GEOMETRY OF QUANTUM ENTANGLEMENT

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On the **set of quantum states**:

What picture does one see, looking at a physical theory from a distance, so that the details disappear?

Since quantum mechanics is a statistical theory, the most universal picture which remains after the details are forgotten is that of a **convex set**.

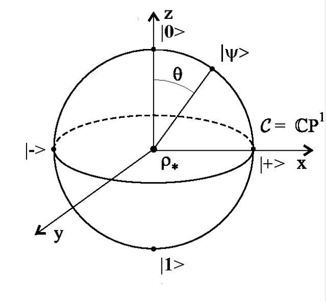
Bogdan Mielnik, (1981)

Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Qubit = quantum bit; $N = 2$, $\langle\psi|\psi\rangle = 1$, $|\psi\rangle \sim e^{i\alpha}|\psi\rangle$

$$|\psi\rangle = \cos\frac{\vartheta}{2}|1\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|0\rangle$$

$\mathbb{C}P^1 = S^2$, Bloch sphere of $N = 2$ pure states



Key feature: Quantum Superposition

define **pure** states $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Unitary evolution in projective space

Space of pure states for an arbitrary (finite) N:

a **complex projective space** $\mathbb{C}P^{N-1}$ of $2N-2$ real dimensions.

Fubini-Study distance in $\mathbb{C}P^{N-1}$

$$D_{FS}\big(|\psi\rangle, |\varphi\rangle\big) \;:=\; \arccos \big|\langle\psi|\varphi\rangle\big|$$

Unitary evolution (determined by a **Hamiltonian** H)

Let $U = \exp(iHt)$. Then $|\psi'\rangle = U|\psi\rangle$.
Since $|\langle\psi|\varphi\rangle|^2 = |\langle\psi|U^\dagger U|\varphi\rangle|^2$ any unitary evolution is an **isometry**
(with respect to any **standard** distance !)

Quantum Chaos: what happends for large N?

How an **isometry** may lead to a classically chaotic dynamics?
The limits $t \to \infty$ and $N \to \infty$ do not commute.

A **'classical'** (non unitary-invariant) distance

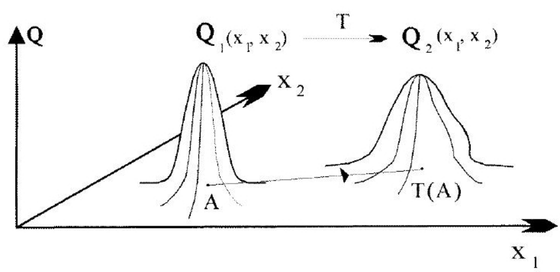
Coherent state $|x\rangle$ defined for any point x of the classical space X

e.g. Displacement operator coherent state, $|x\rangle = T_x|0\rangle$,

Probability distribution: **Husimi distribution (Q-function)**

$Q_\rho(x) = \langle x|\rho|x\rangle$, for a pure state $\rho = |\psi\rangle\langle\psi|$ one has $Q_\psi(x) = |\langle x|\psi\rangle|^2$

Monge distance between quantum states: $D_M(\rho,\sigma) = D_M(Q_\rho, Q_\sigma)$



For two coherent states it gives the **classical distance**,
$D_M(|x\rangle,|y\rangle) = D_M(Q_x, Q_y) = |x-y|$. **W. Słomczyński, K.Ż (1998)**

The set $\mathcal{M}_N$ of mixed states (density matrices) of size N

definition

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N; \rho = \rho^\dagger, \rho \geq 0, \mathrm{Tr}\rho = 1\}$$

Distances in the set of quantum states

a) **Hilbert–Schmidt** distance, $D_{\mathrm{HS}}(\rho, \sigma) := [\mathrm{Tr}(\rho - \sigma)^2]^{1/2}$

b) **trace** distance, $D_{\mathrm{tr}}(\rho, \sigma) := \frac{1}{2}\mathrm{Tr}|\rho - \sigma|$

c) **Bures** distance, $D_{\mathrm{B}}(\rho, \sigma) := \left(2[1 - \sqrt{F(\rho, \sigma)}]\right)^{1/2}$,

where fidelity between two states reads (Uhlmann '76, Jozsa '94),

$$F(\rho, \sigma) := [\mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}|]^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right)^2 .$$

Warning! An alternative definition of fidelity (without the square!) is sometimes used, $F' = \sqrt{F} = \mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}|$

Metrics in the space $\mathcal{M}_N$ of quantum states

properties

a) **Riemannian metric** - related to a geodesic distance
b) **monotone metric** - the corresponding monotone distance $D_{\rm mon}$ does not grow under the action of any **quantum operation** Φ,

$$D_{\rm mon}\big(\rho,\sigma\big) \geq D_{\rm mon}\big(\Phi(\rho),\Phi(\sigma)\big) \tag{1}$$

Metric	Hilbert–Schmidt	Trace	**Bures**
Is it Riemannian ?	Yes	No	**Yes**
Is it monotone ?	No	Yes	**Yes**

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Is it Riemannian ?	Yes	No	**Yes**
Is it monotone ?	No	Yes	**Yes**

Theorem of **Morozova and Chentsov** '90

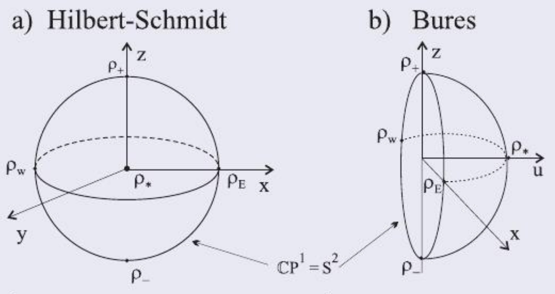
There exist infinitely many monotone Riemannian metrics on $\mathcal{M}_N$...

