The categorical origins of entropy

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Maximum entropy on metric spaces

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Tom Leinster and Mark Meckes, Maximizing diversity in biology and beyond, *Entropy* **18** (2016), 88.

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Entropy in mathematical sciences

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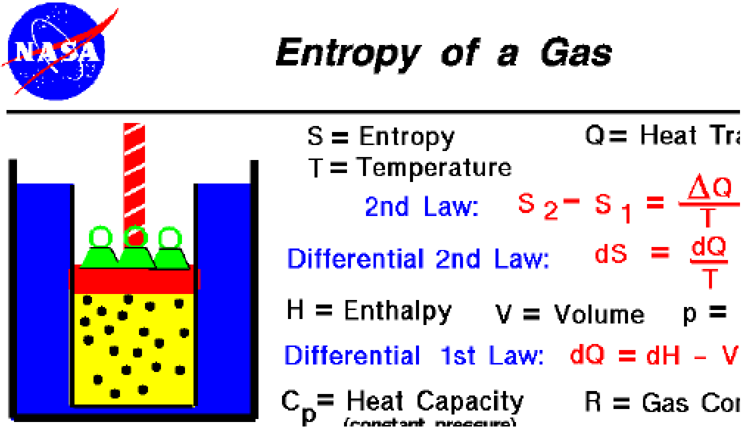

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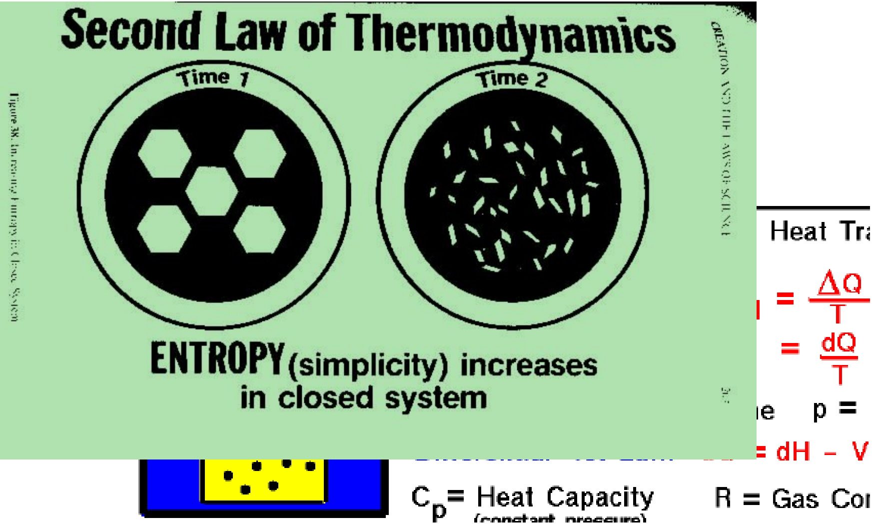

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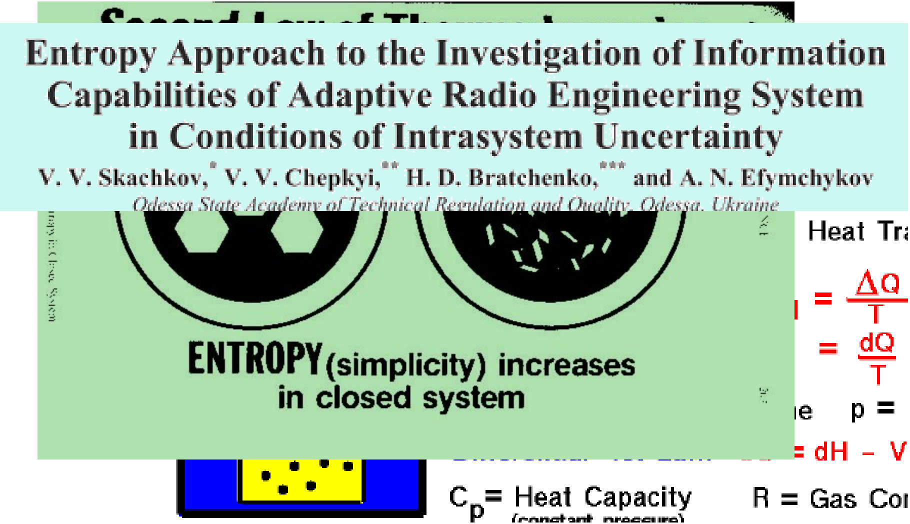

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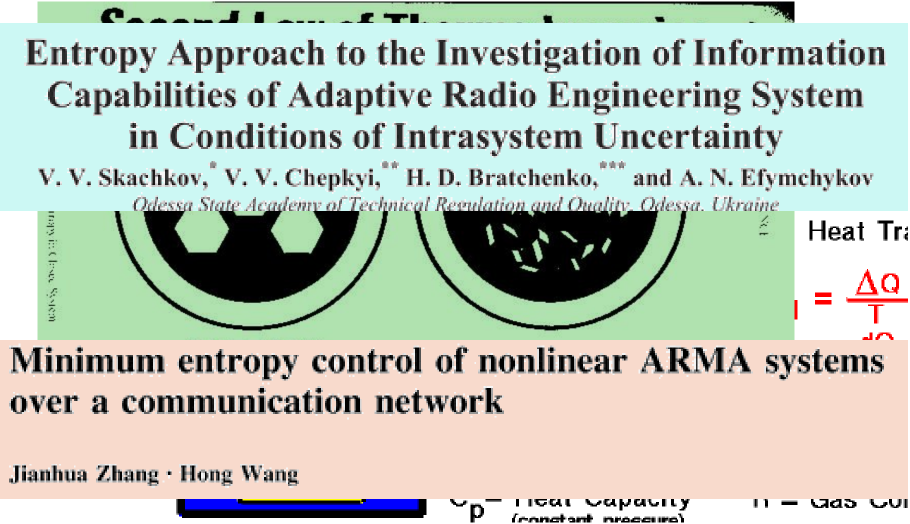

Entropy Approach to the Investigation of Information Capabilities of Adaptive Radio Engineering System in Conditions of Intrasystem Uncertainty

