Localization in time-frequency-analysis

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Definition

Cauchy Integral on Lipschitz curves:

$$Cf(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} dz$$

Question: is this a bounded operator on L^2?

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$$Cf(x) = p.v. \int_{\mathbb{R}} \frac{f(y)}{x - y + i(A(x) - A(y))} dy,$$

with the assumption that $\|A'\|_\infty$ is very small.

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$$C_k(f)(x) := \int_{\mathbb{R}} \frac{(A(x) - A(y))^k}{(x-y)^{k+1}} f(y) dy$$

is Calderón's k^{th} commutator.

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Note

$$C_0(f)(x) = p.v. \int_{\mathbb{R}} \frac{f(y)}{x - y} dy = H(f)(x).$$

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$$= p.v. \int_{\mathbb{R}} \left(\int_0^1 A'(x+\alpha t) d\alpha \right) f(x+t) \frac{dt}{t}$$

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Calderón's idea in the '70s: prove, uniformly in α, that

$$(f, g) \mapsto p.v. \int_{\mathbb{R}} f(x+t) g(x+\alpha t) \frac{dt}{t}$$

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'99: Lacey, Thiele prove $L^2 \times L^\infty \to L^2$ boundedness.
'04: Grafakos, Li: uniform estimates for $L^2 \times L^\infty \to L^2$ boundedness
(seminal ideas due to Thiele: uniform estimates for $L^2 \times L^2 \to L^{1,\infty}$)

The Bilinear Hilbert transform

Definition

$$BHT(f,g)(x) := p.v. \int_{\mathbb{R}} f(x+t)g(x-t)\,\frac{dt}{t}$$
$$= \int_{\mathbb{R}^2} \hat{f}(\xi_1)\,\hat{g}(\xi_2) sgn(\xi_1 - \xi_2)\, e^{2\pi i x(\xi_1+\xi_2)}\, d\xi_1\, d\xi_2$$

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Theorem (Lacey, Thiele '97)

BHT : $L^p \times L^q \to L^s$ *for all* $2 < p, q, s' < \infty$, *with* $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$.

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⚠ Still open: what happens when $\frac{1}{2} < s \leq \frac{2}{3}$?

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- modulation invariance: if $M_a(f)(x) := e^{2\pi i x a} f(x)$, then

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Another modulation-invariant operator: Carleson's operator:

$$\mathcal{C}f(x) := \sup_N \Big| \int_{\xi < N} \hat{f}(\xi) e^{2\pi ix\xi} d\xi \Big|.$$

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Hence time-frequency decomposition; study the model operator

$$BHT_{\mathbb{P}}(f,g)(x) = \sum_{P \in \mathbb{P}} \frac{1}{|I_P|^{1/2}} \langle f, \phi^1_{P_1} \rangle \langle g, \phi^2_{P_2} \rangle \phi^3_{P_3}(x),$$

where $\phi^1_{P_1}, \phi^2_{P_2}, \phi^3_{P_3}$ are L^2-normalized wave packets associated to the tiles $I_P \times \omega_{P_1}, I_P \times \omega_{P_2}$ and $I_P \times \omega_{P_3}$ respectively.
If ω_{P_1} is fixed, then $\omega_{P_2} = \omega_{P_1} + c|\omega_{P_1}|, \pm\omega_{P_3} = \omega_{P_1} + \omega_{P_2}$.

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⚠ It is this model operator that is unbounded for $s \leq \frac{2}{3}$.

Method of the proof (cont.)

Look at the trilinear form:

$$\Lambda_{BHT;\mathbb{P}}(f, g, h) = \sum_{P \in \mathbb{P}} \frac{1}{|I_P|^{1/2}} \langle f, \phi^1_{P_1} \rangle \langle g, \phi^2_{P_2} \rangle \langle h, \phi^3_{P_3} \rangle.$$

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Two ideas emerge:

- define
$$\text{size}_{\mathbb{P}}(f) := \sup_{\substack{T\subseteq \mathbb{P} \\ \text{lac. tree}}} \Big(\frac{1}{|I_T|} \sum_{P\in T} |\langle f, \phi_P\rangle|^2\Big)^{\frac{1}{2}} \lesssim \sup_{P\in\mathbb{P}} \frac{1}{|I_P|} \int_{\mathbb{R}} |f(x)|\, \tilde{\chi}_{I_P}(x) dx := \widetilde{\text{size}}_{\mathbb{P}}(f).$$

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Then

$$\sum_{P \in T} \frac{1}{|I_P|^{1/2}} |\langle f, \phi^1_{P_1}\rangle \langle g, \phi^2_{P_2}\rangle \langle h, \phi^3_{P_3}\rangle| \lesssim \widetilde{\text{size}}_T(f)\, \widetilde{\text{size}}_T(g)\, \widetilde{\text{size}}_T(h)\, |I_T|.$$

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- decompose $\mathbb{P}$ into trees: $\mathbb{P} = \bigcup_{n \in \mathbb{Z}} \bigcup_{T \in \mathbf{T}_n} T$ or $\mathbb{P} = \bigcup_{n_1, n_2, n_3 \in \mathbb{Z}} \bigcup_{T \in \mathbf{T}_{n_1} \cap \mathbf{T}_{n_2} \cap \mathbf{T}_{n_3}} T$.

