# The Geometry of Loop Quantum Gravity

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TOPOLOGICAL & GEOMETRICAL STRUCTURES OF INFORMATION

# The Geometry of Loop Quantum Gravity

- Quantum Gravity
- Loop Quantum Gravity
- Entropy in LQG
- Correlation on spin networks

# The Geometry of Loop Quantum Gravity

- Quantum Gravity:   What is it about ?
- Loop Quantum Gravity
- Entropy in LQG
- Correlation on spin networks

## What is quantum gravity about?

Looking for a **new concept of geometry**,

replacing the paradigm of differentiable manifold,

allowing for general relativity **without singularity**

with finite fluctuations of the geometry

& naturally encompassing **quantum mechanics** and QFT

# Quantum Gravity

## What is quantum gravity about?

**Gravity** → ← **Quantum**

General Relativity:
it's all about geometry

Space-time is curved and
matter is the source of curvature,

The metric evolves and fluctuates.

Quantum states of geometry,
observables become operators,
quantum fuzziness of metric

Path integral: space-time is a
superposition of all possible
histories of the metric

No more fixed background space-time,

geometry is dynamical and quantum.

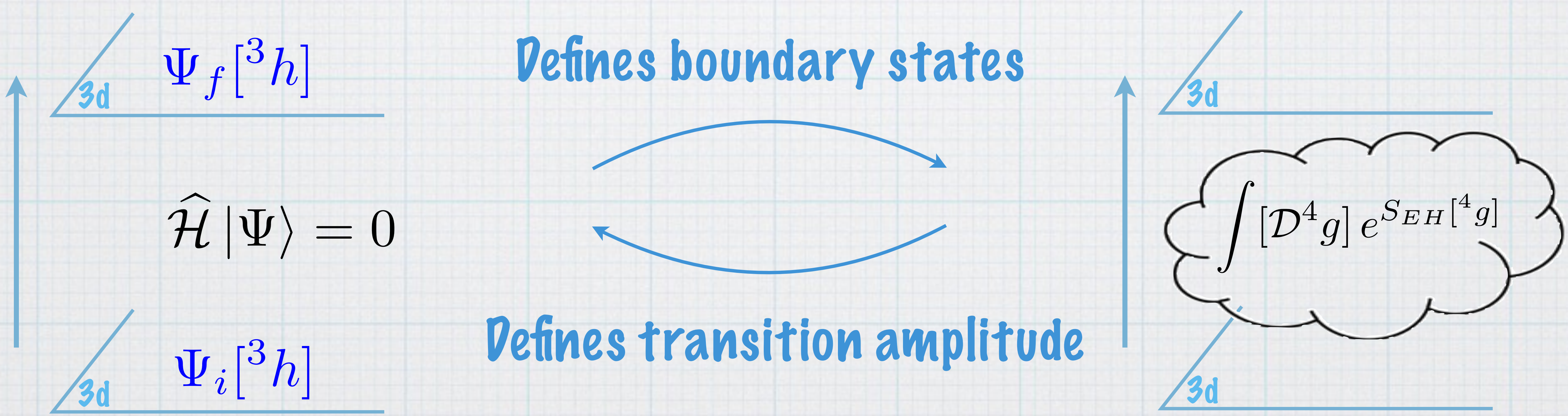
# What is quantum gravity about?

## Canonical Formulation

Wave-functions of the 3d space metric, evolving in time, satisfying Hamiltonian constraints equivalent to quantum Einstein equations

## Covariant Framework

Probability amplitude given by integral over all 4d metrics compatible with boundary of Einstein-Hilbert action



The key ingredient of quantum gravity is also the main problem !

## the Relativity Principle

→ Physics invariant under change of observers, geometry invariant under diffeomorphisms

**Gauge invariance** of general relativity

# Quantum Gravity

The key ingredient of quantum gravity is also the main problem !

the Relativity Principle

→ Physics invariant under change of observers, geometry invariant under diffeomorphisms

Gauge invariance of general relativity

So we can move points around and that's still the same physical geometry...

How to locate systems? Where are the QG d.o.f.'s?

## Where are the QG degrees of freedom?

Even if we can't locate systems,

we can still measure how they interact

### Relational Observables

Reconstruct distances from correlations,
e.g. reversing Newton's law

$$\langle \phi(x)\phi(y)\rangle \sim \frac{1}{d(x,y)^2}$$

### Crucial role of boundaries

Introduce boundaries to stop things from moving around and induce locality from boundary into bulk

# Quantum Gravity

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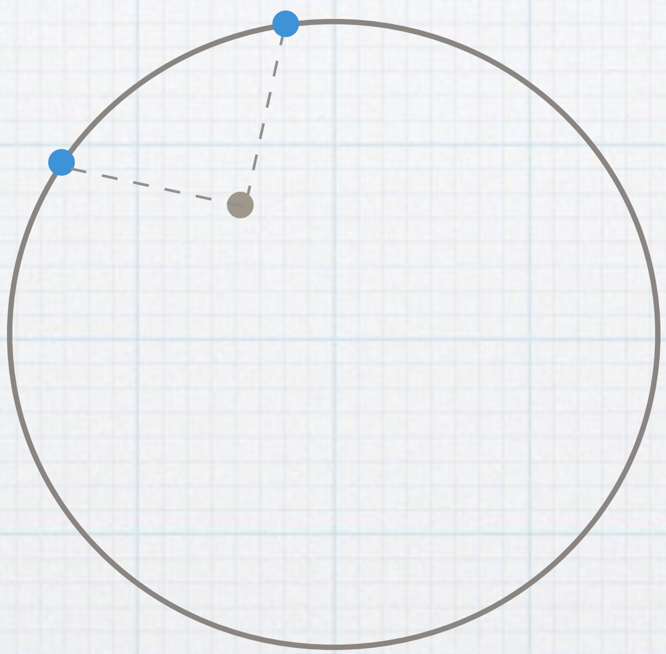
## Non-local degrees of freedom & Holography

# Holography for Quantum Gravity

Boundary state determines structure of bulk geometry,
sometimes? often? admitting
a semi-classical interpretation as an usual manifold



Non-local correlations on boundary to dive
into the bulk and define points

# The Geometry of Loop Quantum Gravity

- Quantum Gravity
- Loop Quantum Gravity:   The Framework
- Entropy in LQG
- Correlation on spin networks