V. V. Skachkov,* V. V. Chepkyi,** H. D. Bratchenko,*** and A. N. Efymchykov

Odessa State Academy of Technical Regulation and Quality, Odessa, Ukraine

Minimum entropy control of nonlinear ARMA systems over a communication network

Jianhua Zhang · Hong Wang

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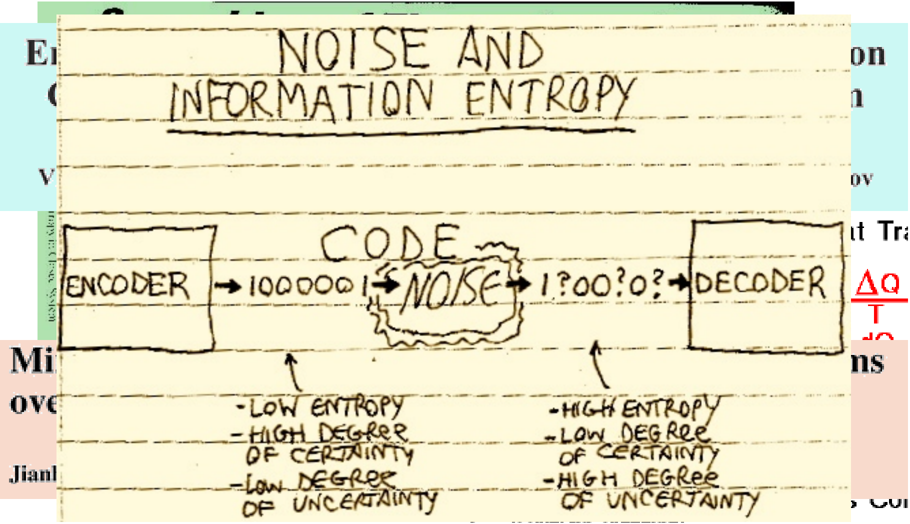

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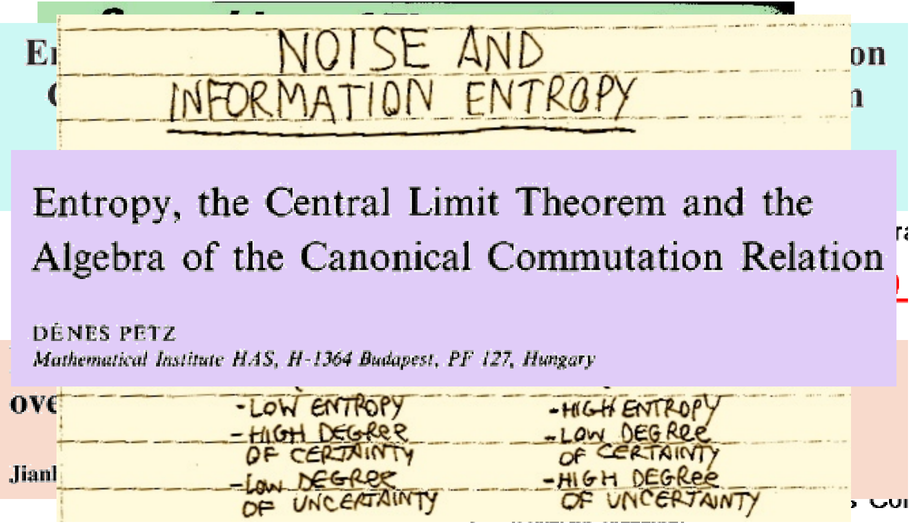

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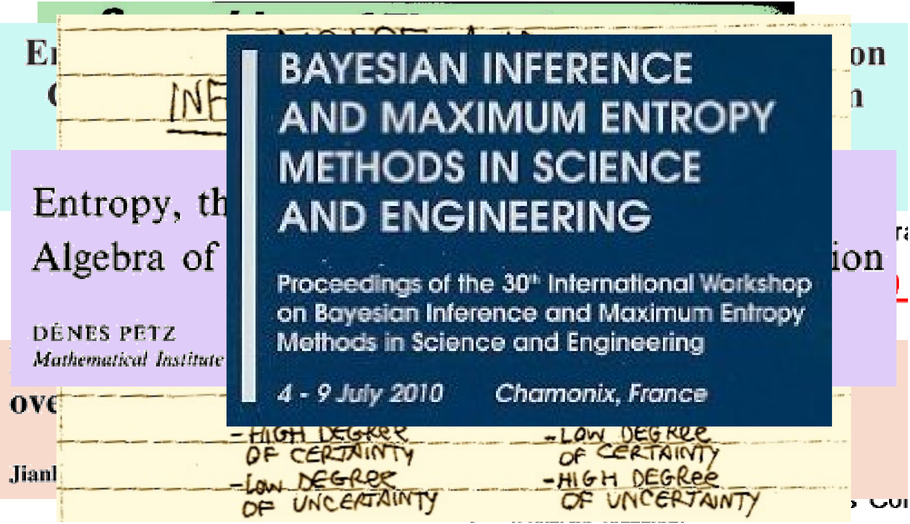

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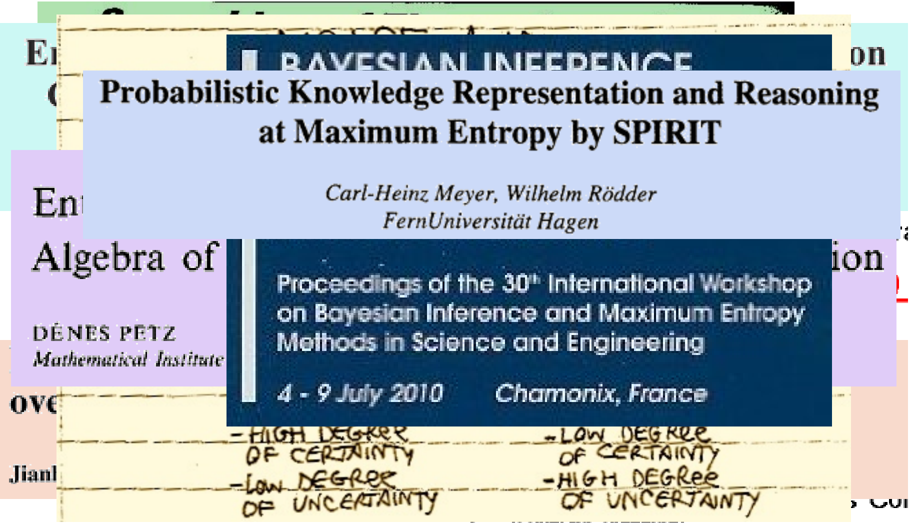

