

The categorical origins of entropy

Tom Leinster

University of Edinburgh

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Maximum entropy on metric spaces

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Maximum entropy on metric spaces

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Tom Leinster and Mark Meckes, **Maximizing diversity in biology and beyond**, *Entropy* **18** (2016), 88.

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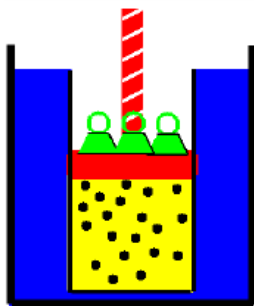
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Entropy in mathematical sciences

Entropy in mathematical sciences



Entropy of a Gas



S = Entropy

Q = Heat Transfer

T = Temperature

2nd Law: $S_2 - S_1 = \frac{\Delta Q}{T}$

Differential 2nd Law: $dS = \frac{dQ}{T}$

H = Enthalpy V = Volume p = Pressure

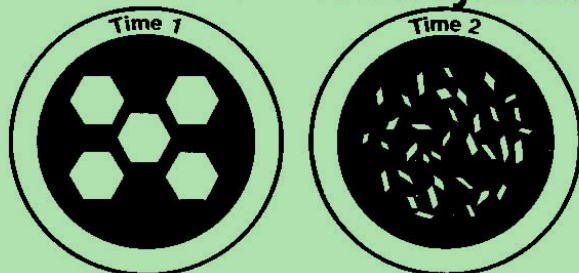
Differential 1st Law: $dQ = dH - V dp$

C_p = Heat Capacity
(constant pressure)

R = Gas Constant

Entropy in mathematical sciences

Second Law of Thermodynamics



ENTROPY (simplicity) increases
in closed system



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(constant pressure)

R = Gas Constant

Heat Transfer

$$Q = \frac{\Delta Q}{T}$$

$$= \frac{dQ}{T}$$

Pressure

$$dH = V dp$$

Entropy in mathematical sciences

Entropy Approach to the Investigation of Information Capabilities of Adaptive Radio Engineering System in Conditions of Intrasystem Uncertainty

V. V. Skachkov,* V. V. Chepkyi,** H. D. Bratchenko,*** and A. N. Efymchykov

Odessa State Academy of Technical Regulation and Quality, Odessa, Ukraine

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Minimum entropy control of nonlinear ARMA systems over a communication network

Jianhua Zhang · Hong Wang

Heat Tra

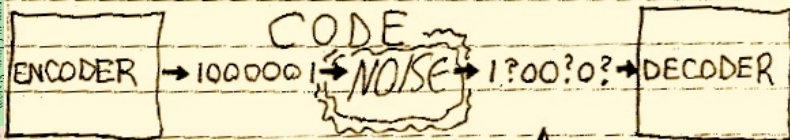
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c_p - heat capacity
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Entropy in mathematical sciences

NOISE AND INFORMATION ENTROPY



- LOW ENTROPY
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- LOW DEGREE OF UNCERTAINTY

- HIGH ENTROPY
- LOW DEGREE OF CERTAINTY
- HIGH DEGREE OF UNCERTAINTY

Entropy in mathematical sciences

NOISE AND INFORMATION ENTROPY

Entropy, the Central Limit Theorem and the Algebra of the Canonical Commutation Relation

DÉNES PETZ

Mathematical Institute HAS, H-1364 Budapest, PF 127, Hungary

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Entropy, the
Algebra of

DÉNES PETZ
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BAYESIAN INFERENCE AND MAXIMUM ENTROPY METHODS IN SCIENCE AND ENGINEERING

Proceedings of the 30th International Workshop
on Bayesian Inference and Maximum Entropy
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4 - 9 July 2010 Chamonix, France

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Entropy in mathematical sciences

Probabilistic Knowledge Representation and Reasoning at Maximum Entropy by SPIRIT

