The Dirichlet problem for sets with higher co-dimensional boundaries

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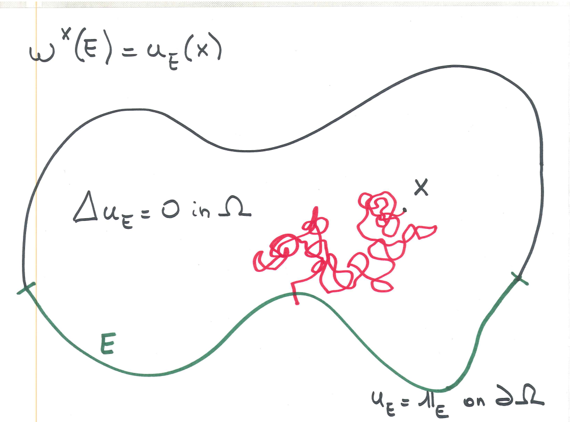
Definition (Harmonic measure)

Let $\Omega \subset \mathbb{R}^n$ open connected bounded, $x \in \Omega$ and $E \subset \Gamma := \partial\Omega$.

- $\omega^x(E)$ = probability that a particule issued at x and subject to Brownian motion goes outside of Ω through E.
- $u_E(x) = \omega^x(E)$ satisfies $\begin{cases} \Delta u_E = 0 & \text{in } \Omega \\ u_E = \mathbb{1}_E & \text{on } \Gamma = \partial\Omega. \end{cases}$



Question: When is ω^x is $\mathcal{A}^\infty$-absolutely continuous with respect to the surface measure $\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}$?

Answers: $\Omega \subset \mathbb{R}^n$

- Dahlberg (1977): $\partial\Omega = \Gamma$ is a Lipschitz graph $\Rightarrow$ $\omega^x \in \mathcal{A}^\infty(\sigma)$.
- David, Jerison, Semmes (1990), Hofmann, Martel (2014): Ω is uniform/NTA, $\partial\Omega = \Gamma$ uniformly rectifiable $\Rightarrow \omega^x \in \mathcal{A}^\infty(\sigma)$.
  - $\hookrightarrow$ Corkscrew balls (quantitative openness). Harnack chains (quantitative connectedness).
  - $\hookrightarrow$ quantitative rectifiability; Γ rectifiable if $\sigma(\Gamma \setminus \bigcup f_i(\mathbb{R}^d)) = 0$, where f_i are Lipschitz.
- Bishop, Jones (1990): connectedness is necessary.
- Azzam, Hofmann, Martell, Mayboroda, Mourgoglou, Tolsa, Volberg (2015): rectifiability is necessary.

Question: Can we ask the same question when $\Gamma \subset \mathbb{R}^n$ - and $\Omega = \mathbb{R}^n \setminus \Gamma$ - is of dimension $d < n - 1$, and t?

Problem: If Γ is of dimension $d < n - 1$, there is no harmonic measure (as defined in the first slide) for Γ. Indeed, Brownian travelers do not see the sets of higher codimension. In terms of weak solutions, the condition

$$\int \nabla u \cdot \nabla v = 0 \qquad \forall v \in C_0^\infty(\mathbb{R}^n \setminus \Gamma)$$

implies the *a priori* stronger condition

$$\int \nabla u \cdot \nabla v = 0 \qquad \forall v \in C_0^\infty(\mathbb{R}^n);$$

which means that the only (weak) solution of $\Delta u = 0$ in $\mathbb{R}^n \setminus \Gamma$ that decays at ∞ is $u \equiv 0$.

Idea: The elliptic operator $-\Delta$ is not adapted to the higher codimension. So we replace $-\Delta$ by L and we find out how L looks like.

If $\Gamma = \mathbb{R}^d \subset \mathbb{R}^n$. We expect the 'good' solution u of

$$\begin{cases} Lu = 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}^d \\ u = g & \text{on } \mathbb{R}^d. \end{cases}$$

to be given by $u(x,t) = u_0(x,|t|)$, where

$$\begin{cases} -\Delta u_0 = 0 & \text{in } \mathbb{R}^{d+1}_+ \\ u_0 = g & \text{on } \mathbb{R}^d. \end{cases}$$

We find

$$L := -\operatorname{div} |t|^{d+1-n} \nabla.$$

Elliptic theory

Let $d < n-1$ and set $\delta_\Gamma(x) := \mathrm{dist}(x, \Gamma)$.