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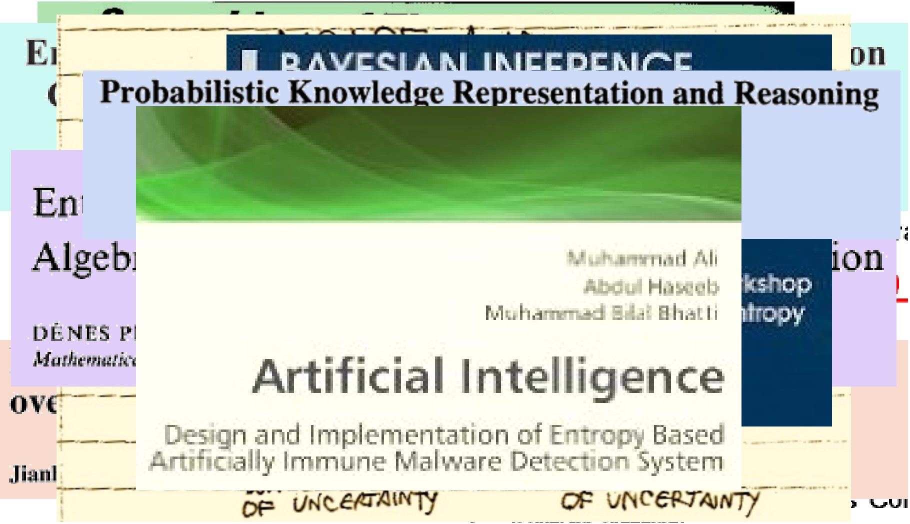

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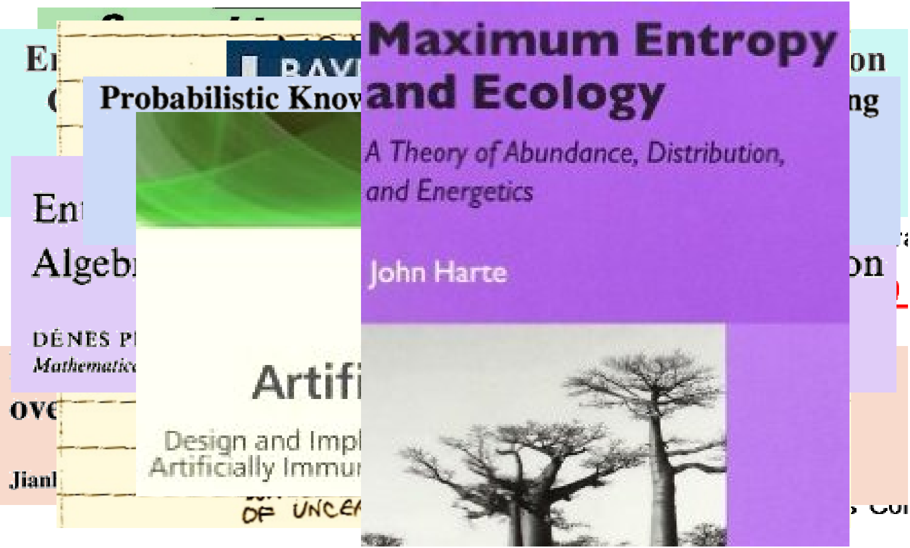

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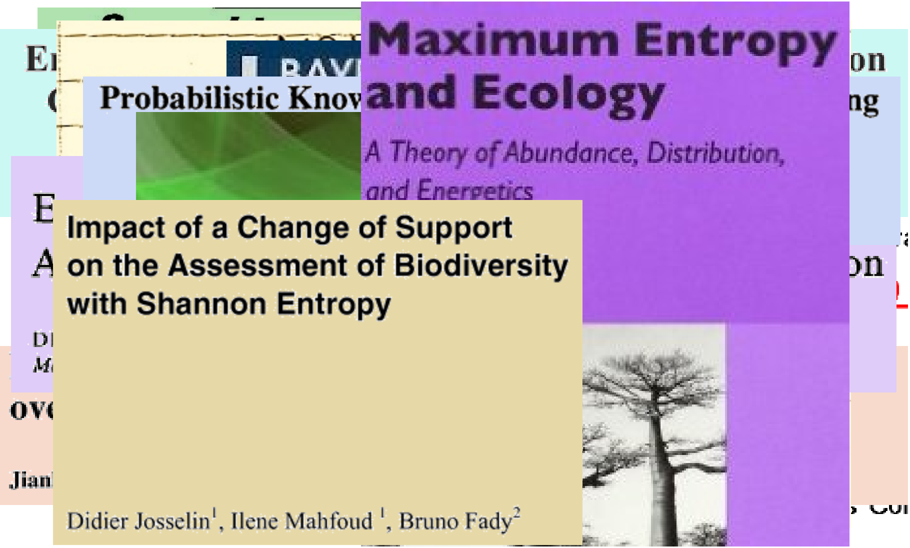

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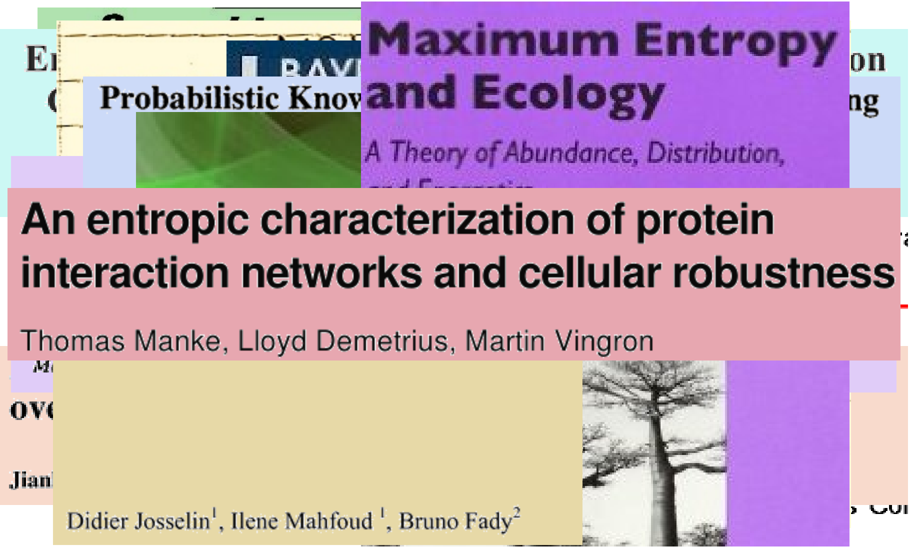

Entropy in mathematical sciences

Roman F. Nalewajski

Reduced communication channels of molecular fragments and their entropy/information bond indices

An entropic characterization of protein interaction networks and cellular robustness

Thomas Manke, Lloyd Demetrius, Martin Vingron

Didier Josselin[1], Ilene Mahfoud [1], Bruno Fady[2]

