compromise between time frequency and time scale analysis

E. Chassande-Mottin

Reconstruction of the gravitational wave



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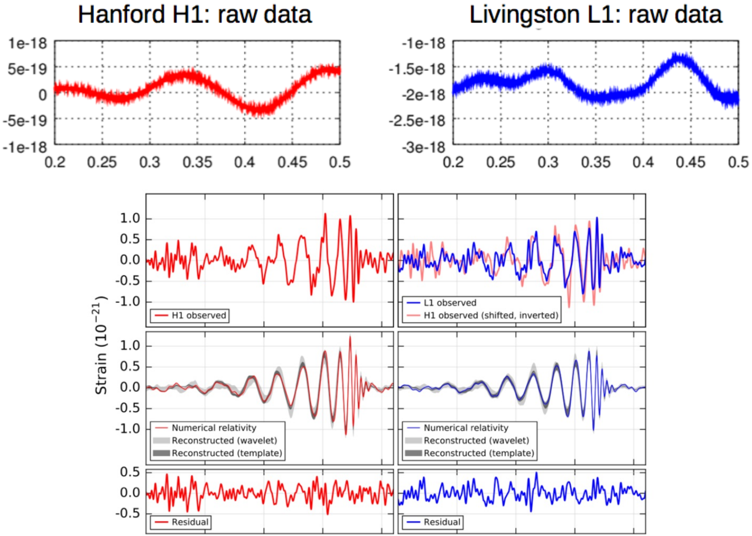

Perfect adequation of the model and of the reconstruction validates general relativity in extreme conditions of mass and velocity

Malvar bases

Time-frequency orthonormal bases of $L^2(\mathbb{R})$:

$$\varphi_{j,k}(t) = \varphi(t-j)\cos\left[\pi\left(k+\frac{1}{2}\right)(t-j)\right]$$

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The MDCT (Modified Discrete Cosinus Transform) is used in audio compression formats, e.g. MP3 or MPEG2 AAC

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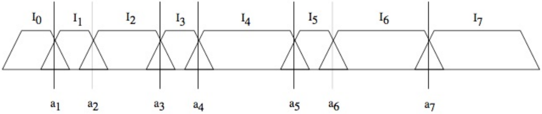
Adaptive Malvar bases

Malvar bases

with arbitrary

windows lengths

R. Coifman and Y. Meyer



$$\varphi_{j,k}(t) = \sqrt{\frac{2}{l_j}}\,\varphi_j(t)\cos\left[\frac{\pi}{l_j}\left(k+\frac{1}{2}\right)(t-a_j)\right]$$

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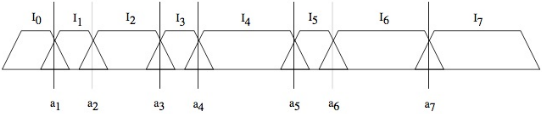

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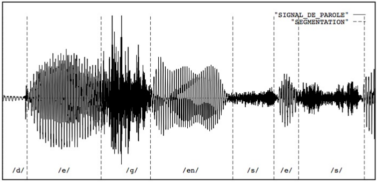

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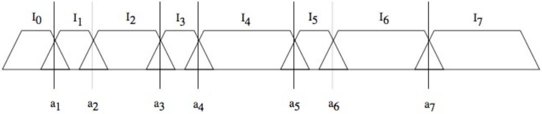

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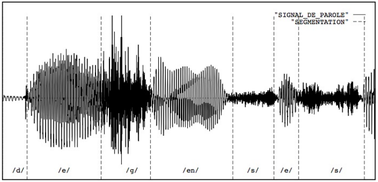

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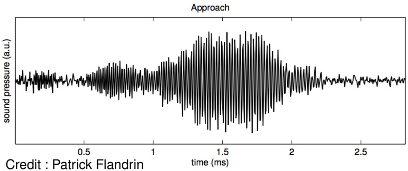
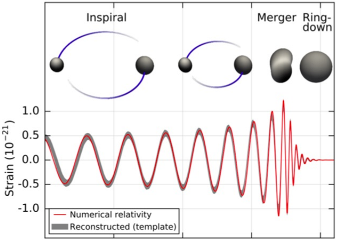
These constructions led to the study of redundant dictionary bases which played a key role in signal processing