Carl-Heinz Meyer, Wilhelm Rödder
FernUniversität Hagen

Proceedings of the 30th International Workshop
on Bayesian Inference and Maximum Entropy
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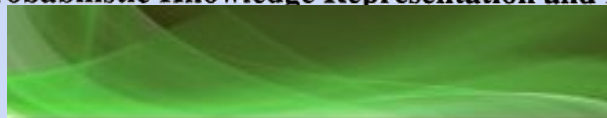
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Entropy in mathematical sciences

PROBABILISTIC KNOWLEDGE REPRESENTATION AND REASONING



Muhammad Ali
Abdul Haseeb
Muhammad Bilal Bhatti

Artificial Intelligence

Design and Implementation of Entropy Based
Artificially Immune Malware Detection System

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Entropy in mathematical sciences

Maximum Entropy and Ecology

A Theory of Abundance, Distribution, and Energetics

John Harte



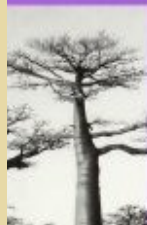
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Maximum Entropy and Ecology

A Theory of Abundance, Distribution, and Energetics

Impact of a Change of Support on the Assessment of Biodiversity with Shannon Entropy

Didier Josselin¹, Ilene Mahfoud¹, Bruno Fady²



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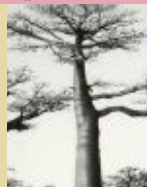
Maximum Entropy and Ecology

A Theory of Abundance, Distribution, and Speciation

An entropic characterization of protein interaction networks and cellular robustness

Thomas Manke, Lloyd Demetrius, Martin Vingron

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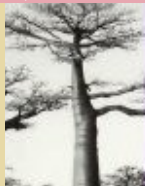
Roman E. Nalewajski

Reduced communication channels of molecular fragments and their entropy/information bond indices

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Resilience and entropy as indices of robustness of water distribution networks

R. Greco, A. Di Nardo and G. Santonastaso

7 Algorithmic entropy and Kolmogorov complexity

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Resilience and entropy as indices of robustness of water distribution networks

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7 Algorithmic entropy and Kolmogorov complexity

Entropy and Quantum Kolmogorov Complexity: A Quantum Brudno's Theorem

Fabio Benatti¹, Tyll Krüger^{2,3}, Markus Müller², Rainer Siegmund-Schultze²,
Arleta Szkoła²

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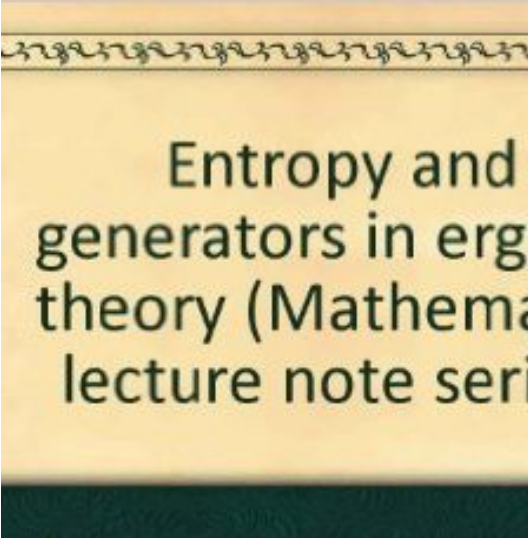
Entropy and
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Entropy and
complexity

complexity:

Siegmund-Schultze²,

Entropy in mathematical sciences



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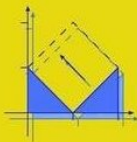
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Graham Everest and Thomas Ward

HEIGHTS OF POLYNOMIALS AND ENTROPY IN ALGEBRAIC DYNAMICS



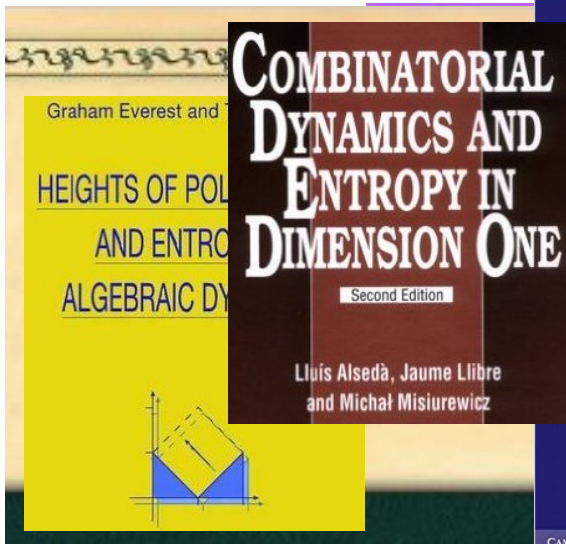
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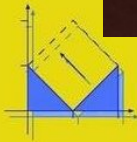
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Graham Everest and