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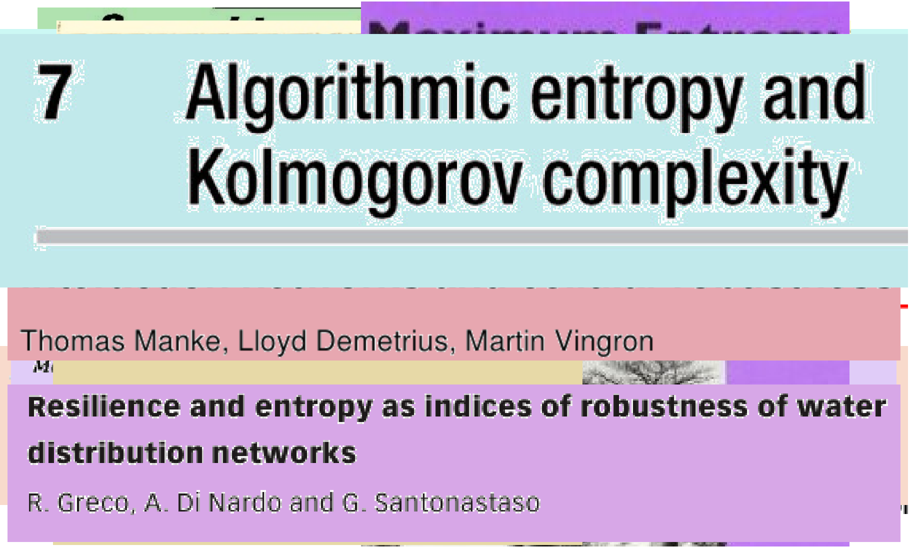

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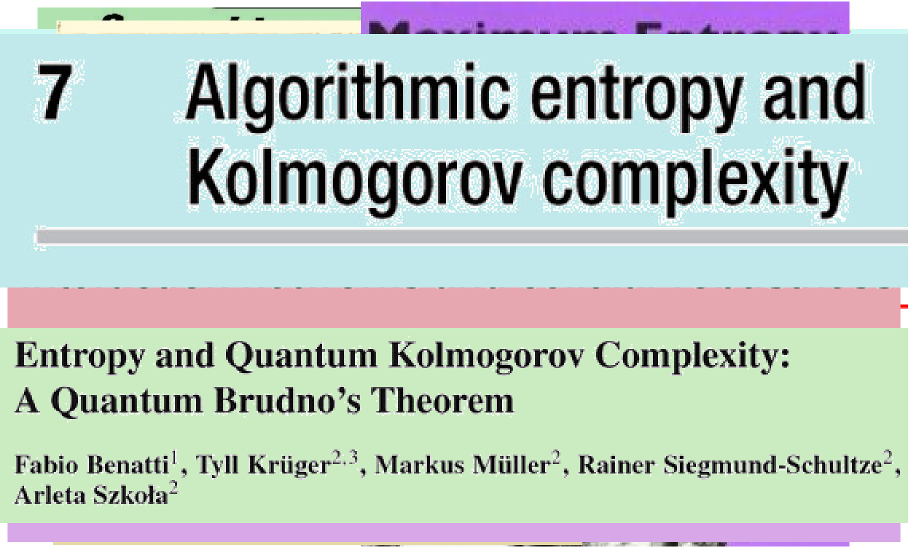
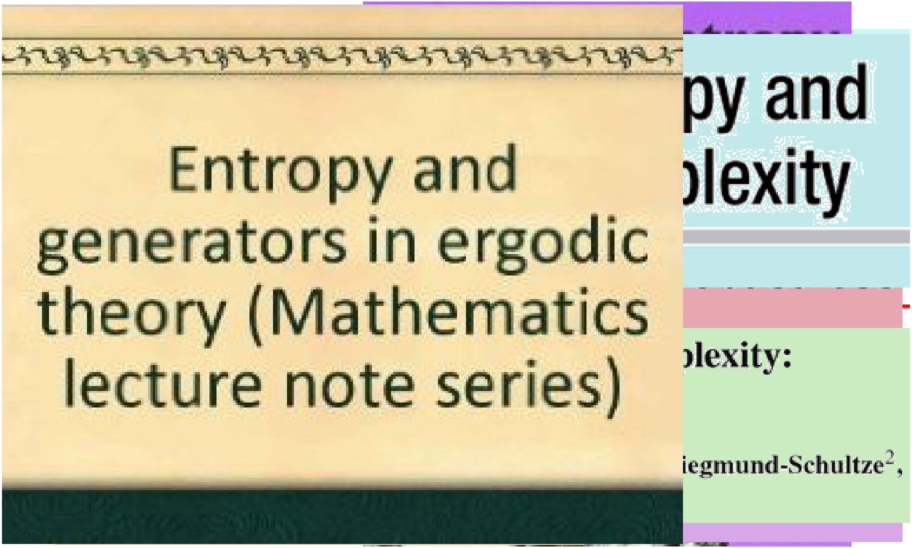


7 Algorithmic entropy and Kolmogorov complexity

Entropy and Quantum Kolmogorov Complexity: A Quantum Brudno's Theorem

Fabio Benatti[1], Tyll Krüger[2,3], Markus Müller[2], Rainer Siegmund-Schultze[2], Arleta Szkoła[2]

Entropy in mathematical sciences



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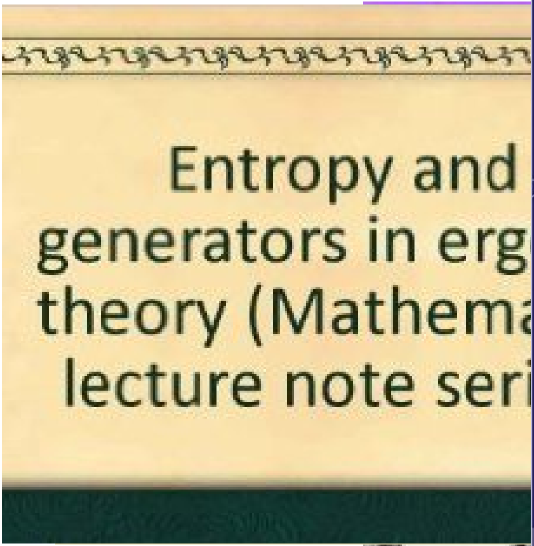