Let Γ be an d-dimensional ADR set, that is for $x \in \Gamma$ and $r > 0$,

$$C^{-1} r^d \leq \sigma(B_\Gamma(x,r)) \leq C r^d.$$

where $\sigma := \mathcal{H}^d|_\Gamma$. We take the degenerated elliptic operator

$$L = -\mathrm{div} A(x) \nabla$$

where $A(x)$ is a $n \times n$ matrix with measurable coefficients and satisfying:

- $A(x)\xi \cdot \xi \geq C^{-1} \delta_\Gamma(x)^{d+1-n}$ for $x \in \Omega := \mathbb{R}^n \setminus \Gamma$ and $\xi \in \mathbb{R}^n$,
- $|A(x)\xi \cdot \zeta| \leq C \delta_\Gamma(x)^{d+1-n}$ for $x \in \Omega$ and $\xi, \zeta \in \mathbb{R}^n$.

Theorem (2017)

We can develop an elliptic theory for L: Trace & Extension theorems; De Giorgi-Nash-Moser estimates inside the domains and at the boundary; comparison principle; definition of the harmonic measure and Green's function.

Take d be an integer such that $d < n-1$. Now, Γ is the graph of a Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}^{n-d}$, and $\Omega = \mathbb{R}^n \setminus \Gamma$.
We set

$$L = -\mathrm{div} D(x)^{d+1-n} \nabla,$$

where

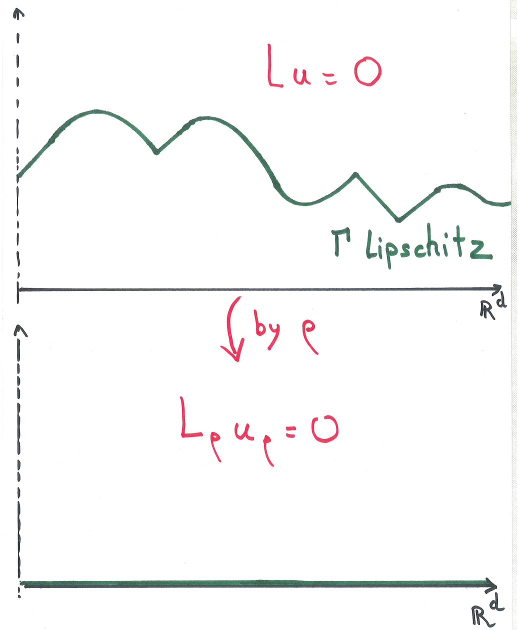
$$D(x) := \left(\int_\Gamma |x-y|^{-d-\alpha} d\sigma(y) \right)^{-\frac{1}{\alpha}},$$

for some $\alpha > 0$. Observe that $D(x) \approx \delta_\Gamma(x)$.

Theorem (2017)

If Γ has a small Lipschitz constant, then the harmonic measure $\omega^x := \omega^x_{\Omega,L}$ is $\mathcal{A}^\infty$-absolutely continuous with respect to the Hausdorff measure σ.

Proof (ideas)



The choice of the change of variable ρ gives $L_\rho = -\operatorname{div} A_\rho \nabla$.

We want to choose ρ such that we can handle A_ρ.

Caffarelli, Fabes, Kenig (1981): there exists A such that $\omega^X \notin \mathcal{A}^\infty(\sigma)$.

Proof (ideas)

Can we take the same change of variable as in codimension 1?

Let $\Gamma = \varphi(\mathbb{R}^d)$, where φ is Lipschitz function.

- Dahlberg (1977), Jerison, Kenig (1981):

$$\rho(x,t) = (x, t + \varphi(x))$$

In this case, $A_\rho(x,t) = A_\rho(x)$ is t-independent and symmetric.
$\hookrightarrow$ Rellich identity
$\hookrightarrow \left(\fint_\Delta k^2 d\sigma\right)^{\frac{1}{2}} \leq C \fint_\Delta k d\sigma$, where $k = k^x := \frac{d\omega^x}{d\sigma}$.

Pb: In codimension higher than 1, $|A(x,t)| \equiv |t|^{d+1-n}$, so it cannot be t-independent.