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Such a decomposition yields, for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$:

$$|\Lambda_{BHT;\mathbb{P}}(f,g,h)| \lesssim \big(\text{size}^1_{\mathbb{P}} f\big)^{\theta_1} \cdot \big(\text{size}^2_{\mathbb{P}} g\big)^{\theta_2} \cdot \big(\text{size}^3_{\mathbb{P}} h\big)^{\theta_3}$$
$$\cdot \big(\text{energy}^1_{\mathbb{P}} f\big)^{1-\theta_1} \cdot \big(\text{energy}^2_{\mathbb{P}} g\big)^{1-\theta_2} \cdot \big(\text{energy}^3_{\mathbb{P}} h\big)^{1-\theta_3}.$$

Here $\text{energy}^j_{\mathbb{P}'}(f) := \sup_{n\in\mathbb{Z}} \sup_{\mathbb{D}} 2^n \big(\sum_{T\in\mathbb{D}} |I_T|\big)^{\frac{1}{2}}$, under some conditions.

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Such a decomposition yields, for any $0 \leq \alpha_1, \alpha_2, \alpha_3 < \frac{1}{2}$ with $\alpha_1+\alpha_2+\alpha_3 = 1$:

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Method of the proof (cont.)

At the heart, a decomposition lemma:

Lemma

If $\mathbb{P}_j$ is a collection of j-tiles with $\text{size}_{\mathbb{P}_j}\big(\langle f_j, \phi_P^j\rangle\big) \leq \lambda$, *then there exists a decomposition* $\mathbb{P}_j = \mathbb{P}_j' \cup \mathbb{P}_j''$ *so that* $\text{size}_{\mathbb{P}_j'}\big(\langle f_j, \phi_P^j\rangle\big) \leq \frac{\lambda}{2}$ *and* $\mathbb{P}_j''$ *is a union* $\mathbf{T} = \bigcup_{T \in \mathbf{T}} T$ *of disjoint trees so that*

$$\sum_{T \in \mathbf{T}} |I_T| \lesssim \lambda^{-2} \|f_j\|_2^2.$$

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$$\sum_{T \in \mathbf{T}} |I_T| \lesssim \lambda^{-2} \|f_j\|_2^2.$$

And also a stopping time: $\lambda = 2^n$ (or three stopping times according to $\lambda = 2^{n_1}, 2^{n_2}, 2^{n_3}$).

The formula

$$|\Lambda_{BHT;\mathbb{P}}(f,g,h)| \lesssim \left(\text{size}^1_{\mathbb{P}} f\right)^{1-2\alpha_1} \left(\text{size}^2_{\mathbb{P}} g\right)^{1-2\alpha_2} \left(\text{size}^3_{\mathbb{P}} h\right)^{1-2\alpha_3} \|f\|_2^{2\alpha_1} \|g\|_2^{2\alpha_2} \|h\|_2^{2\alpha_3}.$$

points out to working with restricted type functions: $|f| \leq \mathbf{1}_F, |g| \leq \mathbf{1}_G, |h| \leq \mathbf{1}_H$.

Method of the proof (ctd.)

The formula

$$|\Lambda_{BHT;\mathbb{P}}(f,g,h)| \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}} \mathbf{1}_F\right)^{1-2\alpha_1} \left(\text{size}_{\mathbb{P}} \mathbf{1}_G\right)^{1-2\alpha_2} \left(\text{size}_{\mathbb{P}} \mathbf{1}_H\right)^{1-2\alpha_3} |F|^{\alpha_1} |G|^{\alpha_2} |H|^{\alpha_3}.$$

points out to working with restricted type functions: $|f| \leq \mathbf{1}_F, |g| \leq \mathbf{1}_G, |h| \leq \mathbf{1}_H$.
Restricted-type interpolation implies the local L^2 case: $2 < p, q, s' < \infty$.

Method of the proof (ctd.)