Geometry of the set Quantum States **depends** on the **metric** used:

Example: $N = 2$ – quantum states for a one–qubit system

$\mathcal{M}_2 \equiv B_3$ – **Bloch ball** for **Hilbert–Schmidt** (Euclidean) metric

$\mathcal{M}_2 \equiv \frac{1}{2}\, S^3$ – **Uhlmann hemisphere** for **Bures** metric



(one dimension supressed)


beeline
beeline

Otton Nikodym & Stefan Banach,

talking at a bench in Planty Garden, **Cracow, summer 1916**

Mixed quantum states = density matrices

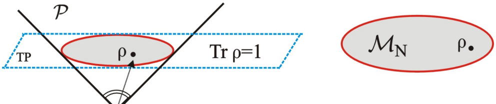
Set $\mathcal{M}_N$ of all mixed states of size N

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N;\ \rho = \rho^\dagger,\ \rho \geq 0,\ \mathrm{Tr}\rho = 1\}$$

Example: $N = 2$, **One–qubit** states + **Hilbert-Schmidt** metric:
$\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ - **Bloch ball** with all pure states at the boundary

The set $\mathcal{M}_N$ is compact and **convex**: $\rho = \sum_i a_i |\psi_i\rangle\langle\psi_i|$, where $a_i \geq 0$ and $\sum_i a_i = 1$.

It has $N^2 - 1$ real dimensions, $\mathcal{M}_N \subset \mathbb{R}^{N^2-1}$.



a) convex cone b) compact set

What the set of all $N = 3$ mixed states looks like?

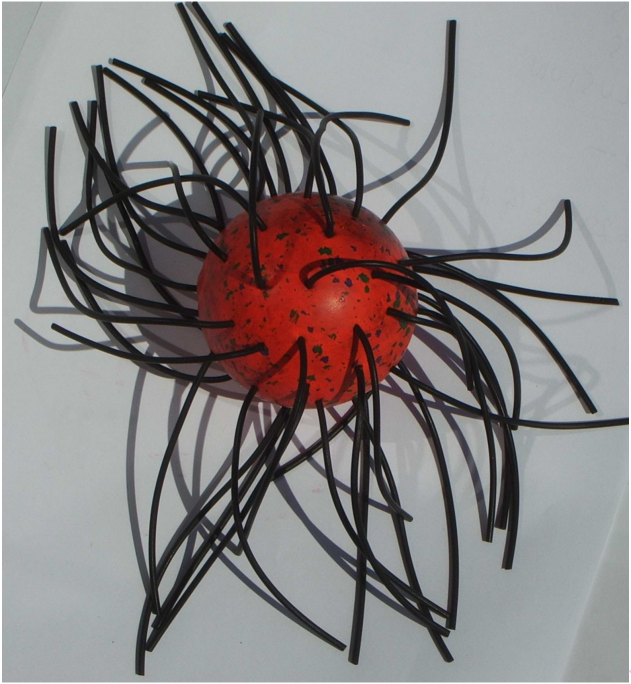
An 8–**dimensional convex** set with only 4–dimensional subset of pure (**extremal**) states, which belong to its 7–dim boundary

The set $\mathcal{M}_N$ of quantum mixed states:

What it looks like for **(for $N \geq 3$)**

?

An apophatic approach :

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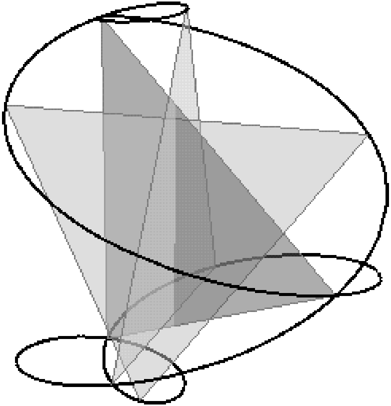
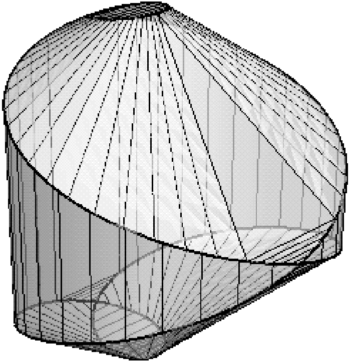
KOORN & CO.
APPEL
THEE
20
Tea Bags
APPEL THEE


BJ





rotated edges of an **equilateral triangle** and its **convex hull**

Vistula river and **Wawel castle** in **Cracow**

The set $\mathcal{M}_N$ of quantum mixed states for $N \geq 3$

A constructive approach:

Analysis of its structure by studying its 2D and 3D cross-sections and projections.

The same tools are useful to investigate the structure of the subsets of $\mathcal{M}_N$, namely sets of

a) **separable** states

and

b) **maximally entangled** states.