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Tímea Haszpra

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Entropy in Dynamical Systems

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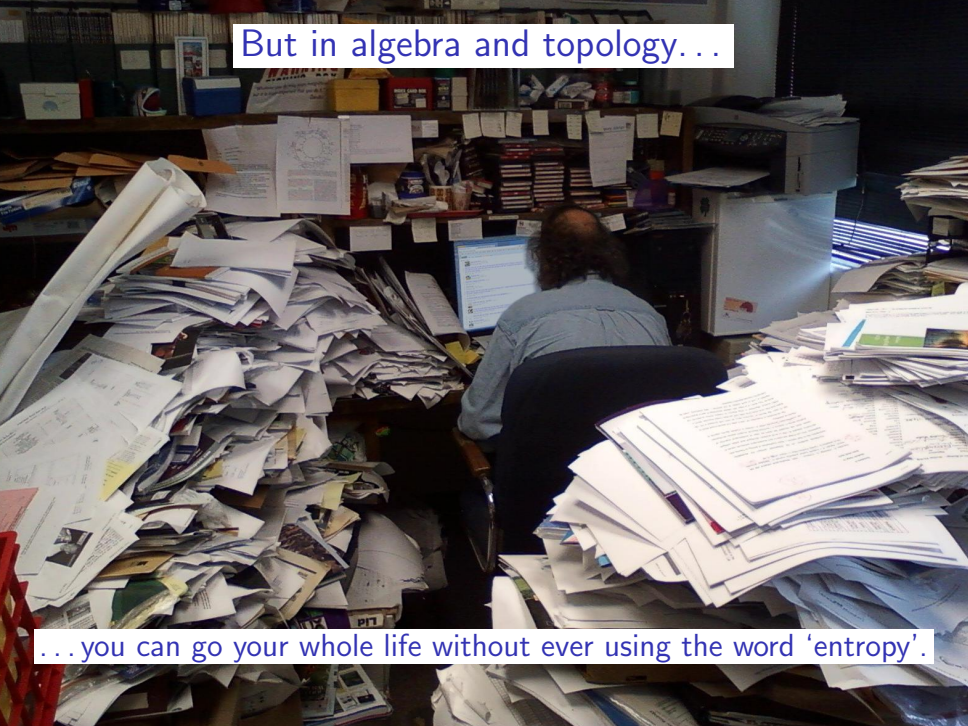
gation of chaotic
in the atmosphere:
f topological entropy

Tímea Haszpra

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But in algebra and topology...

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... you can go your whole life without ever using the word 'entropy'.

The point of this talk

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Entropy is notable by its relative absence from algebra and topology.

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However, we will see that entropy is *inevitable* in these subjects:
it is there whether we like it or not.

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We'll do this using some category theory. . .

What is category theory?

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- starting about half-way through: category, functor, natural transformation, monoidal category.

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What category theory I'll need in this talk:

- starting about half-way through: category, functor, natural transformation, monoidal category.
- right at the end: a bit more.