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Outside the local L^2 range, we need *generalized restricted-type interpolation*, i.e. it is enough to prove for all sets and all functions $|f| \leq \mathbf{1}_F, |g| \leq \mathbf{1}_G, |h| \leq \mathbf{1}_{H'}$

$$|\Lambda_{BHT;\mathbb{P}}(f,g,h)| \lesssim |F|^{\frac{1}{p}} |G|^{\frac{1}{q}} |H|^{\frac{1}{s'}},$$

where $H' := H \setminus \Omega, \quad \Omega = \{x : \mathcal{M}(\mathbf{1}_F) > C\frac{|F|}{|H|}, \mathcal{M}(\mathbf{1}_G) > C\frac{|G|}{|H|}\}.$

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Assuming that for all $P \in \mathbb{P}$ we have $I_P \cap \Omega^c \neq \emptyset$, we deduce

$$\widetilde{\text{size}}_{\mathbb{P}}\mathbf{1}_F \lesssim \min(1, \frac{|F|}{|H|}), \widetilde{\text{size}}_{\mathbb{P}}\mathbf{1}_G \lesssim \min(1, \frac{|G|}{|H|}).$$

This implies, for any $0 \leq a, b \leq 1$:

$$|\Lambda_{BHT;\mathbb{P}}(f,g,h)| \lesssim \big(\frac{|F|}{|H|}\big)^{a(1-2\alpha_1)}\big(\frac{|G|}{|H|}\big)^{b(1-2\alpha_2)}|F|^{\alpha_1}|G|^{\alpha_2}|H|^{\alpha_3}.$$

Vector valued estimates for *BHT*

$$BHT : L^p\left(\ell^{r_1}\right) \times L^q\left(\ell^{r_2}\right) \to L^s\left(\ell^{r}\right), \qquad (*)$$

where $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$.

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Motivation: a Rubio de Francia operator for iterated Fourier integrals (ideally, wanted $r = 1$)

$$T_r(f,g)(x) := \Big(\sum_{k=1}^{N} \Big| \int_{a_k < \xi_1 < \xi_2 < b_k} \hat{f}(\xi_1)\hat{g}(\xi_2) e^{2\pi i x(\xi_1+\xi_2)} d\xi_1 d\xi_2 \Big|^r \Big)^{\frac{1}{r}}.$$

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Partial results of (*), for $r > \frac{4}{3}$, due to P. Silva 2014. He used a vector-valued approach for studying $BHT \otimes \Pi$ (Π is a paraproduct or Fourier multiplier singular at $(0,0)$).

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Theorem (B., Muscalu 2016)

$$BHT : L^p\left(\mathbb{R}; L^{R_1}(\mathcal{W},\mu)\right) \times L^q\left(\mathbb{R}; L^{R_2}(\mathcal{W},\mu)\right) \to L^s\left(\mathbb{R}; L^{R}(\mathcal{W},\mu)\right)$$

for n-tuples R_1, R_2, R *and Lebesgue exponents* p, q, s *satisfying*

$$\frac{1}{p}, \frac{1}{r_1^j} < 1-\alpha_1, \quad \frac{1}{q}, \frac{1}{r_2^j} < 1-\alpha_2, \quad \frac{1}{s'}, \frac{1}{(r^j)'} < 1-\alpha_3,$$

for some $0 < \alpha_1, \alpha_2, \alpha_3 < \frac{1}{2}$ *with* $\alpha_1 + \alpha_2 + \alpha_3 = 1$.
As a consequence, $BHT \otimes \Pi \otimes \ldots \otimes \Pi$ *satisfies the same* L^p *estimates as* BHT.

It is enough to prove for all sets and all vector-valued functions $\left(\sum_k |f_k|^{r_1}\right)^{\frac{1}{r_1}} \leq \mathbf{1}_F$, $\left(\sum_k |g_k|^{r_2}\right)^{\frac{1}{r_2}} \leq \mathbf{1}_G$, $\left(\sum_k |h_k|^{r'}\right)^{\frac{1}{r'}} \leq \mathbf{1}_{H'}$

$$|\sum_k \Lambda_{BHT;\mathbb{P}}(f_k, g_k, h_k)| \lesssim |F|^{\frac{1}{p}} \, |G|^{\frac{1}{q}} \, |H|^{\frac{1}{s'}},$$

where $H' := H \setminus \Omega, \quad \Omega = \{x : \mathcal{M}(\mathbf{1}_F) > C\frac{|F|}{|H|}, \mathcal{M}(\mathbf{1}_G) > C\frac{|G|}{|H|}\}.$

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Recall that

$$|\Lambda_{BHT;\mathbb{P}}(f, g, h)| \lesssim \big(\text{size}_{\mathbb{P}}^1\, f\big)^{1-2\alpha_1} \big(\text{size}_{\mathbb{P}}^2\, g\big)^{1-2\alpha_2} \big(\text{size}_{\mathbb{P}}^3\, h\big)^{1-2\alpha_3} \|f\|_2^{2\alpha_1} \|g\|_2^{2\alpha_2} \|h\|_2^{2\alpha_3}.$$

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We would want to obtain, locally (when spatial information is contained inside I_0)

$$|\Lambda_{BHT;\mathbb{P}(I_0)}(f_k, g_k, h_k)| \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)}\mathbf{1}_F\right)^{1-\alpha_1-\frac{1}{r_1}-\epsilon} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)}\mathbf{1}_G\right)^{1-\alpha_2-\frac{1}{r_2}-\epsilon} \left(\widetilde{\text{size}}_{\mathbb{P}(I_0)}\mathbf{1}_{H'}\right)^{1-\alpha_3-\frac{1}{r'}-\epsilon}$$
$$\cdot \left\|f_k \cdot \tilde{\chi}_{I_0}\right\|_{r_1} \left\|g_k \cdot \tilde{\chi}_{I_0}\right\|_{r_2} \left\|h_k \cdot \tilde{\chi}_{I_0}\right\|_{r'}.$$

This is reasonable (and doable) since also $|f_k| \leq \mathbf{1}_F$.