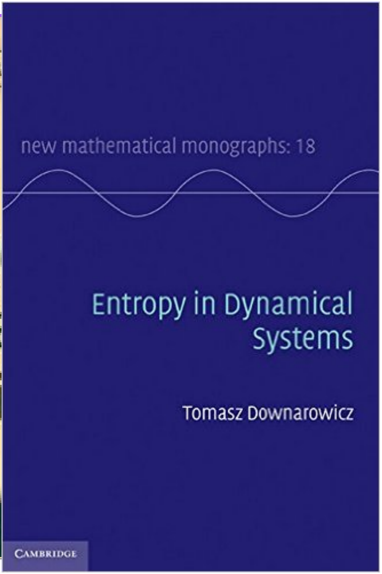

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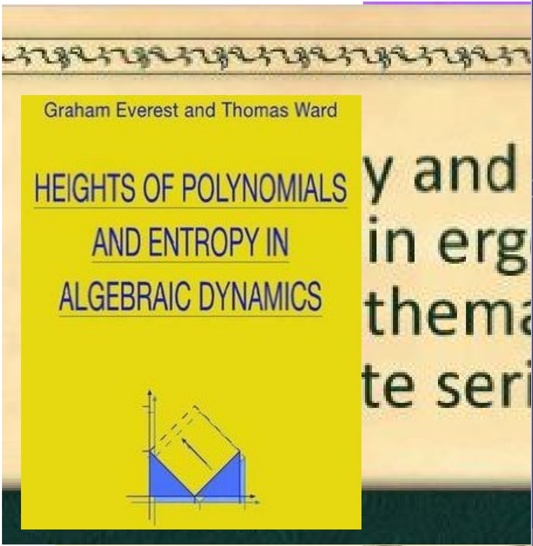

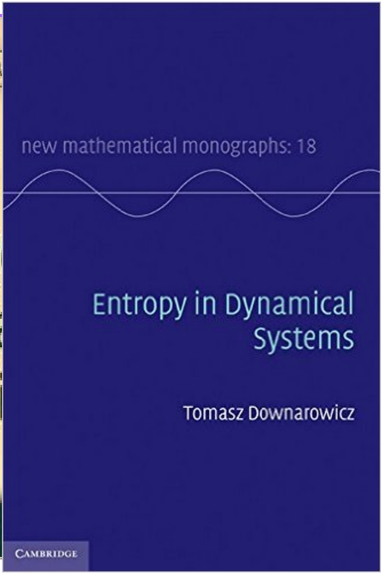

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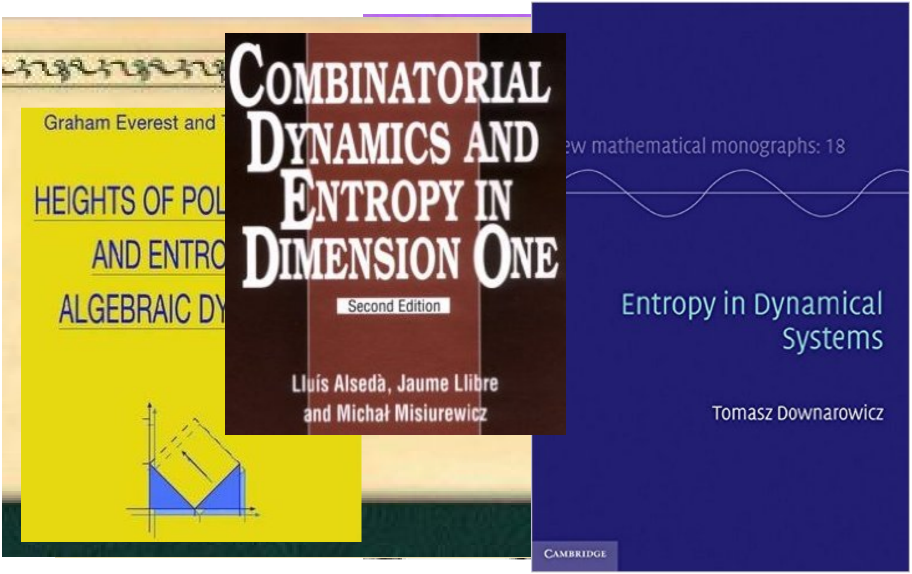

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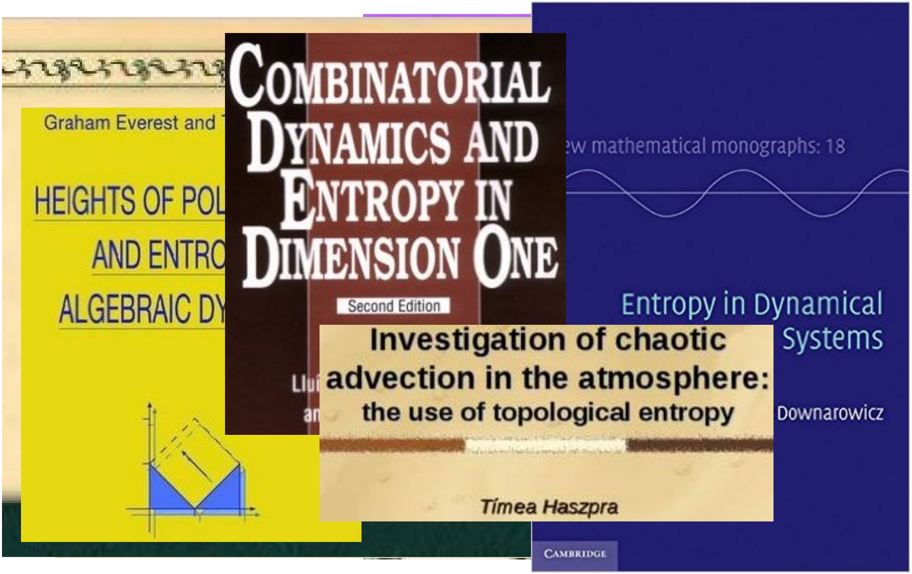

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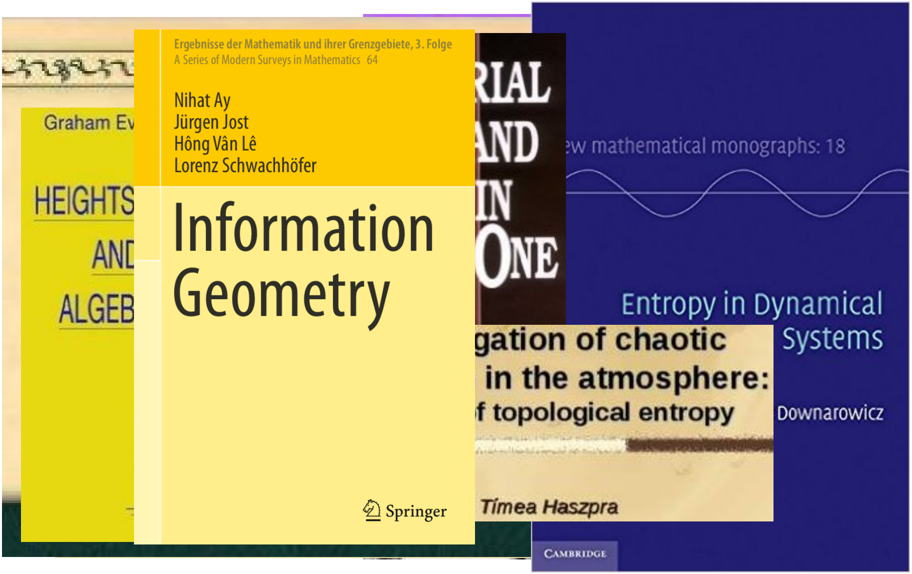