# Loop Quantum Gravity: the Basics

- Use vierbein-connection variables instead of 4d metric

$$g_{\mu\nu} \longrightarrow e^I_\mu, \omega^{IJ}_\mu \qquad\qquad g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$$

- Proceed to a 3+1-d decomposition, with 3d space evolving in time

$$g_{\mu\nu} \longrightarrow h_{ab}, N, N^a$$

3d geometry $E^i_a, A^i_a$

Ashtekar variables:
1-form and su(2)-connection

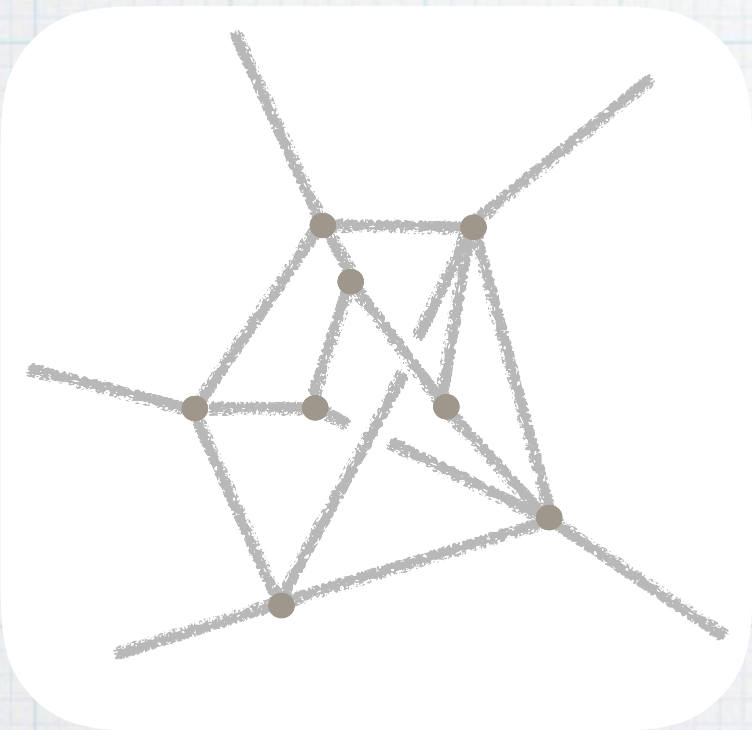
lapse & shift controlling
the evolution of 3d geometry

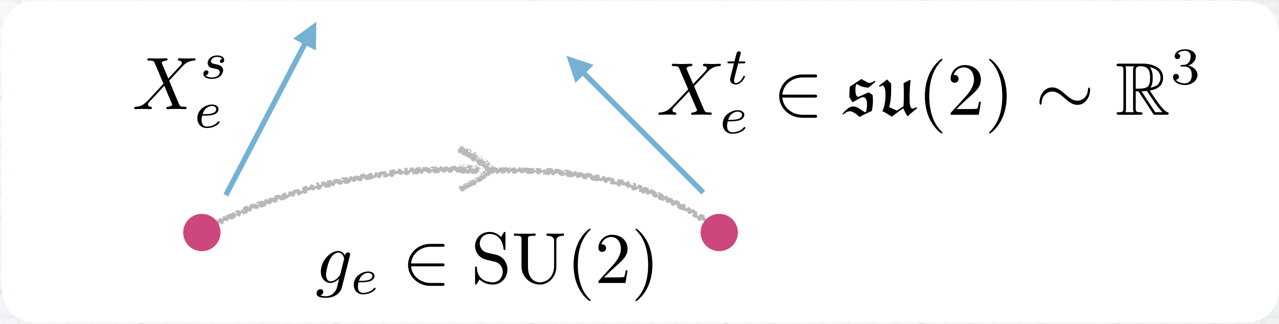
- Discretize & Quantize by using wave-functions $\Psi[A]$

Theory introduced by Ashtekar, Rovelli & Smolin

# Spin Networks

- Work on a graph, to be embedded in 3d space






Phase space: $T^*\mathrm{SU}(2)$ on each edge $e$

$$\{g^{AB}, g^{CD}\} = 0$$

$$\{X^i, X^j\} = \epsilon^{ijk} X^k$$

$$\{X^i, g^{AB}\} = \frac{i}{2}(\sigma^i g)^{AB}$$

Network of relations: 3-vectors with parallel transport

with constraints:

- on each edge $X_e^t = g_e \triangleright X_e^s = g_e X_e^s g_e^{-1}$
- at each node $\sum_{e \ni v} X_e^v = 0$

# Spin Networks

- We quantize all these networks using wave-functions $\Psi[\{g_e\}_{e\in\Gamma}]$

- Closure constraint at node imply invariance under SU(2)

$$\Psi[\{g_e\}] = \Psi[\{h_{s(e)} g_e h_{t(e)}^{-1}\}] \quad \forall h_v \in \mathrm{SU}(2)$$

- Find a nice basis of states: **the Spin Network basis**

with labels:
- on each edge: a spin $j_e \in \frac{\mathbb{N}}{2}$
- at each node: an intertwiner $I_v$

A spin is an irreducible representation of the Lie group SU(2)

$\dim V^j = (2j+1)$ with basis states $|j,m\rangle \longleftrightarrow z_0^{j+m} z_1^{j-m}$

$$-j \le m \le +j$$

# Spin Networks

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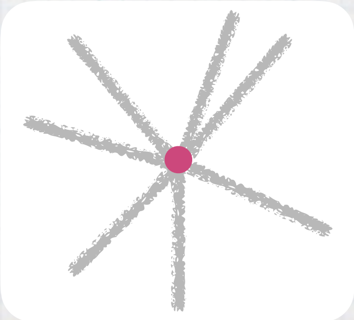
An intertwiner is a SU(2)-invariant state in a tensor product space:

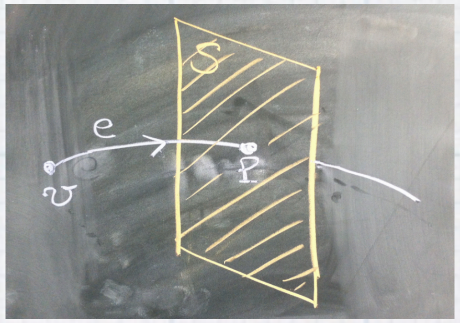
$$I_v : V^{j_1} \otimes .. \otimes V^{j_N} \to \mathbb{C}$$