Projections of the set $\mathcal{M}_N$ onto a plane

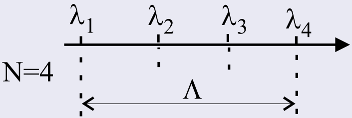
Mathematical tool: **Numerical Range** $\Lambda(A)$ of an operator

For any operator A acting on $\mathcal{H}_N$ one defines its **Numerical Range** (**Wertevorrat**) as a subset of the complex plane defined by:

$$\Lambda(A) = \{\langle x|A|x\rangle : \ |x\rangle \in \mathcal{H}^N, \ \langle x|x\rangle = 1\}. \qquad (2)$$

Hermitian case, $A = A^\dagger$

For any hermitian A with spectrum $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ its **numerical range** forms an interval: the set of all possible **expectation values** of the **observable** A among any normalized pure states, $\Lambda(A) = [\lambda_1, \lambda_N]$.



Numerical range and its properties

Compactness

$\Lambda(A)$ is a **compact** subset of $\mathbb{C}$.

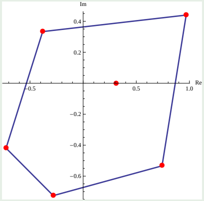
Convexity: Hausdorf-Toeplitz theorem

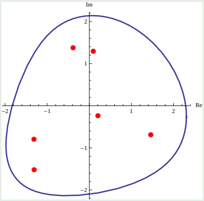
- $\Lambda(A)$ is a **convex** subset of $\mathbb{C}$.

Example

Numerical range for random matrices of order $N = 6$

a) normal, b) generic (non-normal)



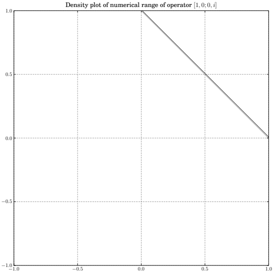


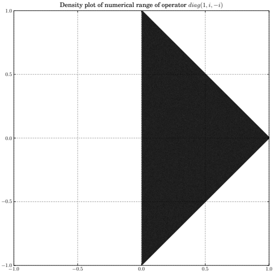
Normal case: a projection of the classical simplex...

Normal matrix, $([A, A^*] = 0)$, of size $N = 2$ with spectrum $\{\lambda_1, \lambda_2\}$

Numerical range $\Lambda(A)$ forms the **interval** $[\lambda_1, \lambda_2]$ on the complex plane,

Examples for diagonal matrices A of size **two** and **three**





Normal matrices of order $N = 3$ with spectrum $\{\lambda_1, \lambda_2, \lambda_3\}$

Numerical range $\Lambda(A)$ forms the **triangle** $\Delta(\lambda_1, \lambda_2, \lambda_3)$ on the complex plane.

Numerical range for matrices of order $N = 2$

with spectrum $\{\lambda_1, \lambda_2\}$.

a) normal matrix, $([A, A^*] = 0)$, $\Rightarrow \Lambda(A) =$ closed interval $[\lambda_1, \lambda_2]$

b) not normal matrix A $\Rightarrow \Lambda(A) =$ elliptical disk with λ_1, λ_2 as focal points and minor axis, $d = \sqrt{\mathrm{Tr} AA^* - |\lambda_1|^2 - |\lambda_2|^2}$

(**Murnaghan, 1932; Li, 1996**).

Example: The **Jordan matrix** $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Its numerical range forms a circular disk, $\Lambda(J) = D(0, r = 1/2)$.

The set of $N = 2$ pure quantum states

A projection of the **Bloch sphere** $S^2 = \mathbb{C}P^1$ onto a plane

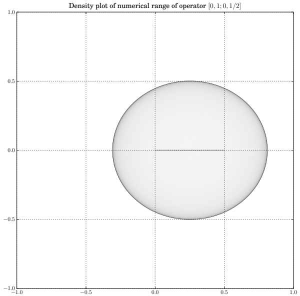
forms an **ellipse**,

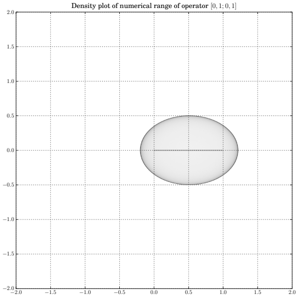
(which could be degenerated to an interval).

Numerical range for $N = 2$

Non–normal matrices of size $N = 2$

Numerical range $\Lambda(A)$ forms an (eliptical) disk on the complex plane: **projection** of (empty!) **Bloch sphere**, $S^2 = \mathbb{C}P^1$ on the complex plane.





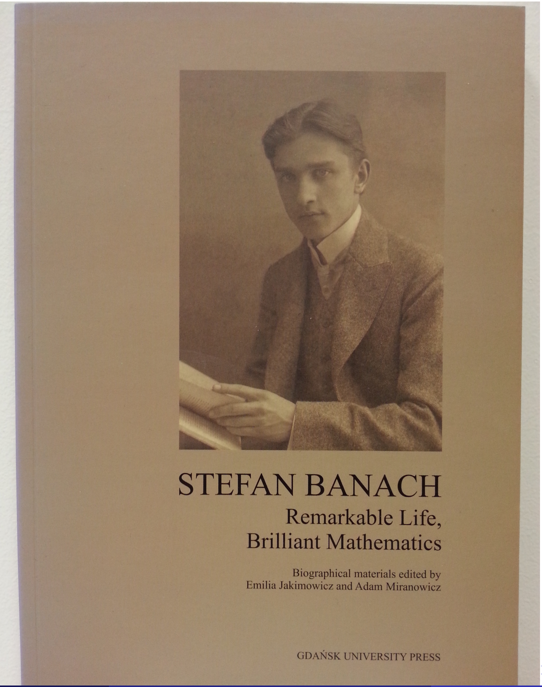
Wawel castle in Cracow

Danuta & Krzysztof Ciesielscy theorem:

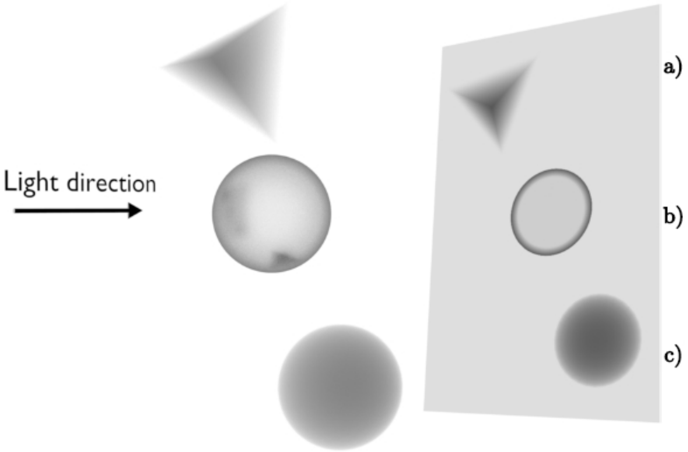
Ciesielscy theorem: With probability $1 - \epsilon$ the bench **Banach** talked to **Nikodym** in 1916 was localized in η-neighbourhood of the **red arrow**.

Plate commemorating the discussion between

Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)


STEFAN BANACH
Remarkable Life,
Brilliant Mathematics
Biographical materials edited by
Emilia Jakimowicz and Adam Miranowicz
GDAŃSK UNIVERSITY PRESS

Shadows of three dimensional objects...



Quantum States and Numerical Range

Classical States & normal matrices

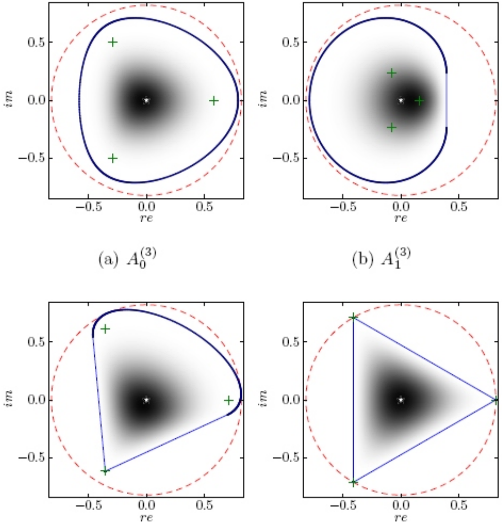
Proposition 1.*Let $\mathcal{C}_N$ denote the set of* **classical states** *of size N, which forms the regular simplex Δ_{N-1} in $\mathbb{R}^{N-1}$. Then the set of similar images of* **orthogonal projections** *of $\mathcal{C}_N$ on a* 2–**plane** *is equivalent to the set of all possible numerical ranges $\Lambda(A)$ of all* **normal matrices** *A of order N*

(such that $AA^* = A^*A$).

Quantum States & non–normal matrices

Proposition 2.*Let $\mathcal{M}_N$ denotes the set of* **quantum states** *size N embedded in $\mathbb{R}^{N^2-1}$ with respect to Euclidean geometry induced by Hilbert-Schmidt distance. Then the set of similar images of* **orthogonal projections** *$\mathcal{M}_N$ on a* 2-**plane** *is equivalent to the set of all possible numerical ranges $\Lambda(A)$ of* **all matrices** *A of order N.*

Projections of the $8D$ set $\mathcal{M}_3$ onto a $2D$ plane



belong to one of **four** different classes specified e.g. by the number s of **flat segments** of the boundary, $s = 0, 1, 2, 3$; **Keeler** *et al.* 1993

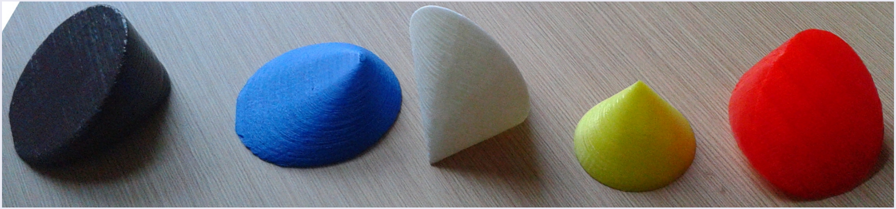
$3D$ projections = Joint Numerical Range $\Lambda(A_1, A_2, A_3)$