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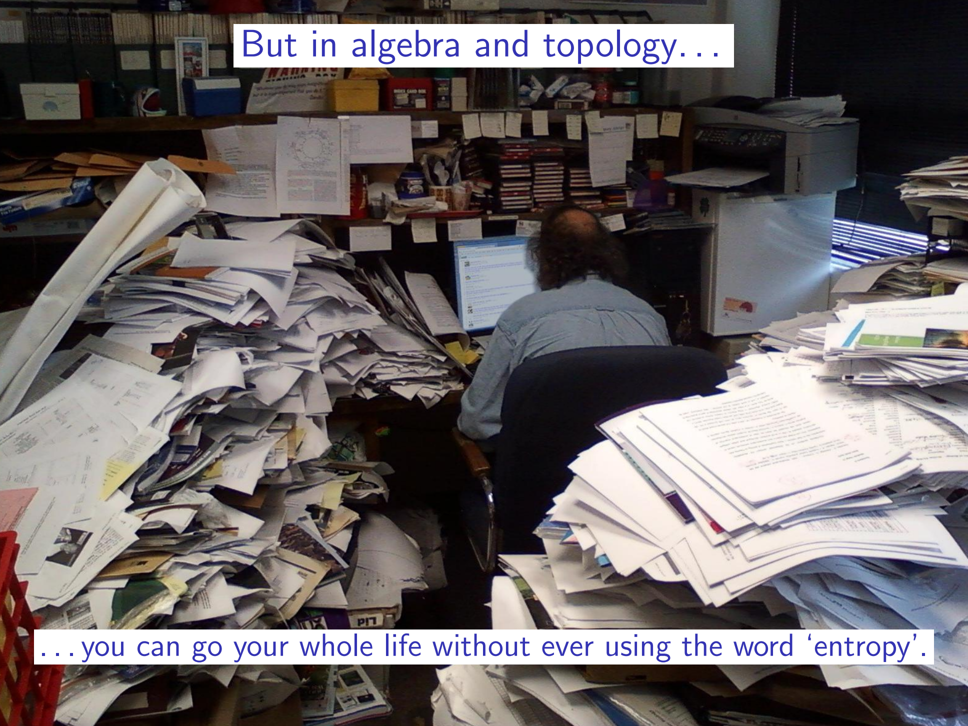

…you can go your whole life without ever using the word ‘entropy’.

The point of this talk

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Entropy is notable by its relative absence from algebra and topology.

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However, we will see that entropy is *inevitable* in these subjects:
it is there whether we like it or not.

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We'll do this using some category theory. . .

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- right at the end: a bit more.

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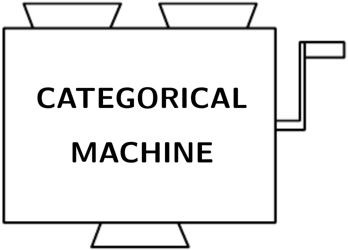

Image: J. Kock

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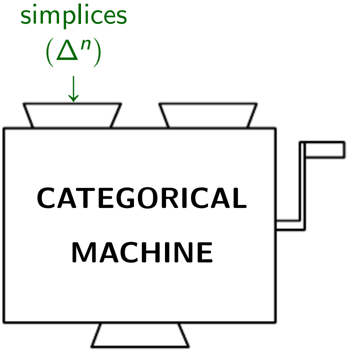

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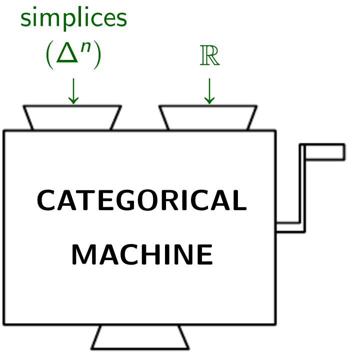

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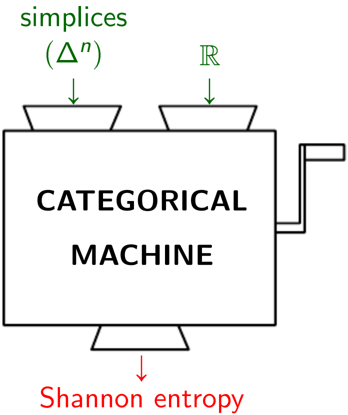

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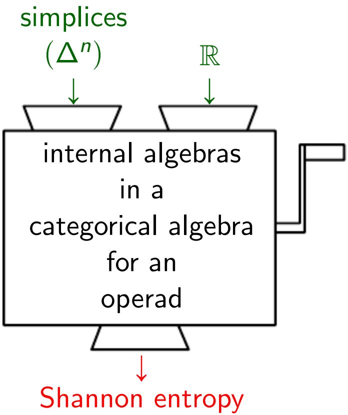

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Plan

Plan

1. Operads and their algebras

Plan

1. Operads and their algebras
2. Internal algebras

Plan

1. Operads and their algebras

2. Internal algebras

3. The theorem

Plan

1. Operads and their algebras
2. Internal algebras
3. The theorem
4. Low-tech corollary

1. Operads and their algebras

The definition of operad

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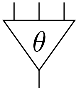
An operad O is a sequence $(O_n)_{n\in\mathbb{N}}$ of sets



$\in O_3$

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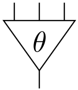
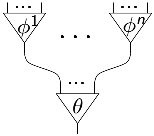


$\in O_3$

(i) composition: for each $k, n_1, \ldots, n_k \in \mathbb{N}$, a function

$$\begin{array}{ccc} O_n \times O_{k_1} \times \cdots \times O_{k_n} & \longrightarrow & O_{k_1+\cdots+k_n} \\ (\theta, \phi^1, \ldots, \phi^n) & \longmapsto & \theta \circ (\phi^1, \ldots, \phi^n) \end{array}$$



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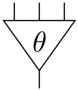

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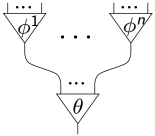

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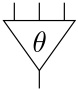

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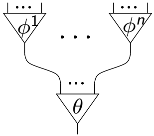

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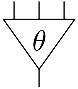

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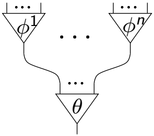

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(Then every tree has a unique composite.)