Method of the proof

It is enough to prove for all sets and all vector-valued functions $\left(\sum_k |f_k|^{r_1}\right)^{\frac{1}{r_1}} \leq \mathbf{1}_F$, $\left(\sum_k |g_k|^{r_2}\right)^{\frac{1}{r_2}} \leq \mathbf{1}_G$, $\left(\sum_k |h_k|^{r'}\right)^{\frac{1}{r'}} \leq \mathbf{1}_{H'}$

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Recall that, for $|f| \leq \mathbf{1}_{E_1}, |g| \leq \mathbf{1}_{E_2}, |h| \leq \mathbf{1}_{E_3}$

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Method of the proof

It is enough to prove for all sets and all vector-valued functions $\left(\sum_k |f_k|^{r_1}\right)^{\frac{1}{r_1}} \leq \mathbf{1}_F$, $\left(\sum_k |g_k|^{r_2}\right)^{\frac{1}{r_2}} \leq \mathbf{1}_G$, $\left(\sum_k |h_k|^{r'}\right)^{\frac{1}{r'}} \leq \mathbf{1}_{H'}$

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It is enough to prove for all sets and all vector-valued functions $\left(\sum_k |f_k|^{r_1}\right)^{\frac{1}{r_1}} \leq \mathbf{1}_F, \left(\sum_k |g_k|^{r_2}\right)^{\frac{1}{r_2}} \leq \mathbf{1}_G, \left(\sum_k |h_k|^{r'}\right)^{\frac{1}{r'}} \leq \mathbf{1}_{H'}$

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Recall that, for $|f| \leq \mathbf{1}_{E_1}, |g| \leq \mathbf{1}_{E_2}, |h| \leq \mathbf{1}_{E_3}$

$$|\Lambda_{BHT;\mathbb{P}(I_0)}(f,g,h)| \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}^+(I_0)}\mathbf{1}_{E_1}\right)^{1-\alpha_1} \left(\widetilde{\text{size}}_{\mathbb{P}^+(I_0)}\mathbf{1}_{E_2}\right)^{1-\alpha_2} \left(\widetilde{\text{size}}_{\mathbb{P}^+(I_0)}\mathbf{1}_{E_3}\right)^{1-\alpha_3} \cdot |I_0|,$$

where

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We would want to obtain, locally (when spatial information is contained inside I_0)

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Method of the proof

It is enough to prove for all sets and all vector-valued functions $\big(\sum_k |f_k|^{r_1}\big)^{\frac{1}{r_1}} \leq \mathbf{1}_F, \big(\sum_k |g_k|^{r_2}\big)^{\frac{1}{r_2}} \leq \mathbf{1}_G, \big(\sum_k |h_k|^{r'}\big)^{\frac{1}{r'}} \leq \mathbf{1}_{H'}$

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where $H' := H \setminus \Omega, \quad \Omega = \{x : \mathcal{M}(\mathbf{1}_F) > C\dfrac{|F|}{|H|}, \mathcal{M}(\mathbf{1}_G) > C\dfrac{|G|}{|H|}\}.$

If $|f_k| \leq \mathbf{1}_{E_1}, |g_k| \leq \mathbf{1}_{E_2}, |h_k| \leq \mathbf{1}_{E_3}$ (then actually $|f_k| \leq \mathbf{1}_{E_1 \cap F}, |g_k| \leq \mathbf{1}_{E_2 \cap G}, |h_k| \leq \mathbf{1}_{E_3 \cap H'}$)

$$|\Lambda_{BHT;\mathbb{P}(I_0)}(f_k, g_k, h_k)| \lesssim \big(\widetilde{\text{size}}_{\mathbb{P}^+(I_0)} \mathbf{1}_{E_1 \cap F}\big)^{1-\alpha_1} \big(\widetilde{\text{size}}_{\mathbb{P}^+(I_0)} \mathbf{1}_{E_2 \cap G}\big)^{1-\alpha_2} \big(\widetilde{\text{size}}_{\mathbb{P}^+(I_0)} \mathbf{1}_{E_3 \cap H'}\big)^{1-\alpha_3} \cdot |I_0|,$$

where

$$\widetilde{\text{size}}_{\mathbb{P}^+(I_0)}(f) := \max\Big(\sup_{P \in \mathbb{P}(I_0)} \frac{1}{|I_P|} \int_{\mathbb{R}} |f| \tilde{\chi}_{I_P} \, dx, \frac{1}{|I_0|} \int_{\mathbb{R}} |f| \tilde{\chi}_{I_0} \, dx\Big)$$

