30 years of T(b) theorems

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The story begins with Coifman-McIntosh-Meyer's theorem (1982): The Cauchy integral operator C on a Lipschitz graph $x \mapsto z(x) = x + i\varphi(x)$ is bounded on L^2.

$$Cf(x) = pv \int \frac{f(y)}{z(x) - z(y)} z'(y) dy.$$

David-Journé (1984) proved the T(1) theorem: a CNS for L^2 boundedness for a singular integral operator T is $T(1), T^*(1) \in BMO$ and T weakly bounded.

$$Cf(x) = pv \int \frac{f(y)}{z(x) - z(y)} z'(y)dy.$$

Observe
1) $C(1) = 0$ but C is not a singular integral : kernel not regular in y.
2) $C = TM_{z'}$ with T singular integral and $T(z') = 0$ in BMO using the residue theorem.
3) $z'(x)$ is non degenerate: accretivity $\mathrm{Re}\, z'(x) \gtrsim 1$.

McIntosh-Meyer (1985) Let T be a singular integral operator and b is a bounded and accretive. Then T is L^2 bdd if $T(b) = 0 = T^*(b) = 0$ in BMO and TM_b is weakly bounded.
Proof: consequence of interpolation result that follows from the proof of the Kato conjecture in 1d.

Applies to $C = T$

Kato conjecture in 1d was proven at the same time as the boundedness of the Cauchy integral, with same method.

Need for another proof

Theorem (Global $T(b)$ theorem. David-Journé-Semmes (1985).)

Given a S.H.T. X (with extra assumptions). Let T be a singular integral and b_1, b_2 are para-accretive functions. Then T bounded on $L^2(X)$ if and only if $T(b_1), T^*(b_2) \in BMO$ and $M_{b_2} T M_{b_1}$ is weakly bounded.

b para-accretivity means that b is bounded and on each cube Q, $|\int_P b| \gtrsim |P|$ on some subcube P of Q with comparable size.

Several ways of extensions.

1) Global T(b) statements for singular integrals

Nazarov-Treil-Volberg (2003) statement on S.N.H.T.
They implemented the random dyadic structures.

They allow BMO type testing condition

2) Global T(b) theorem for square functions (Semmes, 1990):

$$\int_0^\infty \|\theta_t f\|_2^2 \, \frac{dt}{t} \lesssim \|f\|_2^2$$

where θ_t is like a convolution operator: kernel at scale t, but smooth in x only and $\theta_t(b) = 0$ for some b para-accretive.

Proof: show $|\theta_t(1)|^2 \frac{dxdt}{t}$ is a Carleson measure and apply a result of Christ-Journé. Uses the Coifman-Meyer principal approximation to apply Carleson theorem.

3) Local $T(b)$ theorem with L^∞ testing conditions: Christ, 1990.

Theorem

On a S.H.T (with no atoms). Let T be a singular integral operator. If there is a pseudo-accretive system of test functions indexed by balls such that $T(b_B^1)$ and $T^*(b_B^2)$ are bounded uniformly then T is bounded on L^2.

(b_B) pseudo-accretive system if b_B bounded uniformly and $[b_B]_B = 1$ for all balls.

Proof: construct global para-accretive functions adapted to T and apply DJS. Used systems of dyadic cubes.

Motivation: analytic capacity. But one needs to remove the S.H.T. assumption.

Influence of Christ's paper: local $T(b)$ theorems with L^∞ testing conditions for analytic capacity

1) David (1998). Proof of Vitushkin conjecture: Unrectifiable 1-sets have vanishing analytic capacity. needed a local T(b) theorem on S.N.H.T

2) Nazarov-Treil-Volberg (2003) new proof of David's T(b) theorem on S.N.H.T. using random dyadic grids.

3) Tolsa (2003) : proof of Painlevé's conjecture on analytic capacity.

Influence of Christ's and Semmes papers: Kato quare root problem.

Upshot: prove a square function estimate

$$\int_0^\infty \|(1 - t^2 \mathrm{div} A\nabla)^{-1} t \mathrm{div} f\|_2^2 \frac{dt}{t} \lesssim \|f\|_2^2$$

$L = -\mathrm{div} A\nabla$ elliptic operator with bounded measurable complex coefficients on $\mathbf{R}^n$. The square function estimate will ultimately imply the Kato's conjecture:

$$\|L^{1/2} f\|_2 \sim \|\nabla f\|_2,$$

which gives the domain of $L^{1/2}$.