# The discrete geometry of Spin Networks

- Quantum geometry operators: diagonalized by spin network basis

- area operator acting on each edge: elementary area quanta $\mathcal{A} = j_e \, l_P^2$
- volume operator acting at each node: depends on intertwiner




- Quantization of discrete geometries:

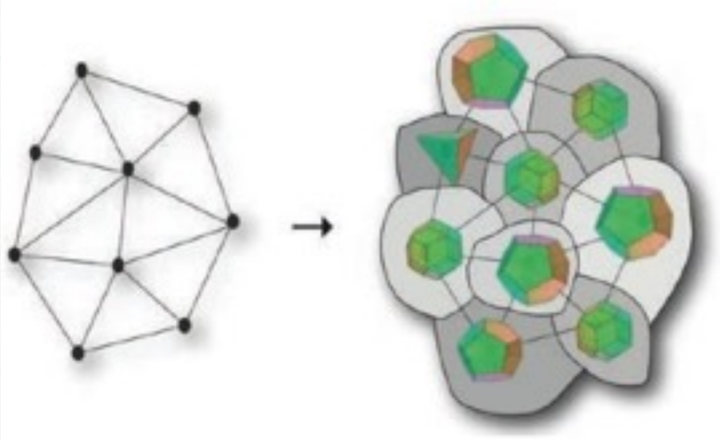
Around each node, we identify the 3-vectors as normals to a unique convex polyhedron

$$\sum_{e \ni v} X_e^v = 0$$

Along each edge, we identify the norm of the normal vectors i.e. the face areas

$$X_e^t = g_e \triangleright X_e^s = g_e X_e^s g_e^{-1}$$

Shape mismatch ⟶ « Twisted geometries »



# What to do with a spin network state ?

- **Spin network is the 3d space:**
  ↪ it defines the geometry itself, there is no background metric
- It evolves and the graph can fluctuate
  ↪ e.g. graphity models: **fluctuating locality**
- We need good observables that probe the large-scale structure of the geometry: correlations? dispersion measure? propagation? ...
  ↪ **correlations between spins and between intertwiners**
- We need to **identify a notion of observers and coordinate systems**

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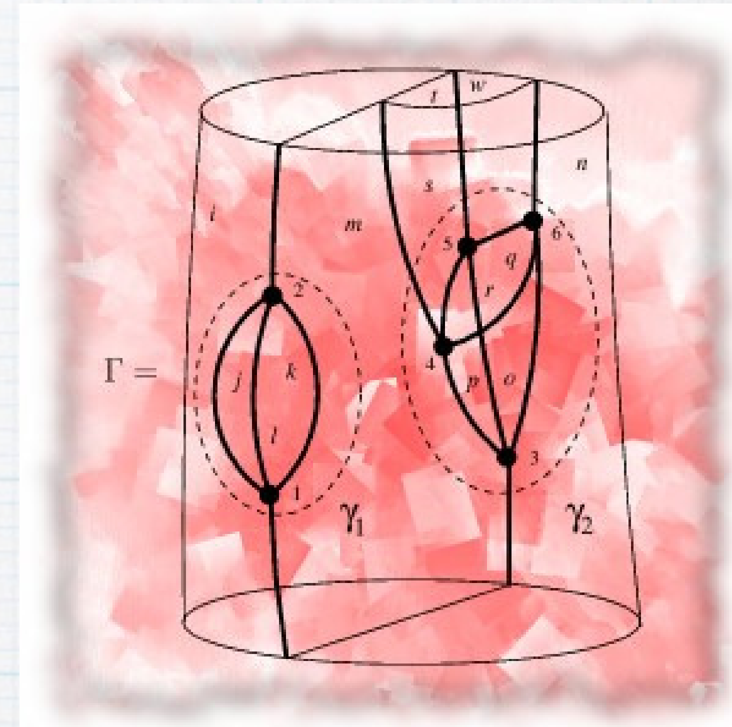
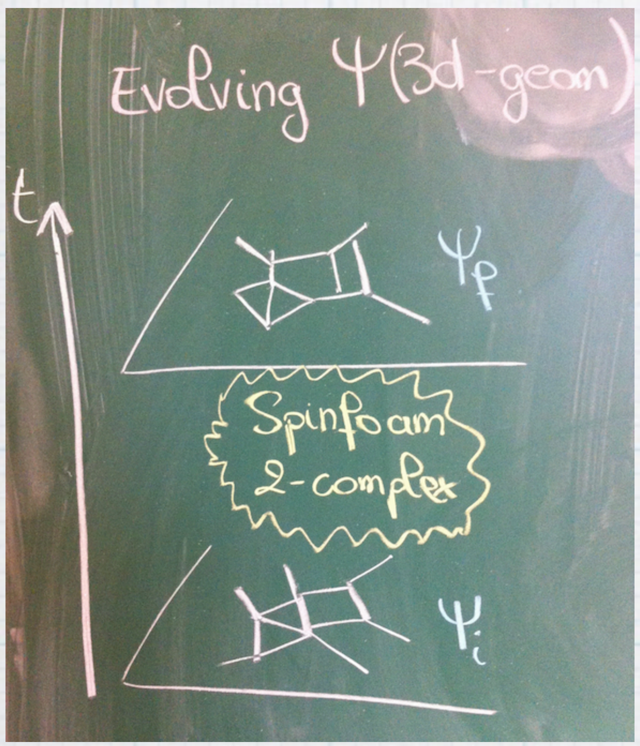
  ↪ **correlations between spins and between intertwiners**
- We need to **identify a notion of observers and coordinate systems**
- We need to **solve Hamiltonian constraints** (quantum Einstein eqn), **understand diffeo inv** (change of graphs), get continuum limit through **coarse-graining flow**, define **boundaries**, describe **evolution**

# Spinfoam: Transition Amplitudes from TQFT

## Histories of Spin Networks as **Spinfoams**




| spin network | | spinfoam | |
|---|---|---|---|
| edge | ⟶ | face | $j_f$ |
| node | ⟶ | edge | $I_e$ |
| evolution | ⟶ | spinfoam vertex | $A_\sigma\left[j_{f\ni\sigma}, I_{e\ni\sigma}\right]$ |

