HARMONIC ANALYSIS AND GEOMETRIC MEASURE THEORY

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Abstracts

Pascal Auscher (Université Paris-Sud)

30 years of T(b) theorems

The T(b) theorem proved 30 years ago by David, Journé and Semmes, following a first result of McIntosh and Meyer, has proved to be a powerful and versatile tool for a number of applications. We will discuss history and main applications including recent ones.

David Bate (University of Helsinki)

Rectifiability in metric and Banach spaces via arbitrarily small perturbations

This talk presents a new characterisation of compact, purely d-unrectifiable metric spaces X with finite d-dimensional Hausdorff measure. It is shown that for any $\epsilon > 0$ there exists a Lipschitz $\sigma\colon X \to \sigma(X)$ with $\mathcal{H}^d(\sigma(X)) < \epsilon$ that perturbs the distances between points in X by at most ϵ. The key point of our construction is that the Lipschitz constant of such a perturbation is independent of ϵ, which allows us to prove the converse statement. In fact, we construct a perturbation $\sigma\colon X \subset B \to B$ for any Banach space B with an unconditional basis. The result for metric spaces is obtained using a suitable embedding into such a B (namely c_0). We obtain improved results if B is a Hilbert space or $d = 1$.

Cristina Benea (Université de Nantes)

Localization in time-frequency analysis

We present a “localization” principle for (multi-)linear operators which admit a wave packet decomposition. This allows us to obtain, for the operator involved, multiple vector-valued extensions. Changing the direction of the stopping time, we deduce from the local estimate a sparse domination (and hence also weighted estimates). Examples include the bilinear Hilbert transform (and generalizations to n-linear multipliers with symbol singular along a k-dimensional subspace, where $k < (n + 1/2)$), paraproducts, Carleson and variational Carleson operators. This is joint work with Camil Muscalu.

Ronald Coifman (Yale University)

Geometry and Harmonic Analysis of Data Clouds

Our goal is to describe new challenges and ideas arising in the analysis of subsets in high dimensional Euclidean spaces. Our aim is to organize their natural internal geometries as they relate to various analytical tasks. In the context of Harmonic Analysis we have seen such approaches, in the study of the restriction of Calderon Zygmund, or potential operators to a subset or to a discrete point cloud, relating the internal geometry to the properties of the operator. In data analysis, internal geometries are revealed through different restriction operators, such as the restriction of the projection on band limited functions, whose Harmonic Analysis, reveals internal structures. We will illustrate by example various analytic methods designed to build natural metrics and geometries on subsets in finite dimension, subsets which are characterized by a family of functions viewed as sensors on the subset, which are can be used to define perceptual distances between point cloud.

José Manuel Conde Alonso (Brown University)

An application of David-Mattila cubes to non-homogeneous Calderón-Zygmund theory

The so-called David-Mattila cubes play the role of the dyadic cubes when the underlying measure μ on $\mathbb{R}^d$ is not the Lebesgue measure. In this talk, we shall illustrate their importance by reviewing some basic results in non-homogeneous Calderón-Zygmund theory. Our approach is simpler than the original one in this context, and new results include extensions to noncommutative L^p spaces and interpolation between L^p and BMO spaces adapted to the measure μ. Based on joint work with Javier Parcet.

Camillo De Lellis (Universität Zürich)

Ill-posedness for Leray solutions of the ipodissipative Navier-Stokes equations

In a joint work with Maria Colombo and Luigi De Rosa we consider the Cauchy problem for the ipodissipative Navier-Stokes equations, where the classical Laplacian $-\Delta$ is substited by a fractional Laplacian $(-\Delta)^\alpha$. Although a classical Hopf approach via a Galerkin approximation shows that there is enough compactness to construct global weak solutions satisfying the energy inequality à la Leray, we show that such solutions are not unique when α is small enough and the initial data are not regular. Our proof is a simple adapation of the methods introduced by László Székelyhidi and myself for the Euler equations. The methods apply for $\alpha < \frac{1}{2}$, but in order to show that they produce Leray solutions some more care is needed and in particular we must take smaller exponents.

Joseph Feneuil (University of Minnesota)

The Dirichlet problem for sets with higher co-dimensional boundaries.