A.-Tchamitchian (98): Kato's conjecture follows from local $T(b)$ criterion with L^2 testing conditions.

Proof of Kato's conjecture in 2002 in a series of paper (involving A., Hofmann, Lacey, Lewis, McIntosh, Tchamtichian): The square function estimate holds for all $-\mathrm{div} A\nabla$.

The method opened two different directions of investigations: motivations are links to PDE's and geometry.

1) Functional analytic approach of square functions

2) study of local T(b) theorem for singular integral and square functions with L^p testing conditions.

Functional analytic approach of square functions

Here the impetus came from a paper of Axelsson, Keith, McIntosh (2006) simplified in 2008 (A., Axelsson, McIntosh). Upshot: prove square function estimates

$$\int_0^\infty \|(1+t^2DBDB)^{-1}tDBf\|_2^2\,\frac{dt}{t} \lesssim \|f\|_2^2$$

in some algebraic context where D is a self-adjoint matrix of first order operators and B is a an accretive multiplication operator on a vector-valued L^2 space. This gives rise to the holomorphic functional calculus of DB on this L^2 space. The Kato problem is seen by specializing.

2 Applications:

1) A.-Axelsson, McIntosh, 2009) There is a choice of D and B which allows to study BVP for elliptic equations $-\mathrm{div}_{x,t} A(x) \nabla_{x,t} u = 0$ on upper half space $\mathbf{R}^n \times (0, \infty)$. One can reobtain Jerison-Kenig and Kenig-Pipher results on L^2 solvability, extend them and get some others.

2) Rosén (2013). The functional calculus of this particular DB encodes the L^2 boundedness properties of the layer potentials (single and double) for real and also complex coefficients in general. In particular this subsumes the boundedness of the double layer potential on Lipschitz domains proved by CMcM.

Extensions of Euclidean Kato to other contexts.

1) Morris (2012) submanifolds with bounded second fundamental form.
2) Bandara, Ter Elst, McIntosh (2013), subelliptic operators on connected Lie groups
3) Bandara, McIntosh (2016), on Riemannian manifolds with Ricci lower bounds and bounded geometry
4) Bandara, McIntosh (2016) on vector bundles with Generalized Bounded Geometry.
5) Egert-Haller-Dintelmann-Tolksdorf (2016) Kato with mixed boundary conditions (Lions' problem)
6) Cruz Uribe-Rios (2015) degenerate elliptic operators with A_2 degeneracy
7) A.-Rosén-Rule (2015) BVP for degenerate elliptic equations.

Parabolic Kato problem

Consider $\mathcal{H} = \partial_t - \mathrm{div}_x A(x,t)\nabla_x$ on $\mathbf{R}^{n+1}$

Theorem (A.Egert, Nyström (2017))

$\mathcal{H}$ can be defined as a maximal-accretive operator on $L^2(\mathbf{R}^{n+1})$ and $\|\mathcal{H}^{1/2} f\|_2^2 \sim \|\nabla_x f\|_2^2 + \|D_t^{1/2} f\|_2^2$.

There is a square function reduction using a “first order” operator “DB” to prove

$$\int \|(1+\lambda^2 DBDB)^{-m}\lambda DB\varphi\|^2 \frac{\lambda}{\lambda} \lesssim \|\varphi\|_2^2$$

This is again a T(b) argument. New difficulties: D is no longer self-adjoint and D contains half-order derivatives in time: non-local estimates.

Study of local $T(b)$ theorems with L^p testing conditions.
Impetus from A.-Hofmann-Muscalu-Tao-Thiele (2002)

Perfect dyadic CZO:

$$|K(x,y)| \lesssim \frac{1}{|x-y|}$$

and the *perfect dyadic Calderón-Zygmund conditions*

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| = 0$$

whenever $x, x' \in I$ and $y \in J$ fo some disjoint dyadic cubes I and J. Equivalently, K is constant on all dyadic rectangles not touching the diagonal.

Theorem (Dyadic local $T(b)$ theorem)

Let T be a perfect CZO, and suppose that for each dyadic cube P we can find functions b_P^1, b_P^2 supported on P obeying the normalization

$$[b_P^1]_P = [b_P^2]_P = 1 \tag{1}$$

and the bounds

$$\int_P |b_P^1|^2 + |Tb_P^1|^2 + |b_P^2|^2 + |T^* b_P^2|^2 \lesssim |P|. \tag{2}$$

Then T is bounded on L^2.