Proof (ideas)

Can we take the same change of variable as in codimension 1?

Let $\Gamma = \varphi(\mathbb{R}^d)$, where φ is Lipschitz function.

- Kenig, Pipher (2001) used a transformation discovered by Dahlberg, Kenig and Stein:

$$\rho(x,t) = (x, ct + \theta_t * \varphi(x))$$

$\hookrightarrow$ $t\nabla A_\rho$ satisfies the Carleson Measure condition (CM).

In higher codimension, we need an isometry in t

$$A_\rho = \begin{pmatrix} \cdots & \cdots \\ \cdots & bI_{n-d} \end{pmatrix}.$$

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$$A_\rho = \begin{pmatrix} \cdots & \cdots \\ \cdots & bI_{n-d} \end{pmatrix}.$$

- u satisfies the Carleson Measure condition if $|u(y,s)|^2 \frac{dy\,ds}{|s|^{n-d}}$ is a Carleson measure.

Proof (ideas)

Can we take the same change of variable as in codimension 1?

Let $\Gamma = \varphi(\mathbb{R}^d)$, where φ is Lipschitz function.

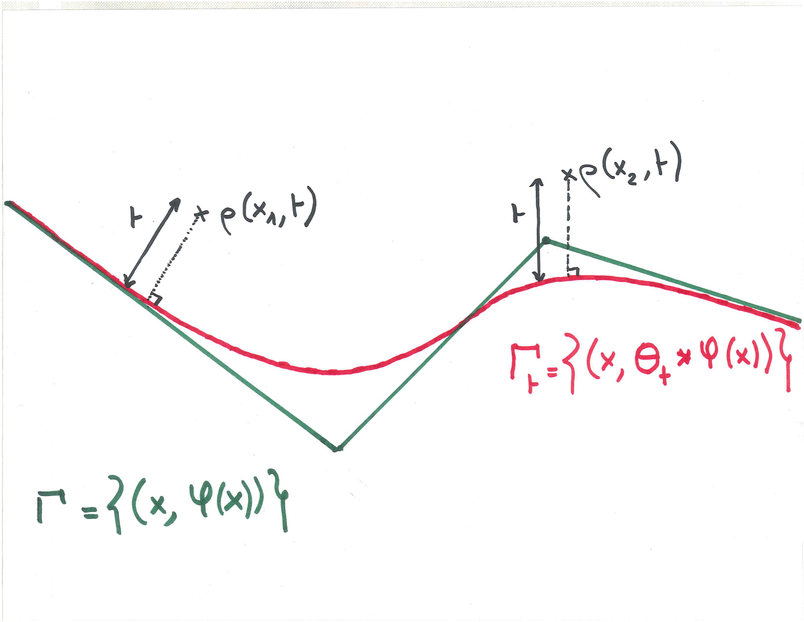
- We have

$$D(X)^{d+1-n} \xrightarrow{\qquad \rho^{-1} \qquad} |t|^{d+1-n} \qquad \underbrace{\lambda(x,t)}_{\text{need to be link to CM.}}$$

We want $\lambda - 1$ to satisfy the Carleson Measure condition.

The function $X \to D(X)$ needs to be smooth enough,
that is why $D(X) = \delta_\Gamma(X)$ doesn't work (except when $d = 1$).

Our change of variable



- The change of variable ρ is new (even in codimension 1).
- We can treat a new class of operators (even in codimension 1).
- ρ is built only when Γ has a small Lipschitz constant.

- A_ρ has the form

$$|t|^{n-d-1} A_\rho = \begin{pmatrix} \mathcal{A}^1 & \mathcal{C}^2 \\ \mathcal{C}^3 & bI_{n-d} + \mathcal{C}^4 \end{pmatrix},$$

  where $\mathcal{C}^2$, $\mathcal{C}^3$, $\mathcal{C}^4$ and $|t|\nabla b$ satisfies CM.

  We use Carleson measure estimates on solutions to get $\mathcal{A}^\infty$, by using ideas from
  - Kenig, Koch, Pipher, Toro (2000),
  - Dindoš, Petermichl, Pipher (2015),
  - Kenig, Kirchheim, Pipher, Toro (2016).