The point of this talk

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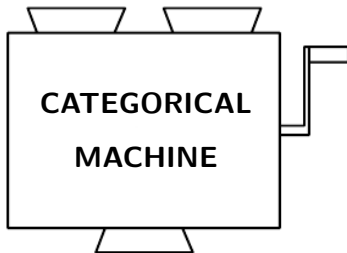


Image: J. Kock

The point of this talk

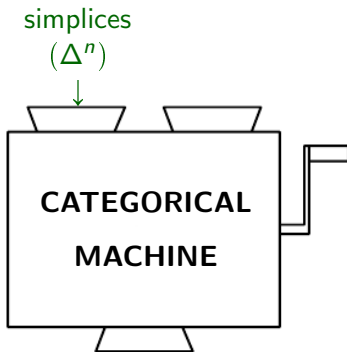


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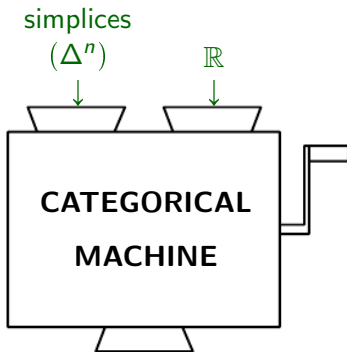


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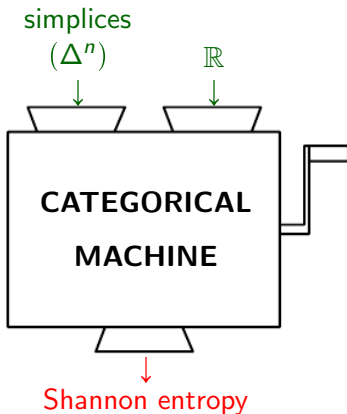


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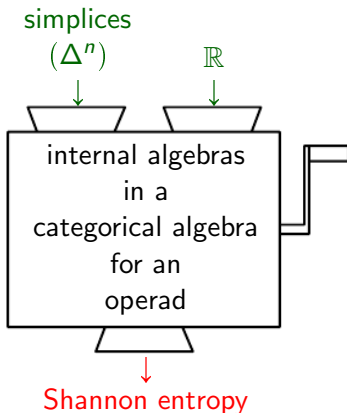


Image: J. Kock

Plan

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1. Operads and their algebras

Plan

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2. Internal algebras

Plan

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2. Internal algebras
3. The theorem

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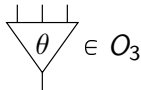
1. Operads and their algebras
2. Internal algebras
3. The theorem
4. Low-tech corollary

1. Operads and their algebras

The definition of operad

The definition of operad

An **operad** O is a sequence $(O_n)_{n \in \mathbb{N}}$ of sets

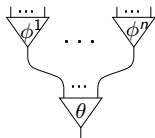
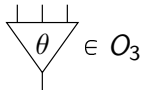


The definition of operad

An **operad** O is a sequence $(O_n)_{n \in \mathbb{N}}$ of sets together with:

- (i) **composition**: for each $k, n_1, \dots, n_k \in \mathbb{N}$, a function

$$\begin{aligned} O_n \times O_{k_1} \times \dots \times O_{k_n} &\longrightarrow O_{k_1 + \dots + k_n} \\ (\theta, \phi^1, \dots, \phi^n) &\longmapsto \theta \circ (\phi^1, \dots, \phi^n) \end{aligned}$$



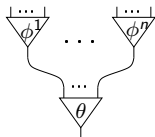
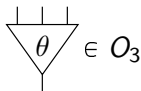
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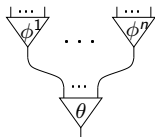
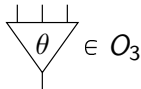
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satisfying associativity and unit axioms.



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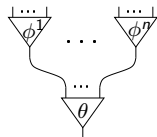
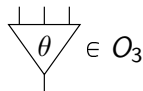
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(Then every tree has a unique composite.)



Examples of operads

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then

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$$\begin{array}{ccc} \bar{\mathbf{p}}: & A^n & \longrightarrow A \\ & (\mathbf{a}^1, \dots, \mathbf{a}^n) & \longmapsto \sum_i p_i \mathbf{a}^i. \end{array}$$

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It is a categorical Δ -algebra via convex combinations, as defined above.

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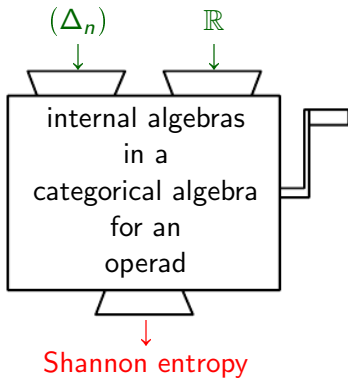
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Explicitly, this means that throughout, we add a condition

(iii) continuity

to the conditions (i) and (ii) that appear repeatedly.