# Spinfoam: Transition Amplitudes from TQFT

We get discrete geometry by taking the dual:

3d triangulations evolved by 4d triangulations

time

- Spin on graph links
- Intertwiner on nodes

→

- Spin on spinfoam faces
- Intertwiner on edges
- Amplitude on spinfoam vertex

topological dual ↓

- Spin for areas
- Intertwiner for volumes

→

- Spin for 2d areas
- Intertwiner for 3d volumes
- Amplitude for 4d-cells

SpaceTime Event
as Spinfoam Vertex

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Spinfoam models defined as Topological Quantum Field Theories

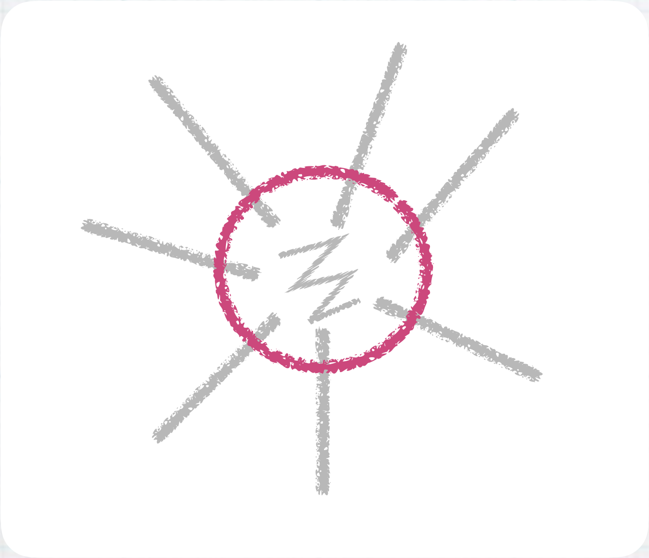
Missing 0d & 1d algebraic structures → active research in Extended TQFT

# The Geometry of Loop Quantum Gravity

- Quantum Gravity
- Loop Quantum Gravity
- **Entropy of Bounded Regions in LQG**
- Correlation on spin networks

# Entropy in LQG: Bulk - Boundary

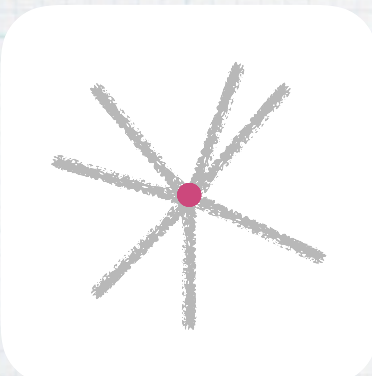
## Bulk graph & surface states



Boundary surface made of N elementary area patches, carrying spins $j_1, j_2, .., j_N$

# The Structure of Intertwiners

Let's start with a single node.



Think of single intertwiner
as prototype BH entropy calculation

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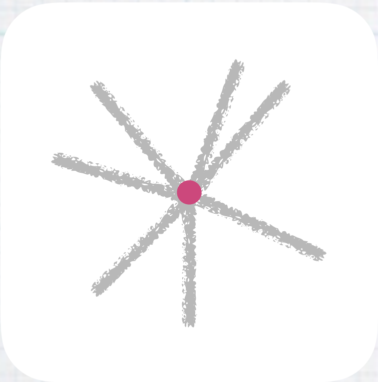

Think of single intertwiner
as prototype BH entropy calculation

Look at all intertwiners for fixed boundary area $J = \sum_{i=1}^{N} j_i$

$$R_N^J = \bigoplus_{J=\sum j_i} \mathrm{Inv}\left[V^{j_1} \otimes .. \otimes V^{j_N}\right]$$

carries an irrep of
the unitary group U(N)

Allows to compute dimension
using hook formula:

$$\dim R_N^J = \frac{1}{J+1} \begin{pmatrix} N+J-1 \\ J \end{pmatrix} \begin{pmatrix} N+J-2 \\ J \end{pmatrix}$$

# The Unitary Group acting on Polyhedra

phase space

| | | |
|---|---|---|
| $\mathbb{C}$ | harmonic oscillator | $\|n\rangle,\ n\in\mathbb{N}$ |
| $\mathbb{C}^2$ | pair of HOs = all SU(2) spins | $\|j,m\rangle$ $\quad \begin{cases} n_1 = (j+m) \\ n_2 = (j-m) \end{cases}$ |
| $\mathbb{C}^{2N}//\mathrm{SU}(2)$ | convex polyhedra | $R_N = \bigoplus_{J\in\mathbb{N}} \mathrm{Inv}\left[V^{j_1}\otimes .. \otimes V^{j_N}\right]$ |

Natural action of $\mathrm{U}(N)$ on $\mathbb{C}^{2N}$ :

$$z_i^A \longmapsto \sum_j^N U_{ij} z_j^A$$

# The Unitary Group acting on Polyhedra

$\mathbb{C}^{2N}//\mathrm{SU}(2)$ convex polyhedra

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Natural action of $\mathrm{U}(N)$ on $\mathbb{C}^{2N}$ :

$$z_i^A \longmapsto \sum_j^N U_{ij} z_j^A$$

Mapping to 3-vectors: $X_i^a = \langle z_i|\sigma^a|z_i\rangle = z_i^A \sigma^a{}_{AB} z_i^B$

Action commutes with closure constraint: $\sum_i X_i^a = 0$

Action leaves total boundary area invariant $A = \sum_i |X_i^a|$

Allows to explore all polyhedra from squashed configuration

# Intertwiner counting: area-entropy law

$$\dim R_N^J = \frac{1}{J+1} \binom{N+J-1}{J} \binom{N+J-2}{J}$$

Compute entropy $S = \ln \dim R_N^J$ for large area and number of edges

**Do we get an area-entropy law?** What should we do with N ?

- We can vary N and J separately, or even assume that they scale together
- Or we can sum over N !... but N goes from 0 to infinity.