Let $\Gamma \subset \mathbb{R}^n$ be a set of dimension $d < n-1$ and $\Omega = \mathbb{R}^n \setminus \Gamma$ be its complement. We develop an elliptic theory adapted to Ω, where "elliptic" operators $L = -\mathrm{div} A \nabla$ are degenerate operators that satisfy

$$C^{-1}\mathrm{dist}(x,\Gamma)^{n-d-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq C\mathrm{dist}(x,\Gamma)^{n-d-1}|\xi|^2.$$

Choose $\Gamma = \mathbb{R}^d$ and write (x,t) for a point in $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d$, so that $\mathrm{dist}(x,\Gamma) = |t|$. Whenever $L = -\mathrm{div}|t|^{d+1-n}\mathcal{A}\nabla$ and $\mathcal{A}$ is close - in terms of Carleson measures - to the identity, we solve the Dirichlet problem (D_p) for any $p \in (1, +\infty)$. This is a joint work with Guy David, Svitlana Mayboroda and Zihui Zhao.

Sandrine Grellier (Université d'Orléans)

An inverse spectral theorem on compact Hankel operators linked to the dynamic of some half wave equation

We establish a one-to-one correspondence between the symbol of a compact Hankel operator on the unit disc and the sequence of its singular values with some additional spectral data. This one-to-one correspondence allows to solve explicitely the cubic Szegö equation, a toy model of totally non-dispersive evolution equations. It allows to exhibit some interesting phenomena on the Szegö dynamic : wave turbulence, almost periodicity, analyticity... From joint works with P. Gérard.

Stéphane Jaffard (UPEC)

Wavelets on the hunt for gravitational waves

On September 14th 2015, LIGO (Laser Interferometer Gravitational-Wave Observatory) performed the first detection of a gravitational wave generated by the coalescence of two black holes. The signal processing algorithm which allowed this detection uses in a crucial way a decomposition of the signal on "Wilson bases" (which constitute an orthonormal "time-frequency" decomposition). We will mention the origin of such bases, which goes back to the seminal work of Gabor in the 50s, and was made more precise by K. Wilson at the beginning of the 80s (motivated by renormalization theory). We will then show why such bases are particularly well adapted to gravitational waves, and which choices were made in the detection algorithm. Finally, we will mention the perspectives opened by this new type of astronomy which, for the first time, is not based on light or electromagnetic waves detection, and the role that such bases, or variants are expected to play in it.

David Jerison (MIT)

Localization of eigenfunctions via an effective potential

We discuss joint work with Doug Arnold, Guy David, Marcel Filoche and Svitlana Mayboroda. Consider the Neumann boundary value problem for the operator $L = \mathrm{div} A\nabla + V$ on a Lipschitz domain Ω and, more generally, on manifolds with and without boundary. The eigenfunctions of L are often localized, as a result of disorder of the potential V, the matrix of coefficients A, irregularities of the boundary, or all of the above. In earlier work, Filoche and Mayboroda introduced the function u solving $Lu = 1$, and showed numerically that it strongly reflects this localization. In this talk, we deepen the connection between the eigenfunctions and this *landscape* function u by proving that its reciprocal $1/u$ acts as an *effective potential*. The effective potential governs the exponential decay of the eigenfunctions of the system and delivers information on the distribution of eigenvalues near the bottom of the spectrum.

Pertti Mattila (University of Helsinki)

Hausdorff dimension of projections and intersections

If the Hausdorff dimension of a subset of a Euclidean space is known, what can we say about the dimensions of its projections, intersections with planes and intersections with rotated and translated copies of some other general set ? The basic generic results have been known for a long time but there are several recent related results, in particular on the size of the sets of exceptional projections, planes and rotations.

Svitlana Mayboroda (University of Minnesota)

Harmonic measure for lower dimensional sets

Harmonic measure and harmonic functions more generally play a unique role in geometric measure theory : boundedness of the harmonic Riesz transform is equivalent to uniform rectifiability of sets, so is the boundedness of the harmonic square function, to mention only a few results. Unfortunately, the concept of a harmonic measure is intrinsically restricted to co-dimension 1. In this talk, we introduce a new notion of a “harmonic” measure, associated to a suitable degenerate PDE, which serves the higher co-dimensions. We discuss its basic properties and interplay between absolute continuity of our harmonic measure with respect to the Hausdorff measure, square function estimates, and rectifiability of a lower-dimensional set. This is joint work with Guy David, Max Engelstein, and Joseph Feneuil.

Jean-Michel Morel (ENS Paris-Saclay)

Visual perception, geometry and the structure of images viewed as 2D functions

It has been long admitted that the structure of 2D functions is described by local characteristics, for example a local Fourier or wavelet expansion, or more trivially a Taylor expansion of some order. The regularity of the function for example would be encoded in the decay of a local expansion or in the boundedness of some energy controlling globally by some integrals the local regularity, like a Sobolev or Besov space or BV.