Re-run the interpolation algorithm to obtain, locally (when spatial information is contained inside I_0)

$$|\Lambda_{BHT;\mathbb{P}(I_0)}(f_k, g_k, h_k)| \lesssim \Big(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_F\Big)^{1-\alpha_1-\frac{1}{r_1}-\epsilon} \Big(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_G\Big)^{1-\alpha_2-\frac{1}{r_2}-\epsilon} \Big(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{H'}\Big)^{1-\alpha_3-\frac{1}{r'}-\epsilon}$$
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Method of the proof

It is enough to prove for all sets and all vector-valued functions $\big(\sum_k |f_k|^{r_1}\big)^{\frac{1}{r_1}} \leq \mathbf{1}_F$, $\big(\sum_k |g_k|^{r_2}\big)^{\frac{1}{r_2}} \leq \mathbf{1}_G$, $\big(\sum_k |h_k|^{r'}\big)^{\frac{1}{r'}} \leq \mathbf{1}_{H'}$

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Recall

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By Hölder's inequality $\big(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r'} = 1\big)$

$$|\sum_k \Lambda_{BHT;\mathbb{P}(I_0)}(f_k, g_k, h_k)| \lesssim \Big(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_F\Big)^{1-\alpha_1 - \frac{1}{r_1} - \epsilon} \Big(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_G\Big)^{1-\alpha_2 - \frac{1}{r_2} - \epsilon} \Big(\widetilde{\text{size}}_{\mathbb{P}(I_0)} \mathbf{1}_{H'}\Big)^{1-\alpha_3 - \frac{1}{r'} - \epsilon}$$
$$\cdot \big\|\mathbf{1}_F \cdot \tilde{\chi}_{I_0}\big\|_{r_1} \big\|\mathbf{1}_G \cdot \tilde{\chi}_{I_0}\big\|_{r_2} \big\|\mathbf{1}_{H'} \cdot \tilde{\chi}_{I_0}\big\|_{r'}.$$

But $\big\|\mathbf{1}_F \cdot \tilde{\chi}_{I_0}\big\|_{r_1} \sim \Big(\dfrac{\big\|\mathbf{1}_F \cdot \tilde{\chi}_{I_0}\big\|_1}{|I_0|}\Big)^{\frac{1}{r_1}} \cdot |I_0|^{\frac{1}{r_1}} \ldots$

It is enough to prove for all sets and all vector-valued functions $\big(\sum_k |f_k|^{r_1}\big)^{\frac{1}{r_1}} \leq \mathbf{1}_F$, $\big(\sum_k |g_k|^{r_2}\big)^{\frac{1}{r_2}} \leq \mathbf{1}_G$, $\big(\sum_k |h_k|^{r'}\big)^{\frac{1}{r'}} \leq \mathbf{1}_{H'}$

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By Hölder's inequality ($\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r'} = 1$)

$$|\sum_k \Lambda_{BHT;\mathbb{P}(I_0)}(f_k, g_k, h_k)| \lesssim \Big(\widetilde{\text{size}}_{\mathbb{P}(I_0)}\mathbf{1}_F\Big)^{1-\alpha_1-\epsilon} \Big(\widetilde{\text{size}}_{\mathbb{P}(I_0)}\mathbf{1}_G\Big)^{1-\alpha_2-\epsilon} \Big(\widetilde{\text{size}}_{\mathbb{P}(I_0)}\mathbf{1}_{H'}\Big)^{1-\alpha_3-\epsilon} \cdot |I_0|.$$

Method of the proof

It is enough to prove for all sets and all vector-valued functions $\big(\sum_k |f_k|^{r_1}\big)^{\frac{1}{r_1}} \leq \mathbf{1}_F,\, \big(\sum_k |g_k|^{r_2}\big)^{\frac{1}{r_2}} \leq \mathbf{1}_G,\, \big(\sum_k |h_k|^{r'}\big)^{\frac{1}{r'}} \leq \mathbf{1}_{H'}$

$$|\sum_k \Lambda_{BHT;\mathbb{P}}(f_k, g_k, h_k)| \lesssim |F|^{\frac{1}{p}}\, |G|^{\frac{1}{q}}\, |H|^{\frac{1}{s'}},$$

where $H' := H \setminus \Omega, \quad \Omega = \{x : \mathcal{M}(\mathbf{1}_F) > C\dfrac{|F|}{|H|}, \mathcal{M}(\mathbf{1}_G) > C\dfrac{|G|}{|H|}\}.$