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- The operad of polynomials P over a field k has $P_n = k[x_1, \ldots, x_n]$. Composition is given by substitution and reindexing: e.g. if

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$$\begin{array}{cccc} \overline{\mathbf{p}}\colon & A^n & \longrightarrow & A \\ & (\mathbf{a}^1, \ldots, \mathbf{a}^n) & \longmapsto & \sum_i p_i \mathbf{a}^i. \end{array}$$

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   It is a categorical Δ-algebra via convex combinations, as defined above.

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2. Internal algebras

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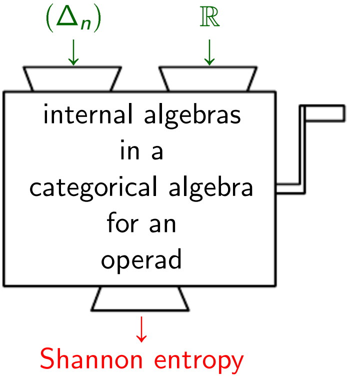
Explicitly, this means that throughout, we add a condition

(iii) continuity

to the conditions (i) and (ii) that appear repeatedly.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3. The theorem

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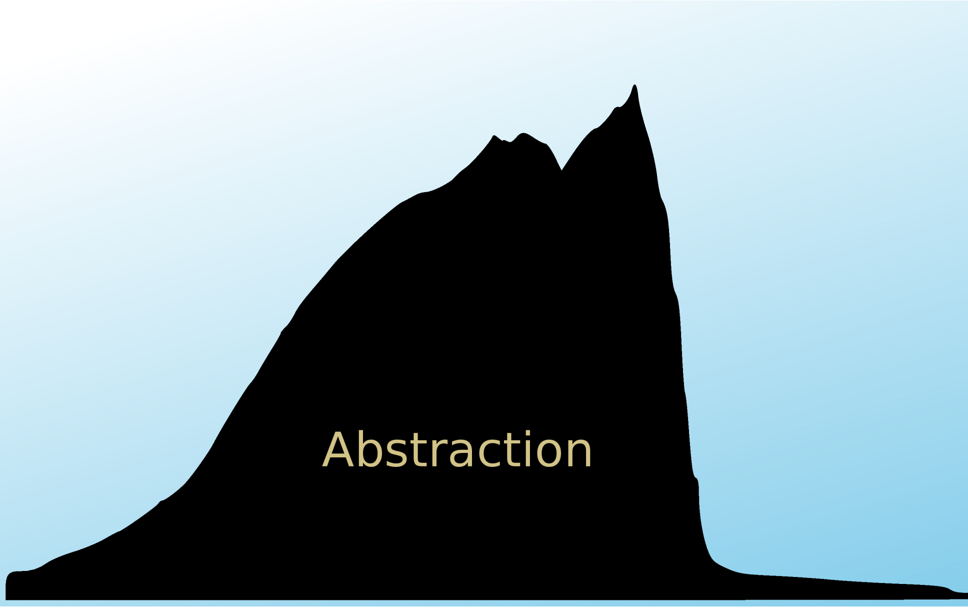

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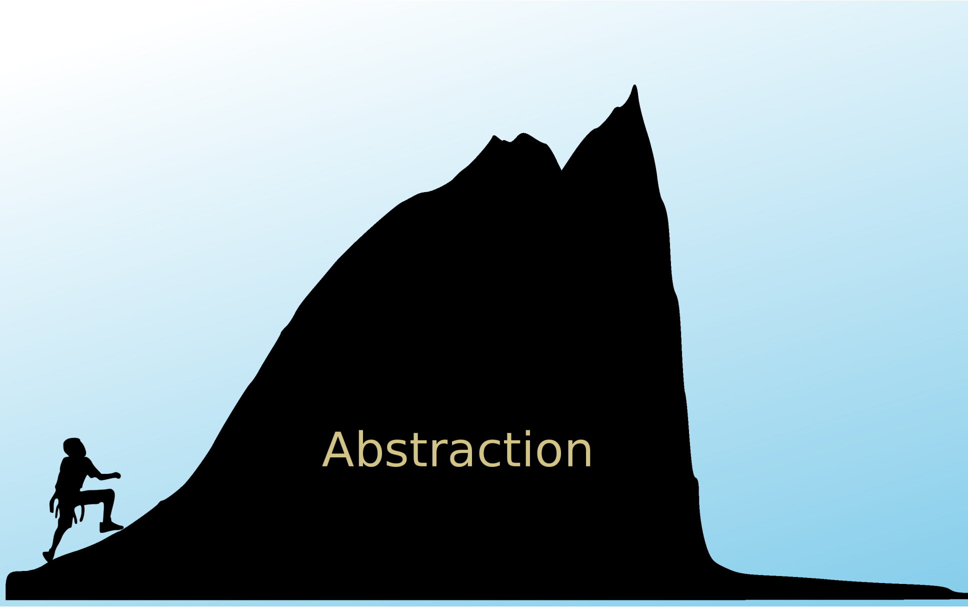

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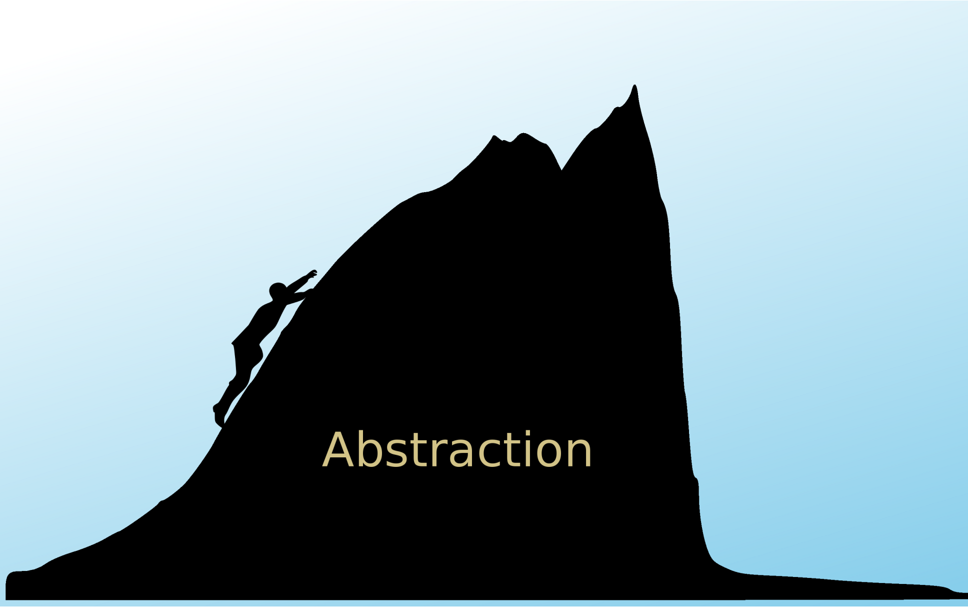