But recent progress in image processing has gone away from this model inherited from calculus and geometric measure theory. It is now focused on the geometry of the space of patches. Patches are simply 8 * 8 or 10 * 10 square images cropped from the image itself. Image characteristics seem to be much better described in the patch space (of dimension say 64 or 100). We can view this as a dimension reduction (from the space of images that would have dimension several million) or instead dimension lifting from a dimension 2 (or 3) to a much higher dimension. So the question is : can we explore the patch space and find some evidence about its geometry ?

This is an experimental question, because we can now store and analyze sets containing, say 10^{10} patches. But, still, this is a priori a far too small number to sample a space in such a high dimension. Fortunately, it so happens that the patch space is itself regular. Only, the ways to encode its regularity are still the object of debate. I'll show on examples that even the sparse information that we have gathered on the patch space seems to resolve some long open problems of image perception, like image restoration or the detection of anomalies.

Raanan Schul (Stony Brook University)

Rectifiability of measures

We say that a locally finite Borel measure μ on $\mathbb{R}^n$ is countably-k-rectifiable if there are countably many Lipschitz maps (with domain $[0,1]^k$) whose image capture all the mass of μ, i.e. there are $f_i : [0,1]^k \to \mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus \cup_i f_i([0,1]^k) = 0$. When is a given measure rectifiable ? Note that μ may be singular to k-dimensional Hausdorff measure. We will discuss joint works with Matthew Badger and with Jonas Azzam as well as a collection of open questions.

Terence Tao (UCLA)

An integration approach to the Toeplitz square peg problem

The Toeplitz square peg problem asks if every simple closed curve in the plane inscribes a square. This is known for sufficiently regular curves (e.g. polygons), but is open in general. We show that the answer is affirmative if the curve consists of two Lipschitz graphs of constant less than 1 using an integration by parts technique, and give some related problems which look more tractable.

Xavier Tolsa (ICREA/Universitat Autònoma de Barcelona)

Harmonic and elliptic measures, and uniform rectifiability

In this talk I will explain a recent characterization of uniform rectifiability in terms of Carleson estimates for bounded harmonic functions. This is based on a corona decomposition involving harmonic measure which also characterizes uniform rectifiability. The analogous problem for functions satisfying an elliptic PDE in divergence form is much more delicate. When the associated matrix is non-symmetric we need to use an appropriate version of the famous Alt-Caffarelli-Friedman monotonicity formula and a new criterion for uniform rectifiability of topological type. This a joint work with J. Azzam, J. Garnett and M. Mourgoglou.

Tatiana Toro (University of Washington)

Parametrizing with Guy

Over the past 20 years we have been interested in finding *good* parameterizations for sets that are well approximated by *nice* sets. In this talk we will discuss the meanings of *good* and *nice*. We will recall some the results from the past and present new results concerning the regularity of sets that can be well approximated by Lipschitz graphs.

Joan Verdera (Universitat Autònoma de Barcelona)

Two theorems on boundary regularity of vortex patches

The vorticity form of the planar Euler equation says that vorticity is constant along particle trajectories. A vortex patch is a weak solution of the vorticity equation with initial condition the characteristic function of a domain D_0. Thus at time t vorticity is the characteristic function of a domain D_t. Simulations show that the evolution of D_t is extremely complicated. In spite of this general fact there are some special domains, called V-states, whose evolution is just rotation around the center of mass with constant angular velocity. Ellipses are examples of V-states. I will discuss Burbea's proof of existence of other V-states and then I will discuss the smoothness of their boundary (joint work with Hmidi and Mateu). For general vortex patches, if the initial condition is the characteristic function of a domain with boundary of class $C^{1+\gamma}$, then the boundary of D_t conserves the regularity for all times (Chemin's theorem). I will mention a similar result for the aggregation equation in higher dimensions (joint work with Bertozzi, Garnett and Laurent).

Alexander Volberg (Michigan State University)
Hamming cubes and martingales : some sort of duality

We will explain a new unified approach to classical isoperimetric inequalities with Gaussian measure (Gaussian isoperimetric inequality, log-Sobolev inequality, Beckners inequality) that allows to create also new isoperimetric inequalities with Gaussian measure. The method involves solving a certain Monge–Ampère equation by the exterior differential systems approach of Bryant–Griffiths. Then we explain that underlying this new method there is a series of inequalities on Hamming cube which are considerably more difficult to obtain. In some situation we show that we can find the Bellman function of “completely unrelated” martingale problem, apply the carefully chosen Legendre transform to it and then this Legendre transform solves the corresponding extremal problem on Hamming cube. Extremal problems on Hamming cubes are closely related to “big data” paradigm, so one can consider these talk (and papers on which they are based) as our attempt to become an applied mathematician.