Recall

$$\widetilde{\text{size}}_{\mathbb{P}^+(I_0)}(f) := \max\Big(\sup_{P\in\mathbb{P}(I_0)} \frac{1}{|I_P|}\int_{\mathbb{R}} |f|\tilde{\chi}_{I_P}\, dx, \frac{1}{|I_0|}\int_{\mathbb{R}} |f|\tilde{\chi}_{I_0}\, dx\Big)$$

By Hölder's inequality $(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r'} = 1)$

$$|\sum_k \Lambda_{BHT;\mathbb{P}(I_0)}(f_k, g_k, h_k)| \lesssim \big(\widetilde{\text{size}}_{\mathbb{P}(I_0)}\mathbf{1}_F\big)^{1-\alpha_1-\epsilon}\, \big(\widetilde{\text{size}}_{\mathbb{P}(I_0)}\mathbf{1}_G\big)^{1-\alpha_2-\epsilon}\, \big(\widetilde{\text{size}}_{\mathbb{P}(I_0)}\mathbf{1}_{H'}\big)^{1-\alpha_3-\epsilon} \cdot |I_0|.$$

Redo the stopping times to obtain

$$|\sum_k \Lambda_{BHT;\mathbb{P}}(f_k, g_k, h_k)| \lesssim |F|^{\frac{1}{p}}\, |G|^{\frac{1}{q}}\, |H|^{\frac{1}{s'}}.$$

Theorem (Culiuc, Di Plinio, Ou (to appear))

For any locally integrable functions f, g, h and any Lebesgue exponents s_1, s_2, s_3 with $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} < 2$, there exists $\mathcal{S}$ a sparse collection of dyadic intervals (depending on f, g, h and s_1, s_2, s_3) to that

$$\left|\Lambda_{BHT}(f, g, h)\right| \lesssim \sum_{Q \in \mathcal{S}} \Big(\fint_Q |f|^{s_1}\Big)^{\frac{1}{s_1}} \Big(\fint_Q |g|^{s_2}\Big)^{\frac{1}{s_2}} \Big(\fint_Q |h|^{s_3}\Big)^{\frac{1}{s_3}} |Q|.$$

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Definition

Let $0 < \eta < 1$. A collection $\mathcal{S}$ of dyadic intervals is called *η-sparse* if one can choose pairwise disjoint measurable sets $E_Q \subseteq Q$ with $|E_Q| \geq \eta |Q|$ for all $Q \in \mathcal{S}$.

Theorem (Culiuc, Di Plinio, Ou (to appear))

For any locally integrable functions f, g, h and any Lebesgue exponents s_1, s_2, s_3 with $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} < 2$, there exists $\mathcal{S}$ a sparse collection of dyadic intervals (depending on f, g, h and s_1, s_2, s_3) to that

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This implies weighted estimates:

Theorem (Culiuc, Di Plinio, Ou (to appear))

If $1 < q_1, q_2, q < \infty$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, any weights w_1, w_2, w with $w = w_1 \cdot w_2$,

$$BHT : L^{q_1}(w_1^{q_1}) \times L^{q_2}(w_2^{q_2}) \to L^q(w^q),$$

provided the vector weight condition

$$\sup_{Q \subset \mathbb{R}} \left(\fint_Q w_1^{\frac{1}{\frac{1}{q_1} - \frac{1}{s_1}}}\right)^{\frac{1}{s_1} - \frac{1}{q_1}} \cdot \left(\fint_Q w_2^{\frac{1}{\frac{1}{q_2} - \frac{1}{s_2}}}\right)^{\frac{1}{s_2} - \frac{1}{q_2}} \cdot \left(\fint_Q w^{s_3}\right)^{\frac{1}{s_3}} < +\infty.$$

The local estimate

$$|\Lambda_{BHT;\mathbb{P}(I_0)}(f,g,h)| \lesssim \left(\widetilde{\text{size}}_{\mathbb{P}^+(I_0)}\mathbf{1}_{E_1}\right)^{1-\alpha_1}\left(\widetilde{\text{size}}_{\mathbb{P}^+(I_0)}\mathbf{1}_{E_2}\right)^{1-\alpha_2}\left(\widetilde{\text{size}}_{\mathbb{P}^+(I_0)}\mathbf{1}_{E_3}\right)^{1-\alpha_3}\cdot|I_0|,$$

proved for $|f| \leq \mathbf{1}_{E_1}, |g| \leq \mathbf{1}_{E_2}, |h| \leq \mathbf{1}_{E_3}$, holds for general functions as well.