**Problem of edges carrying spin-0**

Remove all spin-0's and start with lowest spin 1/2

Then sum over all N's and get finite sum bounded by 2J

# Intertwiner counting: area-entropy law

$$\dim R_N^J = \frac{1}{J+1} \left( \begin{array}{c} J \\ N+J-1 \end{array} \right) \left( \begin{array}{c} J \\ N+J-2 \end{array} \right)$$

Compute entropy $S = \ln \dim R_N^J$ for large area and number of edges

**Problem of edges carrying spin-0**

Remove all spin-0's

$$\dim R_N^J = \sum_{K=0}^{N} \left( \begin{array}{c} K \\ N \end{array} \right) D_K^J$$

$$D_K^J = \sum_{N=0}^{K} (-1)^{K-N} \left( \begin{array}{c} N \\ K \end{array} \right) \dim R_N^J$$

Sum over all N's

$$D_J = \sum_{K=0}^{2J} D_K^J = \sum_N \frac{1}{2^{N+1}} \dim R_N^J$$

$$\ln D_J \sim J \ln(6+4\sqrt{2}) - \frac{3}{2} \ln J + ..$$

**We get area-entropy law !**

could go further and get typical shape ...

# Entropy in LQG: a qubit Toy Model

Let us go further into entropy & correlations ...

... by considering a super-simplified model

We set all boundary spins to smallest value $j_1, j_2, .., j_N = \frac{1}{2}$

- probing the surface with smallest area element, i.e. highest resolution
- avoid over-counting since a spin j is a tensor product of spins 1/2
- area directly proportional to nb of area elements: $J = \frac{N}{2}$
- models a « qubit horizon »

# Entropy in LQG: a qubit Toy Model

Assume observer has no clue about what's inside

↪ Use **totally mixed state on intertwiner space**

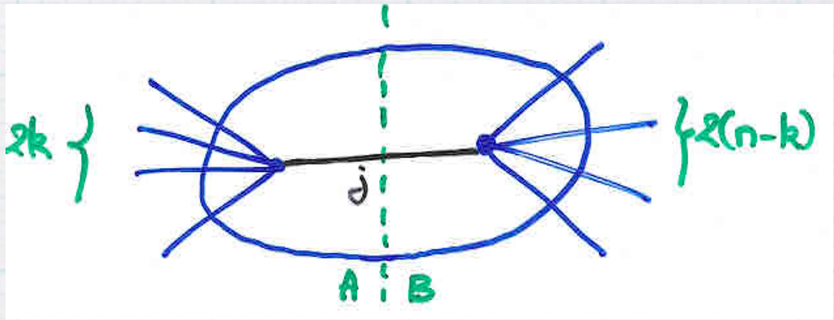
$$\rho = \frac{1}{D}\mathbb{I} \qquad D = \dim \mathrm{Inv}\,(V^{\frac{1}{2}})^{\otimes 2n} = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$$

- **A very simple entropy counting:**

$$S = \ln\, D \sim 2n \ln 2 - \frac{3}{2}\ln n + \dots$$

- Beyond entropy, **look at correlations**:

↪ Separate the surface patches in two sets A & B of size k and (n-k)

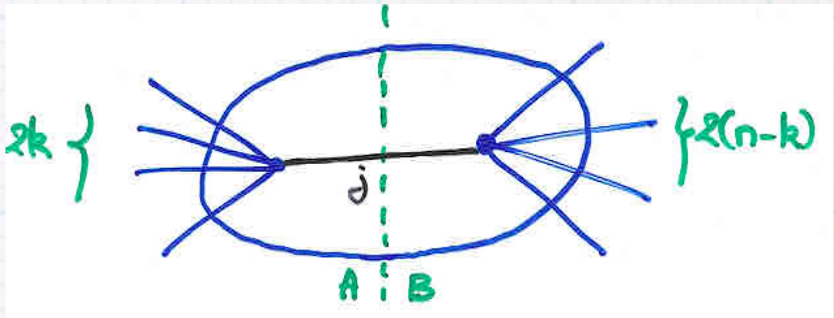



Can introduce intertwiner basis $|j, I_A, I_B\rangle$

# Entropy in LQG: a qubit Toy Model

- Beyond entropy, **look at correlations**:

  ↪ Separate the surface patches in two sets A & B of size 2k and 2(n-k)




Can introduce intertwiner basis $|j, I_A, I_B\rangle$

$$\dim \mathrm{Inv}\,(V^{\frac{1}{2}})^{\otimes 2n} = \sum_j d_j^{(k)} d_j^{(n-k)}$$

$$d_j^{(k)} = \frac{2j+1}{j+k+1}\begin{pmatrix} 2k \\ k+j \end{pmatrix}$$

- For splitting n | n , correlation gives entropy log-correction:

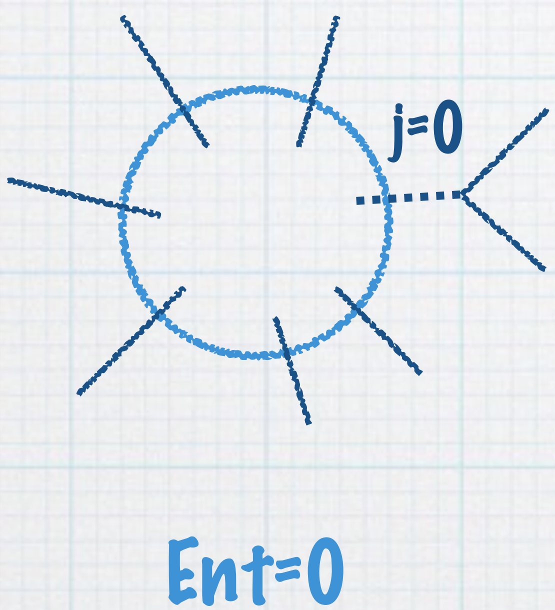
$$\mathrm{Cor}(n|n) = S[\rho_A] + S[\rho_B] - S = S - \sum_j d_j^{(n)\,2} \ln d_j^{(n)\,2} \sim \frac{3}{2}\ln n$$