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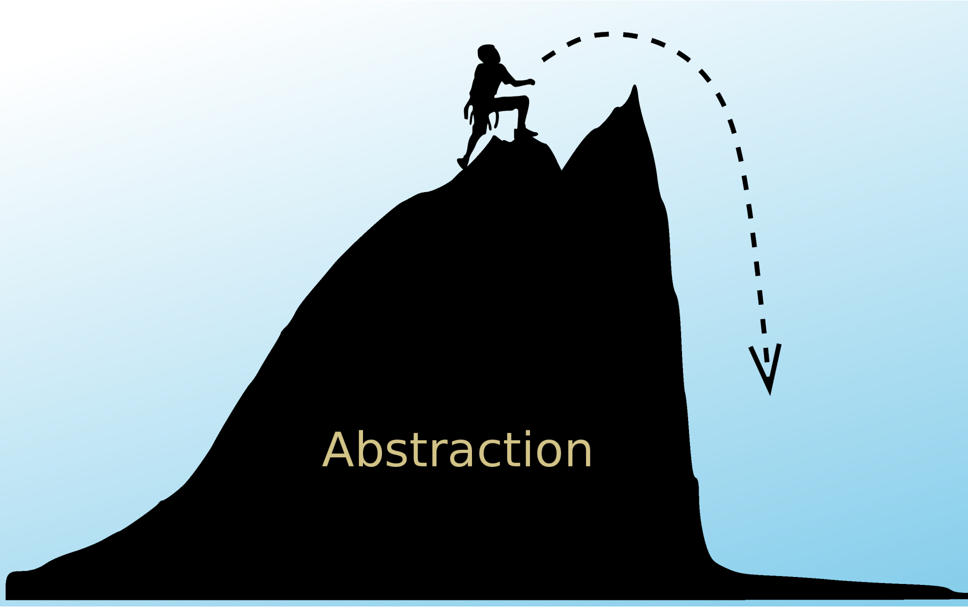

4. Low-tech corollary

(with John Baez and Tobias Fritz)

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So, an internal Δ-algebra in $(\mathbb{R}, +, 0)$ is a functor **FinProb** $\longrightarrow (\mathbb{R}, +, 0)$ satisfying certain axioms.

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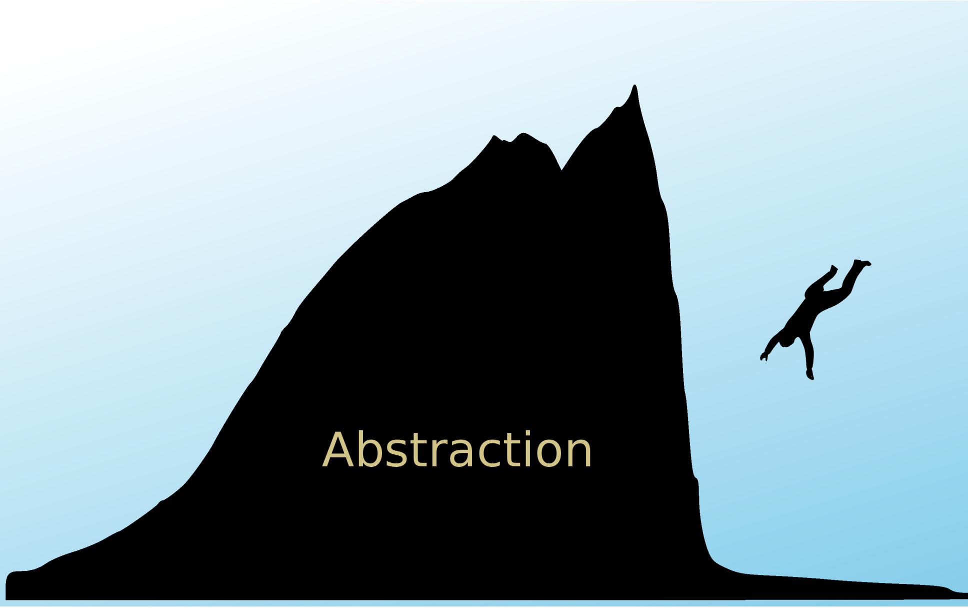

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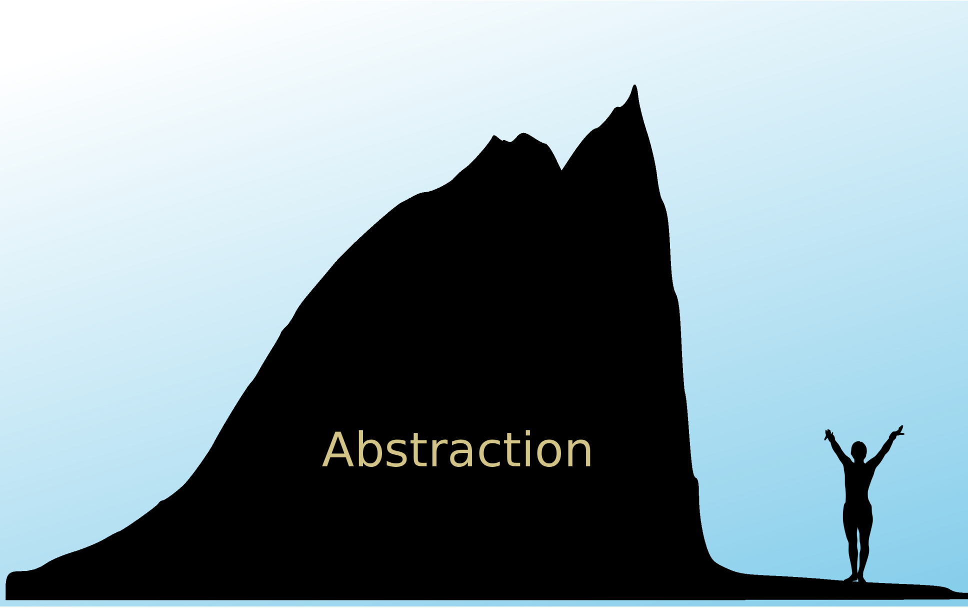

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Then there is some constant $c \in \mathbb{R}$ *such that*

$$L\Big((X, \mathbf{p}) \xrightarrow{f} (Y, \mathbf{q})\Big) = c \cdot \big(H(\mathbf{p}) - H(\mathbf{q})\big)$$

for all f.

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