Local estimate + stopping times =v.v. estimates + sparse domination

$$|\Lambda_{BHT;\mathbb{P}(I_0)}(f,g,h)| \lesssim \Big(\sup_{P\in\mathbb{P}^+(I_0)} \frac{1}{|I_P|}\int_{\mathbb{R}} |f|^{s_1}\tilde{\chi}_{I_P}dx\Big)^{\frac{1}{s_1}} \Big(\sup_{P\in\mathbb{P}^+(I_0)} \frac{1}{|I_P|}\int_{\mathbb{R}} |g|^{s_2}\tilde{\chi}_{I_P}dx\Big)^{\frac{1}{s_2}}$$
$$\cdot \Big(\sup_{P\in\mathbb{P}^+(I_0)} \frac{1}{|I_P|}\int_{\mathbb{R}} |h|^{s_3}\tilde{\chi}_{I_P}dx\Big)^{\frac{1}{s_3}} |I_0|$$

provided $\frac{1}{s_1} < 1-\alpha_1$, $\frac{1}{s_2} < 1-\alpha_2$ and $\frac{1}{s_3} < 1-\alpha_3$.

$$|\Lambda_{BHT;\mathbb{P}(I_0)}(f,g,h)| \lesssim \Big(\sup_{P \in \mathbb{P}^+(I_0)} \frac{1}{|I_P|} \int_{\mathbb{R}} |f|^{s_1} \tilde{\chi}_{I_P} dx \Big)^{\frac{1}{s_1}} \Big(\sup_{P \in \mathbb{P}^+(I_0)} \frac{1}{|I_P|} \int_{\mathbb{R}} |g|^{s_2} \tilde{\chi}_{I_P} dx \Big)^{\frac{1}{s_2}}$$
$$\cdot \Big(\sup_{P \in \mathbb{P}^+(I_0)} \frac{1}{|I_P|} \int_{\mathbb{R}} |h|^{s_3} \tilde{\chi}_{I_P} dx \Big)^{\frac{1}{s_3}} |I_0|$$

provided $\frac{1}{s_1} < 1 - \alpha_1$, $\frac{1}{s_2} < 1 - \alpha_2$ and $\frac{1}{s_3} < 1 - \alpha_3$.
A bottom-top stopping time yields sparse and vector-valued estimates:

Local estimate + stopping times =v.v. estimates + sparse domination

$$|\Lambda_{BHT;\mathbb{P}(I_0)}(f,g,h)| \lesssim \Big(\sup_{P\in\mathbb{P}^+(I_0)} \frac{1}{|I_P|} \int_{\mathbb{R}} |f|^{s_1} \tilde{\chi}_{I_P}\, dx \Big)^{\frac{1}{s_1}} \Big(\sup_{P\in\mathbb{P}^+(I_0)} \frac{1}{|I_P|} \int_{\mathbb{R}} |g|^{s_2} \tilde{\chi}_{I_P}\, dx \Big)^{\frac{1}{s_2}}$$
$$\cdot \Big(\sup_{P\in\mathbb{P}^+(I_0)} \frac{1}{|I_P|} \int_{\mathbb{R}} |h|^{s_3} \tilde{\chi}_{I_P}\, dx \Big)^{\frac{1}{s_3}} |I_0|$$

provided $\frac{1}{s_1} < 1-\alpha_1$, $\frac{1}{s_2} < 1-\alpha_2$ and $\frac{1}{s_3} < 1-\alpha_3$.
A bottom-top stopping time yields sparse and vector-valued estimates:

Theorem (B., Muscalu)

For any $\frac{2}{3} < q$, any vector-valued functions $\vec{f}, \vec{g}$ so that $\|\vec{f}(x,\cdot)\|_{L^{R_1}(\mathcal{W},\mu)}, \|\vec{g}(x,\cdot)\|_{L^{R_2}(\mathcal{W},\mu)}$ are locally integrable, and any v a locally q-integrable function, we can construct a sparse collection $\mathcal{S}$ of dyadic intervals, depending on the functions $\vec{f}, \vec{g}$ and v, and on the exponents s_1, s_2, s_3, q, for which

$$\big\| \|BHT_{\mathbb{P}}(\vec{f},\vec{g})\|_{L^R(\mathcal{W},\mu)} \cdot v \big\|_q^q \lesssim \sum_{Q\in\mathcal{S}} \Big(\frac{1}{|Q|} \int_{\mathbb{R}} \|\vec{f}(x,\cdot)\|_{L^{R_1}(\mathcal{W},\mu)}^{s_1} \cdot \tilde{\chi}_Q\, dx \Big)^{\frac{q}{s_1}} \cdot \Big(\frac{1}{|Q|} \int_{\mathbb{R}} \|\vec{g}(x,\cdot)\|_{L^{R_2}(\mathcal{W},\mu)}^{s_2} \cdot \tilde{\chi}_Q\, dx \Big)$$
$$\cdot \Big(\frac{1}{|Q|} \int_{\mathbb{R}} |v(x)|^{s_3} \cdot \tilde{\chi}_Q\, dx \Big)^{\frac{q}{s_3}} \cdot |Q|.$$