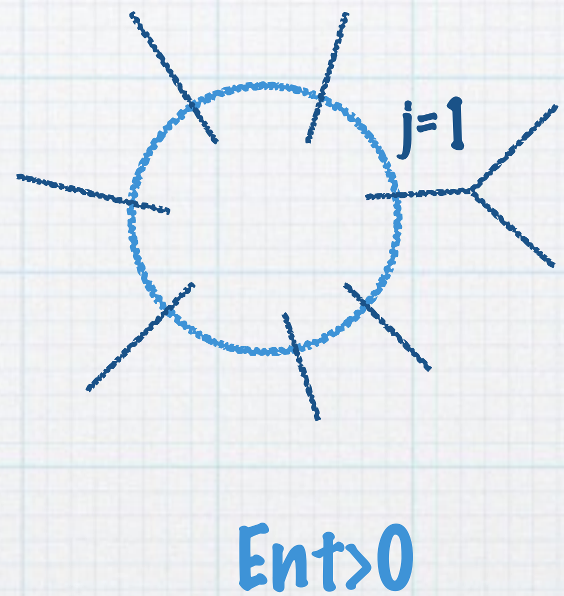
# Entropy in LQG: a qubit Toy Model

- We can also compute entanglement between parts of surface:

$$Ent(A|B) = \frac{1}{D}\sum_j d_j^A d_j^B \ln(2j+1)$$

- Especially interested in the case of a pair versus the rest of the surface




Compute unentangled fraction: $f = \mathrm{Tr}\,\mathrm{Proj}_{A \overset{0}{\leftrightarrow} B}\,\rho = \frac{d_A^0 d_B^0}{D} \sim \frac{1}{4} + \frac{3}{8n}$

# Entropy in LQG: a qubit Toy Model

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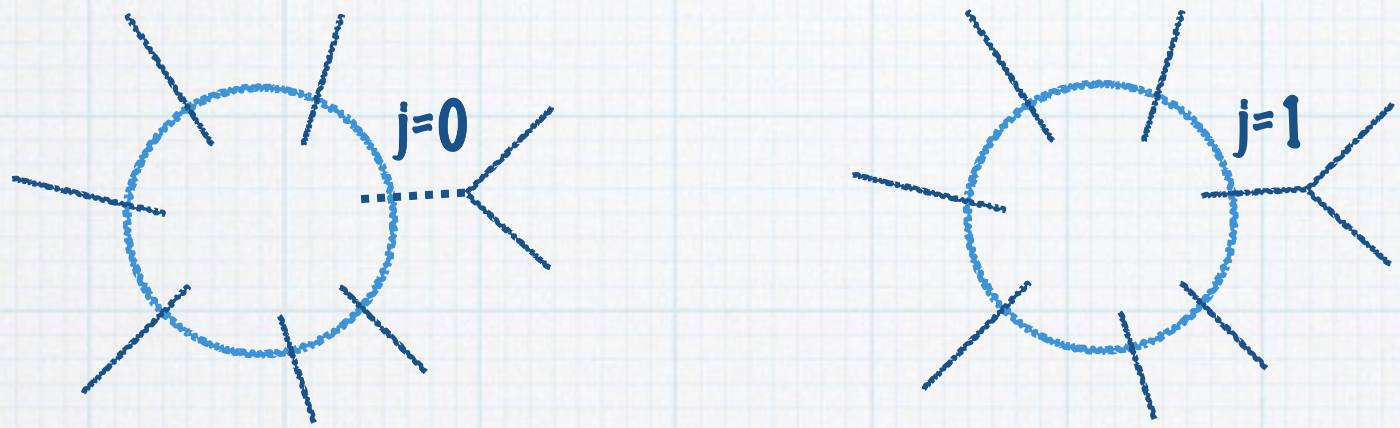



Ent=0 | Ent>0

Compute unentangled fraction: $f = \mathrm{Tr}\,\mathrm{Proj}_{A \overset{0}{\leftrightarrow} B}\,\rho = \frac{d_A^0 d_B^0}{D} \sim \frac{1}{4} + \frac{3}{8n}$

Translate probability into rate using time scale: $\tau \propto \ \mathrm{Mass} \propto \sqrt{n}$

$$\frac{d\mathcal{A}}{dt} \propto \frac{dn}{dt} \propto -\frac{f}{\tau} \propto -\frac{1}{4\sqrt{n}} - \frac{3}{8n^{\frac{3}{2}}}$$

Recover Hawking's evaporation law

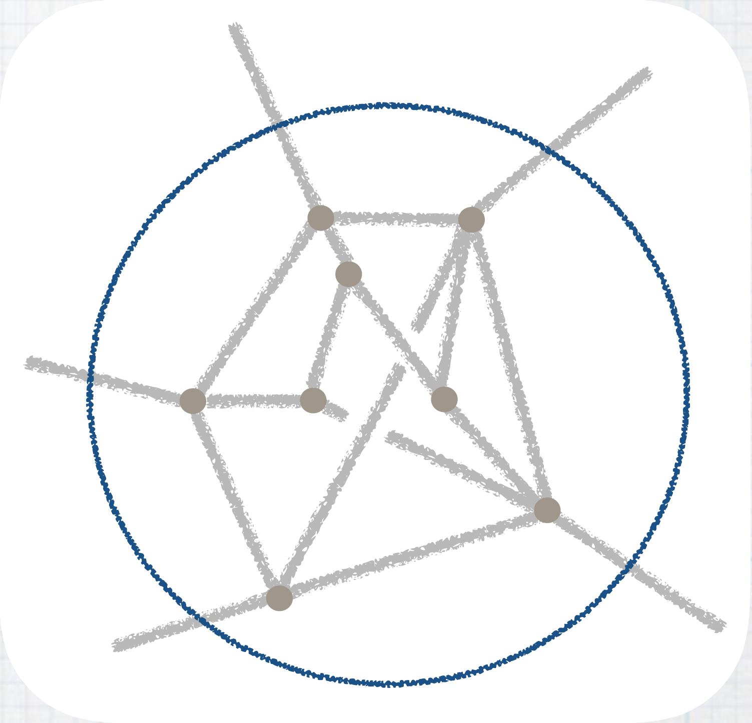
# Entropy in LQG: diving inside the bulk

- A region of a spin network consists in several nodes.