$N = 3$: projections of a one–qutrit mixed states into 3D

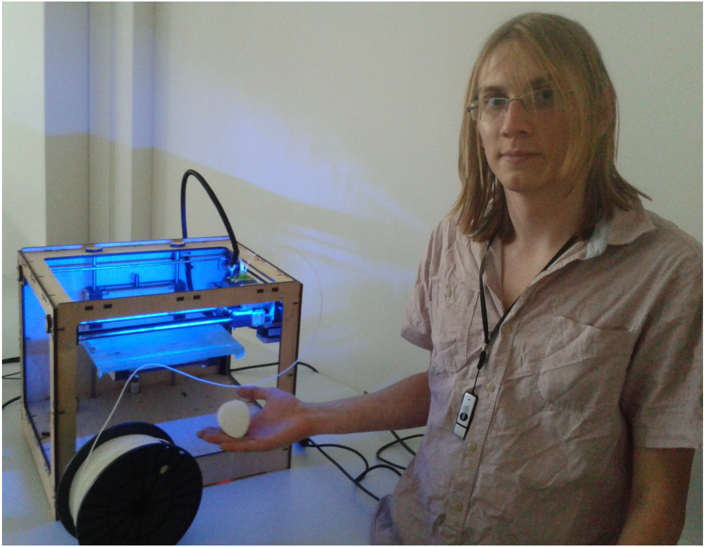
For any triple of hermitian operators $\{A_1, A_2, A_3\}$ of size $N = 3$, find their **expectation values** and define the $3D$ set called **joint numerical range**,

$$\Lambda(A_1, A_2, A_3) = \Big(\langle\psi|A_1|\psi\rangle,\ \langle\psi|A_2|\psi\rangle,\ \langle\psi|A_m|\psi\rangle\Big).$$

It gives a **projection** of the 8D set $\mathcal{M}_3$ of mixed states of a qutrit into **3D**. Examples:



There exist 9 classes of such projections of $\mathcal{M}_3$ into $3D$
(**K. Szymański, S. Weiss, K.Ż**, 2016)



Konrad Szymański producing a 3D joint numerical range

Recall the shadows on the wall of the cave of **Plato**:

we do not understand all details of the $8D$ set $\mathcal{M}_3$ of quantum states of size three, but at least we can study its 2D and 3D **projections**

How to classify possible shapes of JNR of three Hermitian matrices A_1, A_2, A_3 of size $N = 3$?

To classify the $3D$ numerical ranges for each body we count:

a) the number s of flat **segments** in the boundary
b) the number e of flat **faces (ellipses)** in the boundary

a) Ex. 6.2, $s = 0, e = 1$ b) Ex. 6.4 $s = 0, e = 3$ c) Ex. 6.5 $s = 0, e = 4$

d) Ex. 6.6, $s = 1, e = 0$ e) Ex. 6.7 $s = 1, e = 1$ f) Ex. 6.8 $s = 1, e = 2$

g) $s = e = 0$ h) $s = \infty, e = 0$ i) $s = \infty, e = 1$

Convex sets in high dimensions

How a cross-section of a 3–**cube** looks like??

Convex sets in high dimensions

How a cross-section of a 3–**cube** looks like??

How a generic cross-section of a n–**cube** looks like??

Convex sets in high dimensions

How a cross-section of a 3–**cube** looks like??

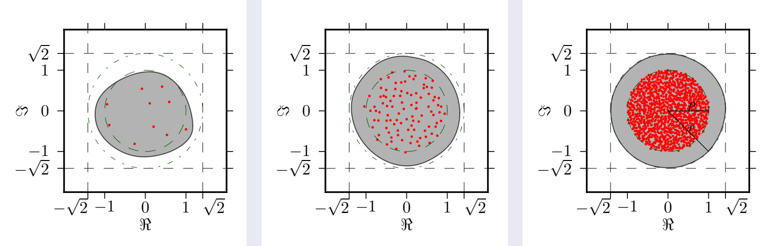
How a generic cross-section of a n–**cube** looks like??

Dvoretzky theorem
A. Dvoretzky, *Some results on convex bodies and Banach spaces* (1961)

(Under some technical assumptions)
a generic cross-section of a compact, high dimensional, convex set almost surely forms a **disk** !

Quand les cubes deviennent ronds
Webpage by **Guillaume Auburn** and **Jos Leys**

Non–hermitian **Ginibre** matrices of size N normalized as $\mathrm{Tr}\, GG^* = N$



$N = 10$ $N = 100$ $N = 1000$

Numerical range and **spectrum** of random Ginibre matrices of size N.
Note circular disk of eigenvalues of Girko and the **non–normality belt**.

Result:

In the limit $N \to \infty$ the numerical range of a **random Ginibre matrix** G converges (in the Hausdorff distance) to the disk of radius $\sqrt{2}$.

Random matrices & quantum states for a large size N

Theorem (Collins, Gawron, Litvak, KŻ, 2014)

Let $R > 0$ and let $\{X_N\}_N$ be a sequence of complex random matrices or order N such that for every $\theta \in \mathbb{R}$ with probability one

$$\lim_{N\to\infty} ||\mathrm{Re}\left(e^{i\theta}X_N\right)|| = R$$

Then with probability one

$$\lim_{N\to\infty} d_H\left(\Lambda(X_N), D(0,R)\right) = 0.$$

Example: For random **Ginibre matrix** G the operator norm $||.||$ does not depend on the phase θ and (upon the normalization used) converges to $R = \sqrt{2}$. Hence the numerical range of G *almost surely* forms a **disk** of radius R.

Relation to the **Dvoretzky theorem** !

$\Longrightarrow$ a generic projection of the set $\mathcal{M}_N$ of mixed states on a plane is close a **disk** (for large N).

Where is physics ?

What is physics ?

What is physics ?

Kick a ball !

It will stop at some point...

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It will stop at some point...

Buy an **ice cream** and wait a while..

It will melt !

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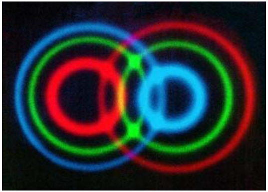
Create an **entangled state** and do nothing...