- u satisfies the Carleson Measure condition $CM(\epsilon)$ if

$$\sup_B \int_{y\in B}\int_{|s|\le r_B} |u(y,s)|^2 \frac{dy\,ds}{|s|^{n-d}} \le \epsilon |r_B|^d,$$

where the supremum is taken over all the boundary balls $B := B(x, r_B)$.

- If A has the form

$$|t|^{n-d-1}A = \begin{pmatrix} \mathcal{A}^1 & \mathcal{A}^2 \\ \mathcal{C}^3 & bI_{n-d} + \mathcal{C}^4 \end{pmatrix},$$

where $\mathcal{C}^3$, $\mathcal{C}^4$ and $|t|\nabla b$ satisfies CM, then for any L-harmonic extension u_g of a continuous function g bounded by 1, we have

$$|t|\nabla u_g \in CM(C_L). \qquad (*)$$

- If L is such that $(*)$ is satisfied, then $\omega^x \in \mathcal{A}^\infty(\sigma)$.

Result: $\omega_L^x \in \mathcal{A}^\infty(\sigma) \Leftrightarrow (\mathrm{D}_p)$ is solvable for some large p.

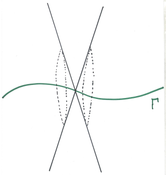
$$\Leftrightarrow k^x = \frac{d\omega^x}{d\sigma} \in \mathcal{B}^q \text{ for some small } q > 1.$$

We say that the Dirichlet problem (D_p) is solvable if for any $g \in C_0^\infty(\mathbb{R}^d)$, there exists a unique solution u satisfying

$$\begin{cases} Lu = 0 \quad \text{in } \mathbb{R}^n \setminus \Gamma \\ u = g \quad \text{on } \Gamma \\ \|N(u)\|_p \leq C\|g\|_p < +\infty. \end{cases}$$

Here, $N(u)(x) := \sup\limits_{(y,s)\in\gamma(x)} |u(y,s)|$,

and $\gamma(x) = \gamma(x', \varphi(x'))$
$= \{(y,s) \in \mathbb{R}^n \setminus \Gamma : |y-x| \leq a|s-\varphi(x)|\}.$



Dirichlet problem in codimension 1:

- Dahlberg (1977): (D_p), $p \geq 2$, for the Laplacian.
- Jerison, Kenig (1981): (D_p), $p \geq 2$, for symmetric t-independent operators.
- Kenig, Koch, Pipher, Toro (2000) and Kenig, Pipher (2001): (D_p) for some $p < +\infty$ if $L = -\operatorname{div} A\nabla$ is such that $r\nabla A$ satisfies CM.
- Dindoš, Petermichl, Pipher (2007): (D_p) for any $p \in (1, +\infty)$ if Γ has small Lipschitz constant and $L = -\operatorname{div} A\nabla$ is such that $\delta_\Gamma \nabla A$ satisfies $CM(\epsilon)$ with $\epsilon = \epsilon(p)$ small.
- Dindoš, Pipher (2017, preprint): extension to the case where A has complex coefficients.

What is needed to get (D_p) for all $p \in (1, +\infty)$:

- a <u>small</u> Lipschitz constant for the boundary Γ.
- <u>small</u> oscillations, for instance $\delta_\Gamma \nabla A$ satisfies $CM(\epsilon)$ with small ϵ.

In codimension higher than 1, we had:

Theorem

Assume that $L = -\mathrm{div} A \nabla$ *is an operator on* $\mathbb{R}^n \setminus \mathbb{R}^d$ *satisfying the weighted elliptic conditions and such that*

$$|t|^{n-d-1} A = \begin{pmatrix} \mathcal{A}^1 & \mathcal{A}^2 \\ \mathcal{C}^3 & b.I_{n-d} + \mathcal{C}^4 \end{pmatrix},$$

where $\mathcal{C}^3$, $\mathcal{C}^4$, $|t|\nabla b$ *satisfies CM.*
Then, for some $p < +\infty$, (D_p) *is solvable.*