Can show that loops can carry arbitrary spins

↓

Infinite bulk entropy



But still need to use physical states, satisfying Hamiltonian constraints

# Entropy in LQG: from bulk to boundary dynamics

Two current directions of research

## typicality of boundary states

Large bulk Hilbert space implies quasi thermal boundary state (concentration of measure)

## surface dynamics

Boundary geometry is determined by bulk dynamics

↓

Quantum surface dynamics with interaction between area quanta, dissipation & decoherence

towards Thermodynamics

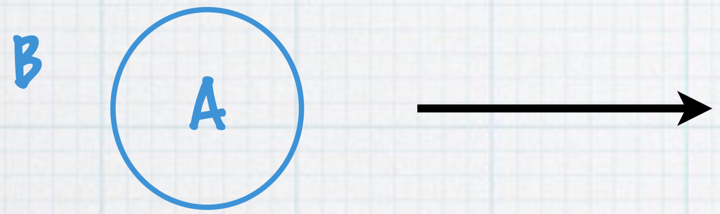
# The Geometry of Loop Quantum Gravity

- Quantum Gravity
- Loop Quantum Gravity
- Entropy in LQG
- Correlation on spin networks: Holographic states

# Searching for Holographic States

Let us look at correlations between two parts of spin network

- For a bi-partite partition: we want area-entropy law




**Holographic states** e.g. topological phases

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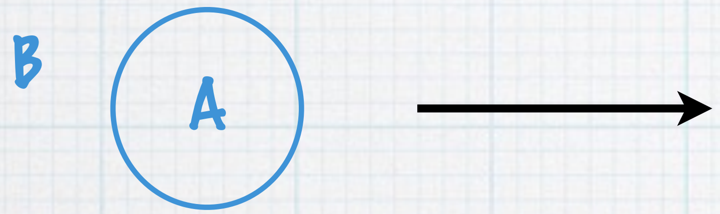
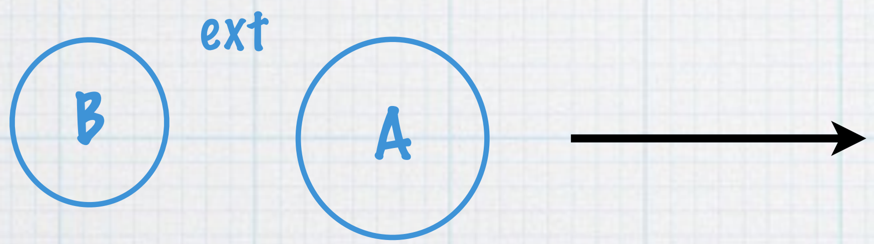



**Holographic states** e.g. topological phases

- For two subsystems: get notion of distance from correlation




**Interplay with condensed matter** e.g. in critical regime

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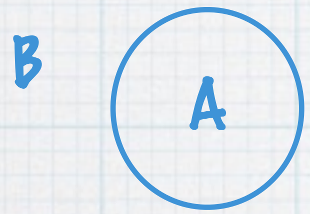


→ **Holographic states** e.g. topological phases

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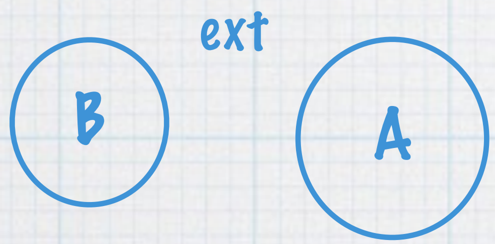



→ **Interplay with condensed matter** e.g. in critical regime

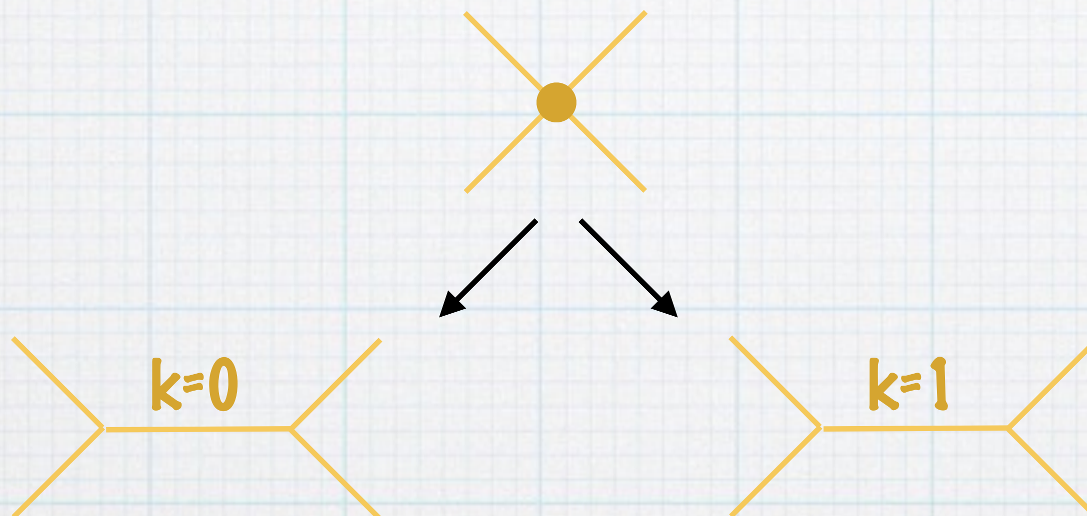
- For three subsystems: get notion of curvature from correlation ?

**Get « good » states, with large-scale structure corresponding semi-classically to usual manifold ?**

# Qubit reduction of Spin networks

In order to **simplify structure of spin networks**,

- Fix all spins to $j = \frac{1}{2}$
- Consider only 4-valent nodes: $\dim \mathrm{Inv}\, (V^{\frac{1}{2}})^{\otimes 4} = 2$