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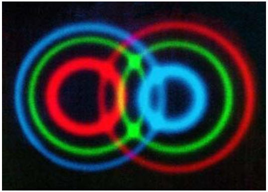


It will **decohere** to a separable one.

Composed systems & entangled states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states not of the above product form.

Two–qubit system: $N = 2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

Entanglement measures

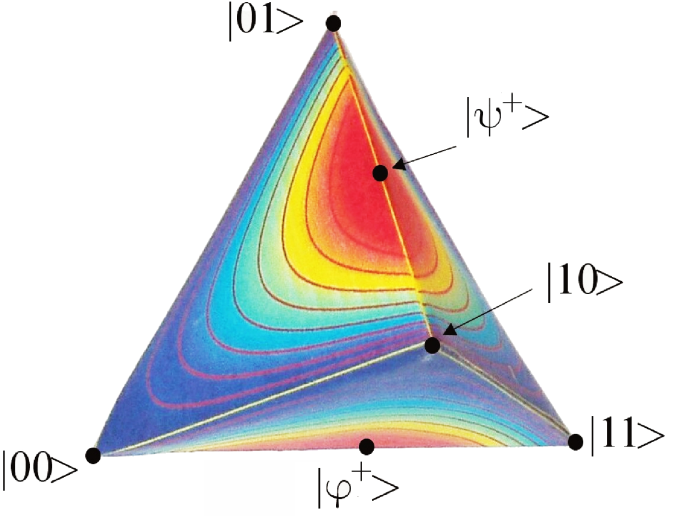
For any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ define its partial trace $\sigma = \mathrm{Tr}_B|\psi\rangle\langle\psi|$.
Definition: Entanglement **entropy** of $|\psi\rangle$ is equal to **von Neuman entropy** of the partial trace

$$E(|\psi\rangle) := \ -\mathrm{Tr}\ \sigma \ln \sigma$$

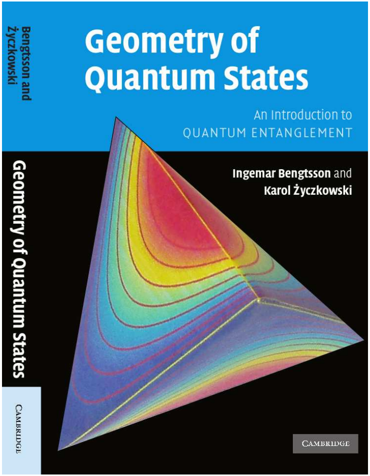
The more **mixed** partial trace, the more **entangled** initial pure state...

Entanglement of two real qubits

Entanglement **entropy** at the thetrahedron of $N = 4$ real pure states



Book published by Cambridge University Press in **2006**,



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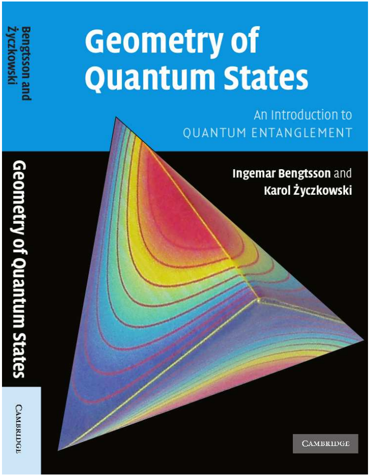
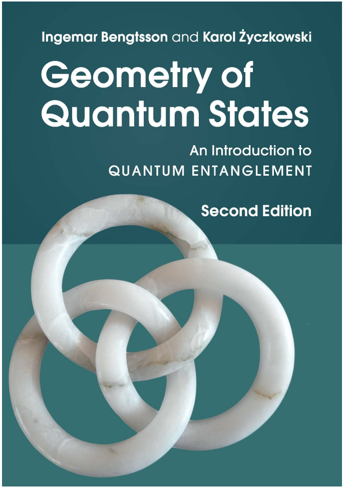




II edition (with new chapters on multipartite entanglement & discrete structures in the Hilbert space), **August 2017**

Entanglement of mixed quantum states

Mixed states of a bi-partite system, (A, B)

- **separable mixed states:** $\rho_{\rm sep} = \sum_j p_j \rho_j^A \otimes \rho_j^B \qquad (**)$
- **entangled mixed states:** all states not of the above product form.

How to find,
whether a given density matrix ρ can be written in the form (**)
and is **separable ?**

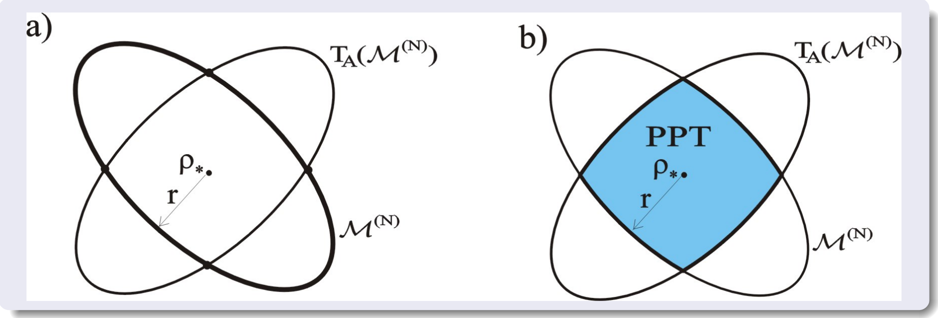
The **separability problem** is solved only for the simplest cases of 2×2 and 2×3 problems...