We proved

Theorem

Assume that $L = -\mathrm{div} A \nabla$ *is an operator on* $\mathbb{R}^n \setminus \mathbb{R}^d$ *satisfying the weighted elliptic conditions and such that*

$$|t|^{n-d-1} A = \begin{pmatrix} \mathcal{A}^1 & \mathcal{A}^2 \\ \mathcal{B}^3 + \mathcal{C}^3 & b.I_{n-d} + \mathcal{C}^4 \end{pmatrix},$$

where $\mathcal{C}^3$, $\mathcal{C}^4$, $|t|\nabla\mathcal{B}^3$, $|t|\nabla b$ *satisfies* $CM(\epsilon)$.
Then, for any $1 < p < +\infty$, *if* ϵ *is small enough,* (D_p) *is solvable.*

Ideas for the proof

Similarly to Dindoš, Petermichl, Pipher (2007), we introduce the p-adapted square functional $S_p(u)$ defined on $\mathbb{R}^d$ by

$$S_p(u)(x) = \left(\iint_{(y,s)\in\gamma(x)} |\nabla u|^2 |u|^{p-2} \frac{dy\, ds}{|s|^{n-2}} \right)^{\frac{1}{p}},$$

and we prove

1. $\|S_2(u)\|_p \le C \|N(u)\|_p^{1-p/2} \|S_p(u)\|_p^{p/2}$ for any $p \in (1,2]$,
2. $\|N(u)\|_p \le C_\epsilon \|S_2(u)\|_p$ for any $p > 0$,
3. $\|S_p(u)\|_p \le C \|\operatorname{Tr} u\|_p + C\epsilon^{\frac{1}{p}} \|N(u)\|_p$ for any $p > 1$.

Now, 1 and 2 gives that

4. $\|N(u)\|_p \le C_\epsilon \|S_p(u)\|_p$ for any $p \in (1,2]$.

Besides, 3 and 4 implies that,

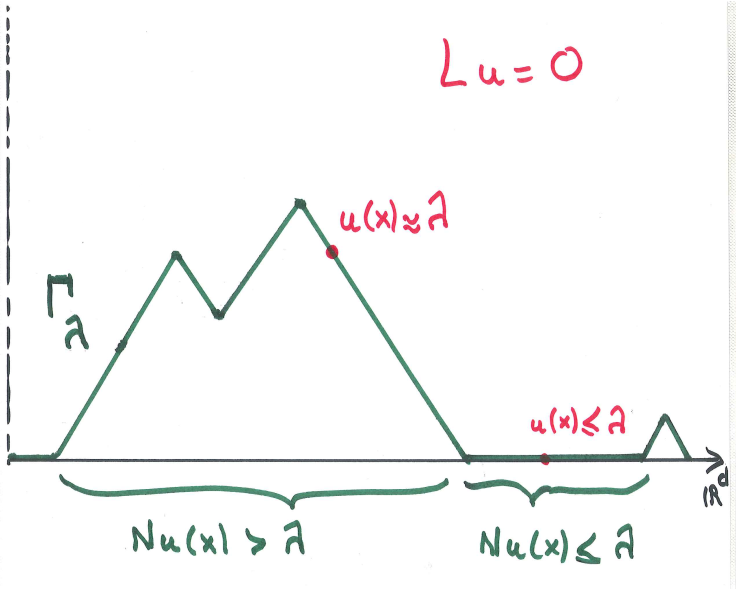
5. $\|N(u)\|_p \le C_\epsilon \|\operatorname{Tr} u\|_p + \epsilon^{1/p} C_\epsilon \|N(u)\|_p$ for any $p \in (1,2]$.

Comments:

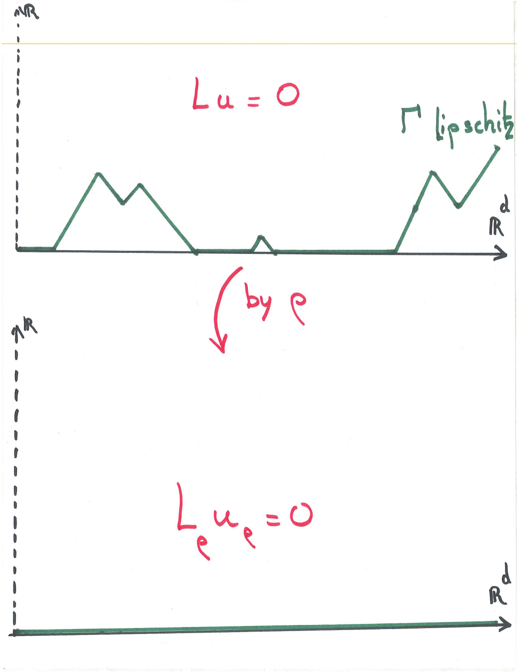
- We actually prove local estimates (new even in codimension 1).
- S_p is a good intermediate for computation, since

$$\|S_p(u)\|_p^p = \iint_{(y,s)\in\mathbb{R}^n} |\nabla u|^2 |u|^{p-2} \frac{dy\,ds}{|s|^{n-2}}.$$