Chirps everywhere



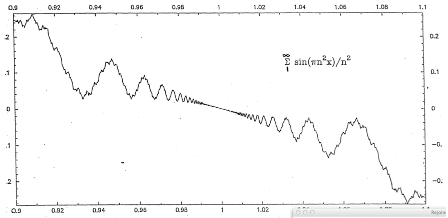
Credit : Patrick Flandrin

Ultrasound emitted by a bat

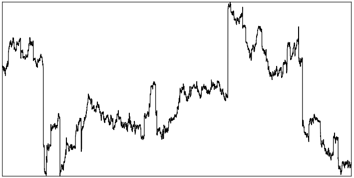


Gravitational wave

$$\sim |t - t_0|^{-1/4} \cos(\omega |t - t_0|^{5/8} + \varphi)$$



$$\mathcal{R}(x) = \sum_{1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2}$$



Lévy process

The instantaneous frequency evolves with time

$$f(t) = Re\left(a(t) e^{i\varphi(t)}\right) \quad \text{where} \quad \left|\frac{a'(t)}{a(t)}\right| << \varphi'(t)$$

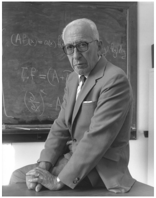
Time-scale analysis : Wavelets

If the wavelet ψ is well localized, of vanishing integral, even or odd, the continuous wavelet transform of a function f defined on $\mathbb{R}$ is

$$C_f(a,b) = \frac{1}{a}\int_{\mathbb{R}} f(t)\,\psi\left(\frac{t-b}{a}\right)dt$$

Calderón's reconstruction formula :

$$f(x) = C\int_{a>0}\int_{b\in\mathbb{R}} C_f(a,b)\,\psi\left(\frac{x-b}{a}\right)\frac{da\,db}{a^2}$$



A. Calderón

A. Grossmann

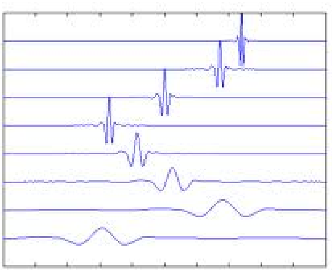
J. Morlet

Orthonormal wavelet bases

A wavelet basis on $\mathbb{R}$ is generated by one smooth well localized, oscillating wavelet ψ such that the

$2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}$

form an orthonormal basis of $L^2(\mathbb{R})$



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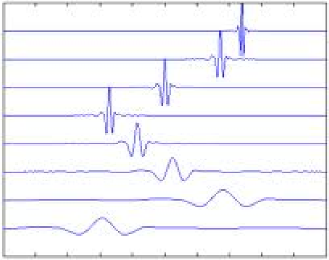


Advantages :

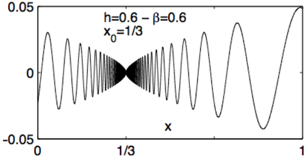
- Fast decomposition algorithms
- Simple characterization of Besov spaces
- Sparse representations for large classes of signals and images

May 24th 2017 : Scientific day following the Abel prize ceremony

Chirps as pointwise singularities

$$C_{H,\beta} = |x - x_0|^H \sin\left(\frac{1}{|x - x_0|^\beta}\right)$$

Wavelet characterization of chirps
(B. Torresani, Y. Meyer)



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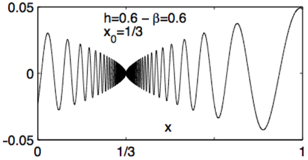
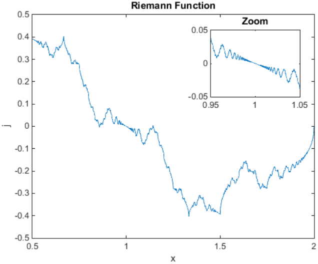
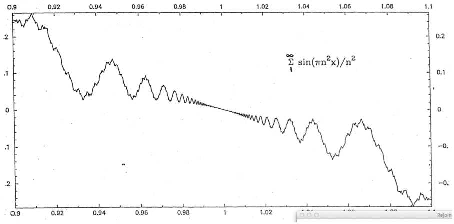


Trigonometric chirp in the Riemann series
(Y. Meyer)

$$\mathcal{R}(x) = \sum_{1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2}$$





$$\mathcal{R}(\pi + x) = -\frac{x}{2} + \sum_{k\geq 1} |x|^{k+1/2} g_k\left(\frac{1}{x}\right) \quad \text{where} \quad g_k \sim \mathcal{R}^{(-k)}$$

Chirps in signals : Pointwise exponents

$f \in C^\alpha(x_0)$ it there exist $C > 0$ and a polynomial P of degree $< \alpha$:

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha : \quad f \in C^\alpha(x_0)\}$$