Positive partial transpose: Two–qubit mixed states

Peres – Horodeccy criterion (1996):

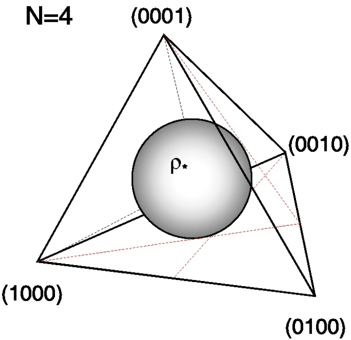
$$(\mathbb{I} \otimes T)\rho \;=\; \rho^{T_2} \geq 0 \;\Leftrightarrow\; \rho \text{ is separable.}$$

The set of separable states of two–qubit system arises as an intersection of $\mathcal{M}^{(4)}$ and its mirror image with respect to partial transposition $T_A\big(\mathcal{M}^{(4)}\big)$.



Two–qubit mixed states

The maximal ball inscribed into $\mathcal{M}^{(4)}$ of radius $r_4 = 1/\sqrt{12}$ centred at $\rho_* = \mathbb{1}/4$ is separable !

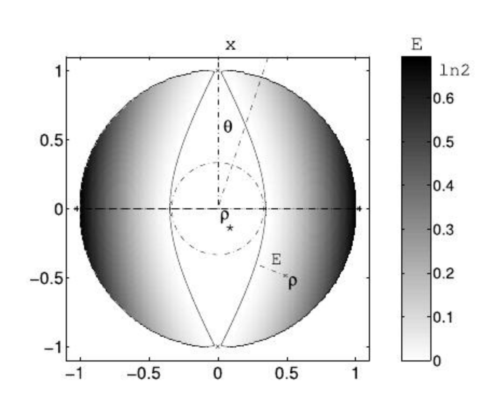


thetrahedron of eigenvalues

K. Ż, P. Horodecki, M. Lewenstein, A. Sanpera, 1998

Two–qubit mixed states

Degree of entanglement: a **distance** to the closest **separable state**



(E = **Entanglement** of formation)

K. Ż, M. Kuś, 2001

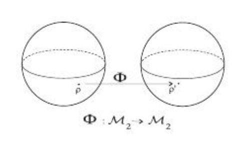
Stefan Banach sitting at his bench close to the Wawel Castle

Sculpture: Stefan Dousa

Fot. Andrzej Kobos

Quantum maps

Quantum operation: linear, completely positive trace preserving map



Enviromental form

$$\rho' = \Phi(\rho) = \mathrm{Tr}_E[U\,(\rho \otimes \omega_E)\;U^\dagger]\;.$$

where ω_E is an initial state of the environment while $UU^\dagger = \mathbb{1}$.

Kraus form

$$\rho' = \Phi(\rho) = \sum\nolimits_i A_i \rho A_i^\dagger \;,$$

where the Kraus operators satisfy $\sum_i A_i^\dagger A_i = \mathbb{1}$.

A model discrete quantum dynamics

a) unitary dynamics (rotation), $\rho' = U\rho U^\dagger$
b) decoherence (contraction), $\rho'' = \sum_i^k A_i \rho' A_i^\dagger$

Two qubit model - $N = 2 \times 2 = 4$

a) free evolution: $U = \exp(itH)$ where $H = \sigma_x \otimes \sigma_y$
(non-local unitary dynamics !)

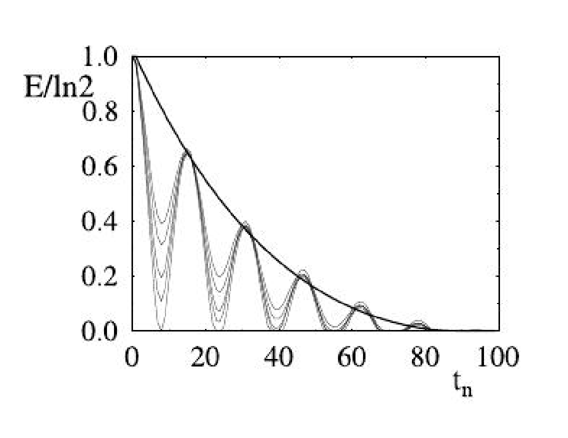
variant b1) bistochastic channel: $\Phi(\mathbb{1}/N) = \mathbb{1}/N$,
One–qubit **Pauli channel**: $k = 4$, $A_1 = \sqrt{1-\epsilon}\ \mathbb{1} \otimes \mathbb{1}$,
$A_2 = \sqrt{\epsilon/3}\ \mathbb{1} \otimes \sigma_x$, $A_3 = \sqrt{\epsilon/3}\ \mathbb{1} \otimes \sigma_y$, $A_4 = \sqrt{\epsilon/3}\ \mathbb{1} \otimes \sigma_z$.

variant b2) non bistochastic channel:
One qubit **amplitude damping channel**, *(decaying channel)*, $k = 2$,
where $A_1 = \mathbb{1} \otimes B_1$ and $A_2 = \mathbb{1} \otimes B_2$
with $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix}$

Dynamics of entanglement

Entanglement of formation E as a function of time t_n

for some initially pure states of a two–qubit system.