- We only established that (D_p) is solvable for $p \in (1,2]$. However, by the maximum principle, the solvability of (D_p) implies the one of (D_q) for $q > p$.

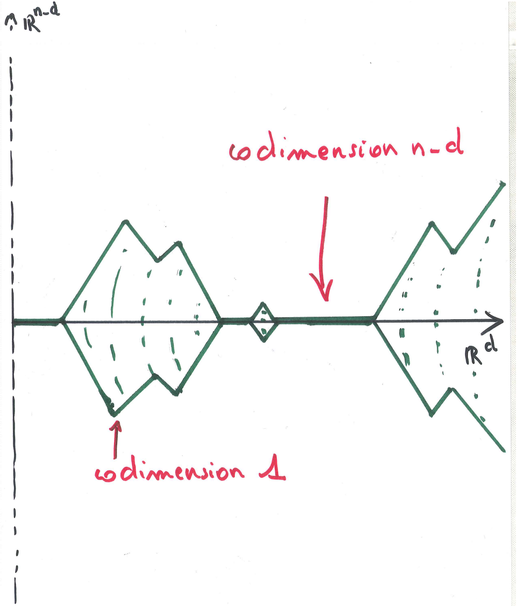


At some point, we need to work with saw-tooth domains.



In codimension 1,

we use a change of variable to come back to the case where the boundary is $\mathbb{R}^d$.



In higher codimension,

Saw tooth domains have mixed codimensions boundaries, and the solutions are not radially invariant,

$\hookrightarrow$ there is no way to flatten,

$\hookrightarrow$ we had to prove the estimates directly.

Elliptic operators with complex coefficients.

- We say that the Dirichlet problem (D_p) is solvable if for any $g \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$, there exists a unique solution u satisfying

$$\begin{cases} Lu = 0 \quad \text{in } \mathbb{R}^n \setminus \mathbb{R}^d \\ \operatorname{Tr} u = g \quad \text{on } \mathbb{R}^d \\ \|\widetilde{N}(u)\|_p \leq C\|g\|_p < +\infty, \end{cases}$$

  where

$$\widetilde{N}(u)(x) := \sup_{X \in \gamma(x)} \left(\fint_{B_{\delta(X)/2}(X)} |u|^2 \right).$$

- We say that $L = -\operatorname{div} A\nabla$ is p-elliptic if there exists $\lambda_p > 0$ such that

$$\lambda_p |t|^{n-d-1} |\xi|^2 \leq \mathcal{R}e\, \langle A\xi, \mathcal{J}_p \xi \rangle \qquad \text{for } \xi \in \mathbb{C}^n,$$

  where $\mathcal{J}_p(\alpha + i\beta) = \frac{1}{p}\alpha + \frac{i}{p'}\beta$.

Theorem

Let $p \in (1, +\infty)$. Assume that $L = -\mathrm{div} A\nabla$ is a weighted p-elliptic operator and such that

$$|t|^{n-d-1} A = \begin{pmatrix} \mathcal{A}^1 & \mathcal{A}^2 \\ \mathcal{B}^3 + \mathcal{C}^3 & b.I_{n-d} + \mathcal{C}^4 \end{pmatrix},$$

where

- $\mathcal{B}^3$, *b are real valued,*
- $\mathcal{C}^3$, $\mathcal{C}^4$, $|t|\nabla\mathcal{B}^3$, $|t|\nabla b$ *satisfies* $CM(\epsilon)$.

Then, if ϵ is small enough, the Dirichlet problem (D_p) is solvable.

Comment: since the maximum principle doesn't hold here, contrary to the real case, the solvability of (D_p) for $p > 2$ doesn't follow from (D_2).

Thank you for your attention.