(some conditions missing)

Local estimate + stopping times =v.v. estimates + sparse domination

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$$\big\| \|BHT_{\mathbb{P}}(\vec{f},\vec{g})\|_{L^R(\mathcal{W},\mu)} \cdot v \big\|_q^q \lesssim \sum_{Q \in \mathcal{S}} \Big(\frac{1}{|Q|}\int_{\mathbb{R}} \|\vec{f}(x,\cdot)\|_{L^{R_1}(\mathcal{W},\mu)}^{s_1} \cdot \tilde{\chi}_Q dx\Big)^{\frac{q}{s_1}} \cdot \Big(\frac{1}{|Q|}\int_{\mathbb{R}} \|\vec{g}(x,\cdot)\|_{L^{R_2}(\mathcal{W},\mu)}^{s_2} \cdot \tilde{\chi}_Q dx\Big)^{\frac{q}{s_2}} \cdot \Big(\frac{1}{|Q|}\int_{\mathbb{R}} |v(x)|^{s_3} \cdot \tilde{\chi}_Q dx\Big)^{\frac{q}{s_3}} \cdot |Q|.$$

Corollary

If the Lebesgue exponents q, s_1, s_2 and the n-tuples R_1, R_2, R' are as in the theorem above (that is, they are conditioned by certain $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$), then

$$\Big\| \|BHT_{\mathbb{P}}(\vec{f},\vec{g})\|_{L^R(\mathcal{W},\mu)} \Big\|_q \lesssim \Big\| \mathcal{M}_{s_1}\big(\|\vec{f}(x,\cdot)\|_{L^{R_1}(\mathcal{W},\mu)}\big) \cdot \mathcal{M}_{s_2}\big(\|\vec{g}(x,\cdot)\|_{L^{R_2}(\mathcal{W},\mu)}\big) \Big\|_q,$$

where $\mathcal{M}_s(f) := \big(\mathcal{M}(|f|^s)\big)^{\frac{1}{s}}$.

Local estimate + stopping times =v.v. estimates + sparse domination

Theorem (B., Muscalu)

For any $\frac{2}{3} < q$, any vector-valued functions $\vec{f}, \vec{g}$ so that $\|\vec{f}(x,\cdot)\|_{L^{R_1}(\mathcal{W},\mu)}, \|\vec{g}(x,\cdot)\|_{L^{R_2}(\mathcal{W},\mu)}$ are locally integrable, and any v a locally q-integrable function, we can construct a sparse collection $\mathcal{S}$ of dyadic intervals, depending on the functions $\vec{f}, \vec{g}$ and v, and on the exponents s_1, s_2, s_3, q, for which

$$\big\| \|BHT_{\mathbb{P}}(\vec{f}, \vec{g})\|_{L^R(\mathcal{W},\mu)} \cdot v \big\|_q^q \lesssim \sum_{Q \in \mathcal{S}} \Big(\frac{1}{|Q|}\int_{\mathbb{R}} \|\vec{f}(x,\cdot)\|_{L^{R_1}(\mathcal{W},\mu)}^{s_1} \cdot \tilde{\chi}_Q dx\Big)^{\frac{q}{s_1}} \cdot \Big(\frac{1}{|Q|}\int_{\mathbb{R}} \|\vec{g}(x,\cdot)\|_{L^{R_2}(\mathcal{W},\mu)}^{s_2} \cdot \tilde{\chi}_Q dx\Big)^{\frac{q}{s_2}}$$

$$\cdot \Big(\frac{1}{|Q|}\int_{\mathbb{R}} |v(x)|^{s_3} \cdot \tilde{\chi}_Q dx\Big)^{\frac{q}{s_3}} \cdot |Q|.$$

Corollary

If the Lebesgue exponents q, s_1, s_2 and the n-tuples R_1, R_2, R' are as in the theorem above, then

$$\big\| \|BHT_{\mathbb{P}}(\vec{f}, \vec{g})\|_{L^R(\mathcal{W},\mu)} \big\|_q \lesssim \big\| \vec{\mathcal{M}}_{s_1,s_2}^{R_1,R_2}(\vec{f}, \vec{g}) \big\|_q, \quad \textit{where}$$

$$\vec{\mathcal{M}}_{s_1,s_2}(\vec{f}, \vec{g})(x) := \sup_{Q \ni x} \Big(\frac{1}{|Q|}\int_Q \|\vec{f}(y,\cdot)\|_{L^{R_1}(\mathcal{W},\mu)}^{s_1} dy\Big)^{\frac{1}{s_1}} \Big(\frac{1}{|Q|}\int_Q \|\vec{g}(y,\cdot)\|_{L^{R_2}(\mathcal{W},\mu)}^{s_2} dy\Big)^{\frac{1}{s_2}}.$$

Thank you slide

Thank you!