(b_P^1) and (b_P^2) are called accretive systems (adapted to T).
Note: L^2 testing conditions: compare to Christ's result.

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Then T is bounded on L^2.

Strategy: Chrsit's argument does not work. Estimate $\|T(1_P)\|_{L^1(P)}$ and $\|T^*(1_P)\|_{L^1(P)}$ and apply local $T(1)$ theorem.

One can relax (2), to

$$\int_P |b_P^1|^p + |Tb_P^1|^{q'} + |b_P^2|^q + |T^* b_P^2|^{p'} \lesssim |P|$$

with $1 < p, q < \infty$.

AHMTT: "It is also straightforward to generalize [this result] to Calderón-Zygmund operators which do not obey the perfect dyadic cancellation condition, and instead obey a more classical cancellation condition such as

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \lesssim 1/|x-y|^2."$$

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Problem: control on "boundary" integrals $\int Tb^1_P(x) b^2_{P'}(x)\, dx$ whenever P, P' are adjacent disjoint dyadic intervals?

Hofmann '08 (Icm address). Thm true if

$$\int_P |b_P^1|^{2+\varepsilon} + |Tb_P^1|^2 + |b_P^2|^{2+\varepsilon} + |T^* b_P^2|^2 \lesssim |P|.$$

Motivation: layer potentials S_t for elliptic equations: $Lu = -\mathrm{div} A \nabla u$ on upper half space. For $t \neq 0$,

$$Tf(x) := \partial_t S_t f(x) = \int \partial_t \Gamma(x, t, y, 0) f(y)\, dy, \ x \in \mathbf{R}^n.$$

A.-Alfonseca-Axelsson-Hofmann-Kim (2011): If A is real symmetric then T is bounded on $L^2(\mathbf{R}^n)$, uniformly in t. Consequently, S_t is bounded from L^2 to $\dot{H}^1$. (Compare Rosén)

Idea: use local T(b) above with $b_P^1 = b_P^2 = c|P| 1_P k^{(c_P, -\ell(P))}$, where k^X is the Poisson kernel at pole X, c_P is the center of P. Size estimates come from the solvability of the Dirichlet problem in L^2 proved by Jerison-Kenig. Mean condition is from the CFMS estimate.

Hofmann (2008 Icm address). Thm true for CZ0 if

$$\int_P |b_P^1|^{2+\varepsilon} + |Tb_P^1|^2 + |b_P^2|^{2+\varepsilon} + |T^* b_P^2|^2 \lesssim |P|$$

A.-Yang (2009) True in the subdual case $1/p + 1/q \geq 1$ (includes $p = q = 2$).
Idea: use the Beylkin-Coifman-Rokhlin algorithm to show any CZO is a perfect CZO plus an L^p bounded operator (any $1 < p < \infty$).
Subdual condition arises from the fact that $\int Tb_P^1(x) b_{P'}^2(x)\, dx$ can be handled by Hölder's inequality.

Hofmann (2008 Icm address). Thm true for CZO if

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A.-Yang (2009) True in the subdual case $1/p + 1/q \leq 1$ (includes $p = q = 2$).
A.-Routin (2013) True, with additional technical condition in supdual case $1/p + 1/q > 1$.
Direct proof following AHMTT and additional technical conditions to handle the boundary integrals and compensate the impossibility of using Hardy's inequality

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Hytönen-Nazarov (2013) True for any p, q under stronger condition

$$\int_{2P} |b_P^1|^p + |Tb_P^1|^{q'} + |b_P^2|^q + |T^* b_P^2|^{p'} \lesssim |P|$$

with $1 < p, q < \infty$, and $2P$ concentric enlargement of P.
idea:

Hofmann (2008 Icm address). Thm true for CZO if

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Hytönen-Nazarov (2013) True for any p, q under stronger buffered condition

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Martikainen-Mourgoglou-Tolsa (2015) buffered condition not needed in some range of supdual exponents for antisymmetric kernels (S.N.H.T.).
Hytönen-Martikainen (2012 : S.N.H.T. version of Hofmann's result: random dyadic structures

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Lacey-Vähäkangas (2014-2016) new arguments for perfect CZ0.

Bon anniversaire, Guy.