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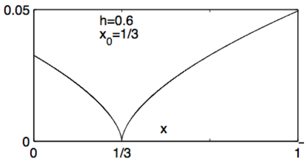
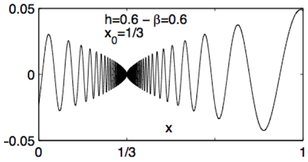


$C_H = |x - x_0|^H$



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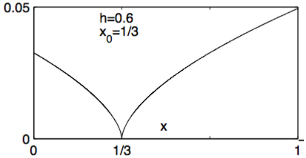

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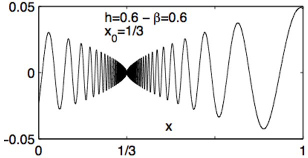

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Hölder exponents coincide

Hölder exponents of the primitive $f^{(-1)}$:

$h_{f^{(-1)}}(x_0) = H + 1$ $\qquad$ $h_{f^{(-1)}}(x_0) = H + \beta + 1$

Exponents associated with pointwise regularity

How to associate an oscillation exponent that would not change with the addition of a smoother noise ?

The fractional integral of order s of f is

$$\widehat{f^{(-s)}}(\xi) = (1+|\xi|^2)^{-s/2}\ \widehat{f}\ (\xi)$$

The fractional Hölder exponent of f at x_0 is $h^s_f(x_0) = h_{f^{(-s)}}(x_0)$

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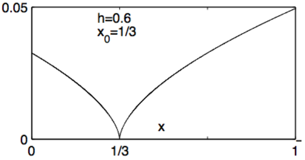
f has an oscillating singularity at x_0 if $h_f^s(x_0) \neq h_f(x_0) + s$

Oscillation exponent : $\mathcal{O}s_f(x_0) = \left(\frac{\partial\ h_f^s(x_0)}{\partial s}\right)_{s=0^+} - 1$

- Takes the value β for the chirp
$$C_{H,\beta} = |x - x_0|^H \sin\left(\frac{1}{|x-x_0|^\beta}\right)$$
- Takes the value 1 for the Riemann chirp

Cusps vs. Oscillating singularities



Cusps $\quad C_H(x) = |x - x_0|^H$

A cusp satisfies : $\mathcal{O}s_f(x_0) = 0$

An exotic example of cusps expected in Gravitational waves analysis :

According to T. Damour and A. Vilenkin, cosmic strings (if they exist !) should emit gravitational waves which are cusps

Cusps vs. Oscillating singularities

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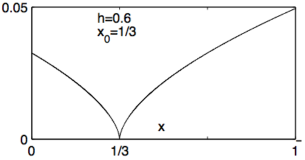

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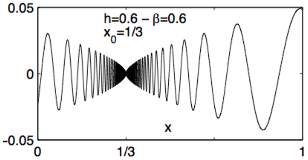

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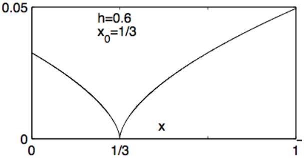

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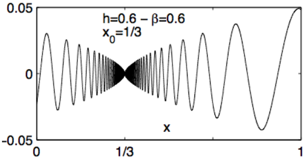

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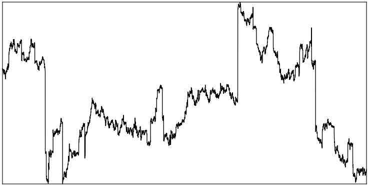
f has an oscillating singularity at x_0

if $\mathcal{O}s_f(x_0) \neq 0$



Sample paths of some Lévy processes (P. Balança)

Where are the oscillating singularities ? ? ?

A new astronomy

What we learned from the first detection :

- Binary black holes exist and merge
- General relativity remains valid in extreme velocity and energy conditions
- The knowledge of the theoretical shape of a gravitational wave yields the corresponding physics parameters (masses, spins, etc.,) which give information on the scenario that led to its emission

A new astronomy

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Challenges in applied harmonic analysis :

- Improve Wilson graphs methods
- New statistical tools for estimating the probability of false alarms
- Reconstruction algorithms for coherent bases

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The future :

- Validate models that predict the repartition of black holes in the universe
- Accelerate algorithms in order to perform real-time detection
- Improve glitch detection

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Detect other types of gravitational waves such as :

- binaries with a large excentricity
- γ-ray bursts
- binaries containing neutrons stars
- explosions of super novas
- residual cosmologic noise
- Detect something unexpected


Thank you for your attention