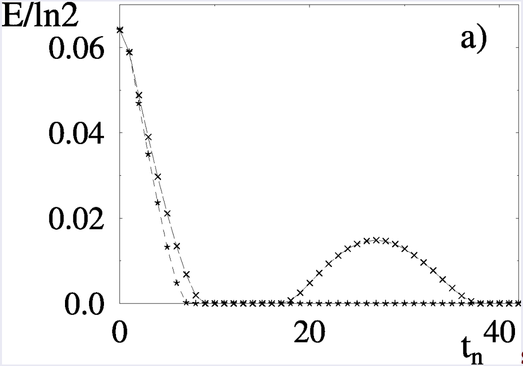


revivals of entanglement

Dynamics of entanglement

Entanglement of formation E as a function of time t_n

for some initially pure states of a two–qubit system.

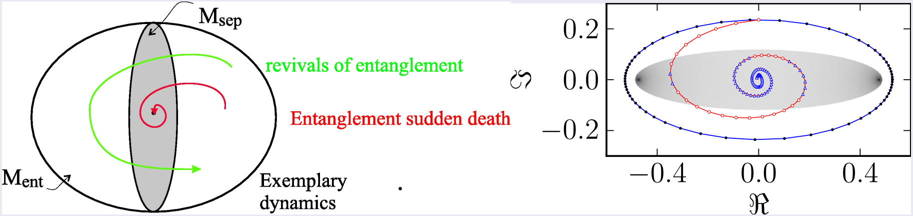


sudden death of entanglement

K. Ż, P. Horodecki, M. Horodecki, R. Horodecki, Phys. Rev. **A 2001**
the name **sudden death** coined by **Yau and Eberly**,
who reported this effect in 2003.

Trajectories of quantum dynamics on the complex plane

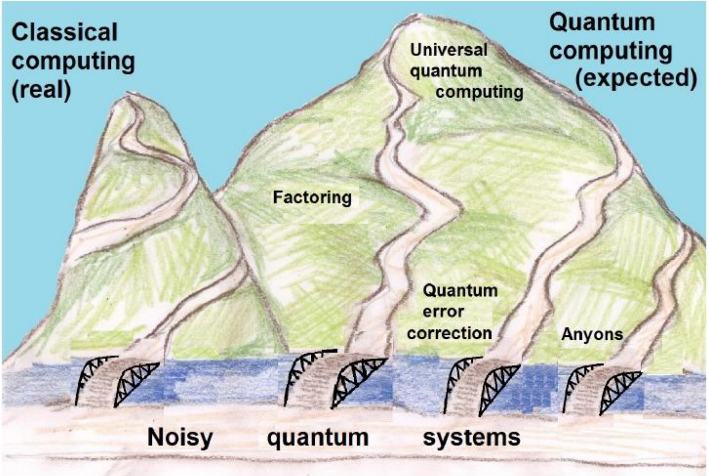
$$z(t) = \langle\psi(t)|A|\psi(t)\rangle$$



a) sketch of the problem; b) data for 2×2 system with initial separable pure state $|\psi(0)\rangle$ and suitably chosen (non–Hermitian !) operator A of size $N = 4$ **visualize** possible behaviour of quantum entanglement...

Quantum computing and coping with noise

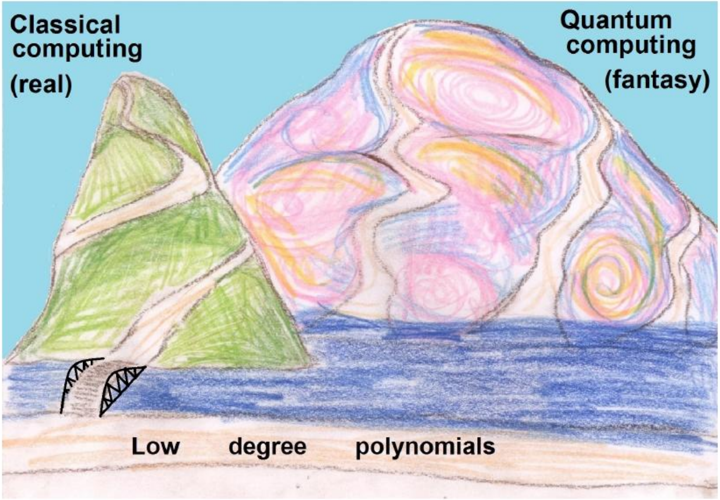
Alternative 1 (optimistic)



Gil Kalai (2016)

Quantum computing and coping with noise

Alternative 2 (pessimistic)



Gil Kalai (2016)

Concluding Remarks

- The set $\mathcal{M}_N$ of **mixed quantum states** of size N forms a scene for which the screenplays of **quantum information** processing are written.
  It is useful for any author to learn about the **structure & geometry** of the scene.
- As the set $\mathcal{M}_N$ has $N^2 - 1$ dimensions for $N \geq 3$ it is possible to investigate it by studying the **numerical range**:
  its projections onto a $2-$planes or $3-$hyper-planes.
- **Geometric approach** is usefull to study **quantum entanglement** and its dynamics. It allows one to explain the effects of **entanglement revival** and **entanglement sudden death**.
- More work is still required to understand the consequences of noise and decoherence for known schemes of **quantum computation**.

Bench commemorating the discussion between

Otton Nikodym and **Stefan Banach** (**Kraków, summer 1916**)

Sculpture: Stefan Dousa Fot. Andrzej Kobos

opened in Planty Garden, Cracow, Oct. 14, 2016