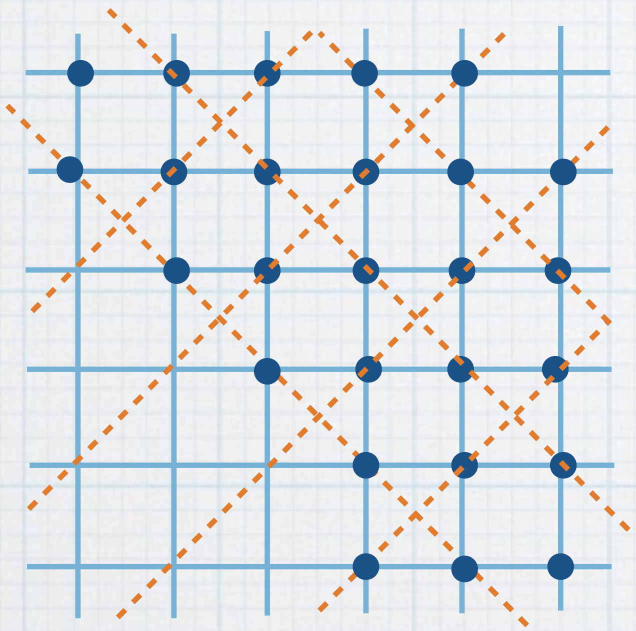

e.g. on a 2d square lattice,
or 3d diamond crystal

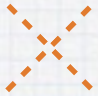
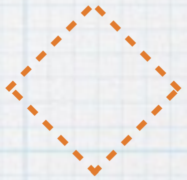
**Can compare**
**LQG states & dynamics**
**to condensed matter models**

# Entanglement on a Spin Network

## Let us look at bipartite entanglement on spin network

↪ prototype model: **Kitaev's toric code**



- Switch to qubits on edges $\mathcal{H} = \mathbb{C}^{2E}$
- Impose 4-qubit constraints on  and 

$$A_v = \sigma_x^{e_1}\sigma_x^{e_2}\sigma_x^{e_3}\sigma_x^{e_4} = 1 \qquad \text{closure constraint}$$

$$B_p = \sigma_z^{e_1}\sigma_z^{e_2}\sigma_z^{e_3}\sigma_z^{e_4} = 1 \qquad \text{flatness constraint}$$

## Solution states are condensate of loop operators:

$$|\Psi\rangle = \frac{1}{\#\mathcal{G}} \sum_{g\in\mathcal{G}} g\,|\uparrow .. \uparrow\rangle$$

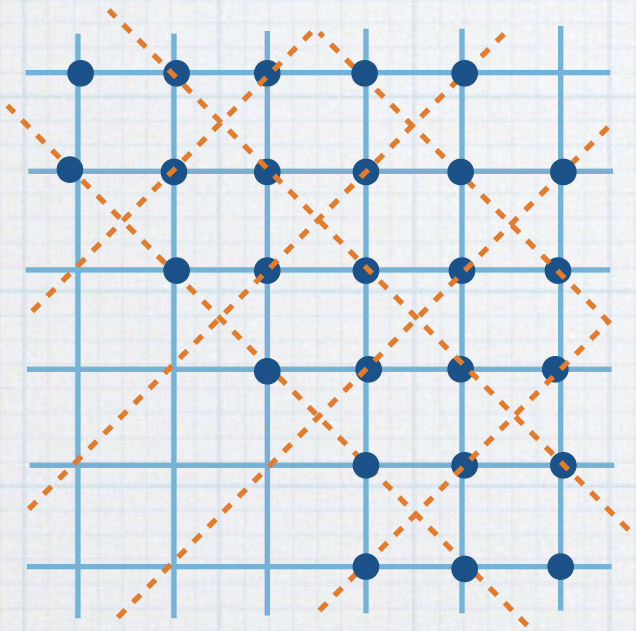
$\mathcal{G}$ group generated by star operators $A_v$

$g$ are $\sigma_x$ loops on dual lattice

# Entanglement on a Spin Network

Let us look at bipartite entanglement on spin network

↪ prototype model: **Kitaev's toric code**



- Holographic behavior for bipartite splitting:

$$S[\rho_R] = (E_{\partial R} - 1) \ln 2$$

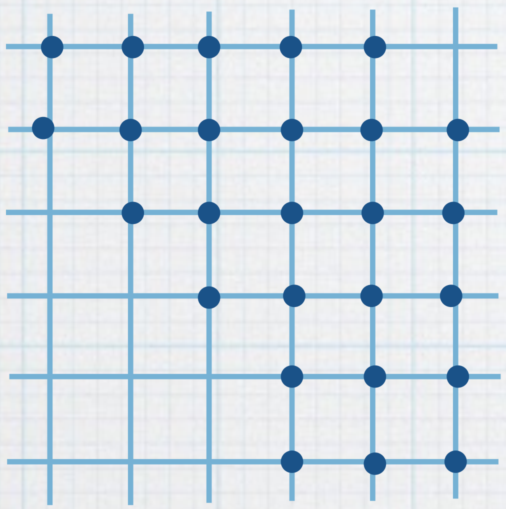
- Works on any lattice: counts nb of independent loops crossing the boundary
- Works for « BF theory » on any group: Kitaev model is $\mathbb{Z}_2$ case

# Correlations on a Spin Network

Let us look at 2-point functions,

i.e. correlations between two subsystems

↪ prototype model: **use known statistical physics models e.g. Ising**



$$|\Psi\rangle = \sum_{\epsilon_v} e^{\frac{\beta}{2}\sum_e \epsilon_{s(e)}\epsilon_{t(e)}} |\{\epsilon_v\}_v\rangle$$

- Known phase diagram and critical points
- Algebraic decay of correlation in critical regime

$$\langle \epsilon_v \epsilon_w \rangle \sim \mathrm{dist}(v,w)^{-\frac{1}{4}}$$

↪ can get distance from 2-pt function

- Also admits expression as loop condensate
- Generalizable to more interesting

# Searching for Holographic States

two starting points for the search for « good » states

Ansatz?

**loop condensate states**

↪ statistical physics models, long-range structure

**MERA-like states**

↪ from tensor network renormalisation, using local entangling operators, fine-tune local structure

↪ QC techniques, holography as quantum error-correction

Thank you for your attention !

# Entropy in LQG: a qubit Toy Model

- We can also compute entanglement between parts of horizon:

$$E(A|B) = \frac{1}{D} \sum_j d_j^A d_j^B \ln(2j+1)$$

↪ Turns out that $C(n|n) \sim 3E(n|n) > 2E(n|n)$

interpreted in QI , as cryptographic power of reference frames !

# Outlook

## Holography in 3d Quantum Gravity

Ponzano-Regge on the twisted torus

## Some key ideas for 4d (L)QG :

- Continuum limit from refining boundary state (not bulk state)
- Duality with condensed matter models (CFTs) on the boundary
- How to coarse-grain to asympt symmetries?
- Actually huge cross-over with « string theory » QG research