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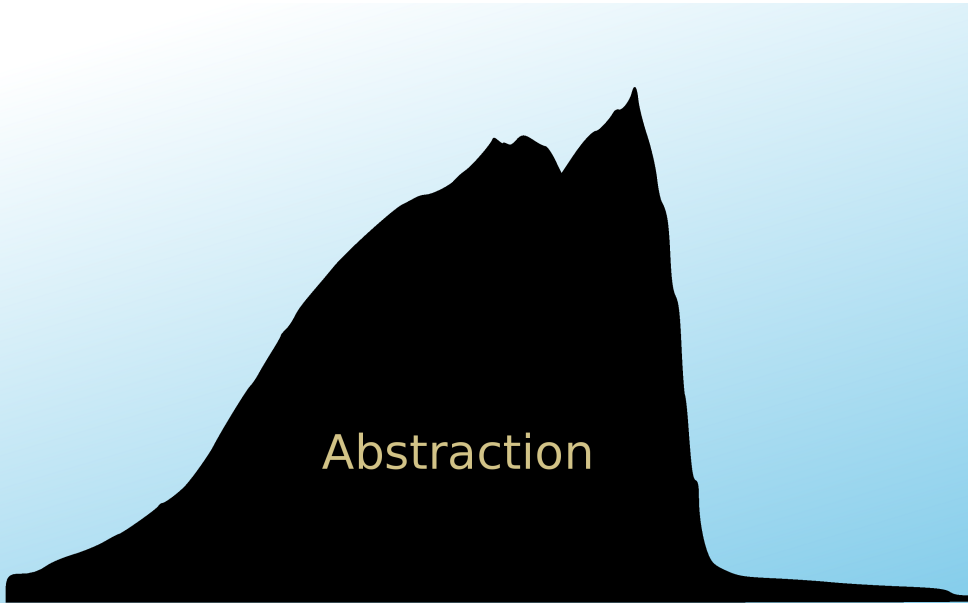
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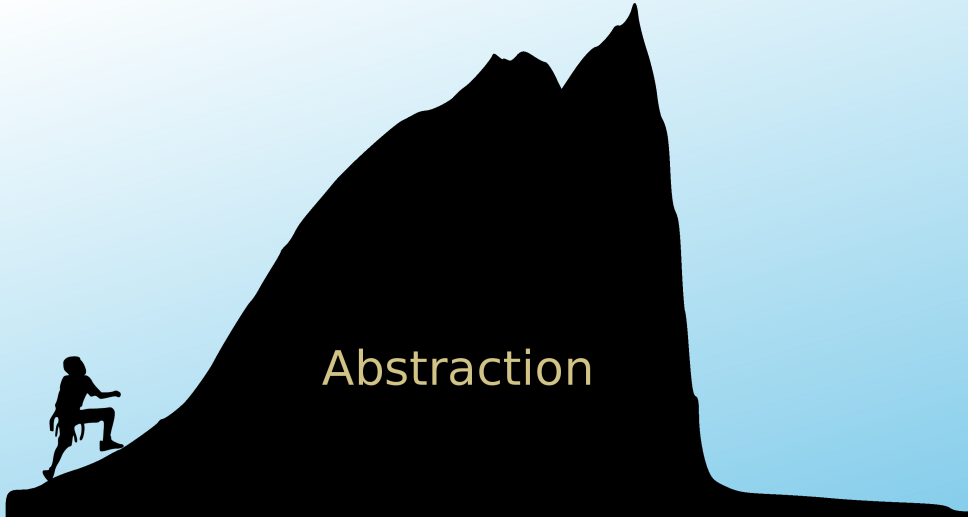
Proof: This explicit form is almost equivalent to a 1956 theorem of Faddeev, except that he also imposed a symmetry axiom (which here is redundant). $\square$

Pause: where are we?

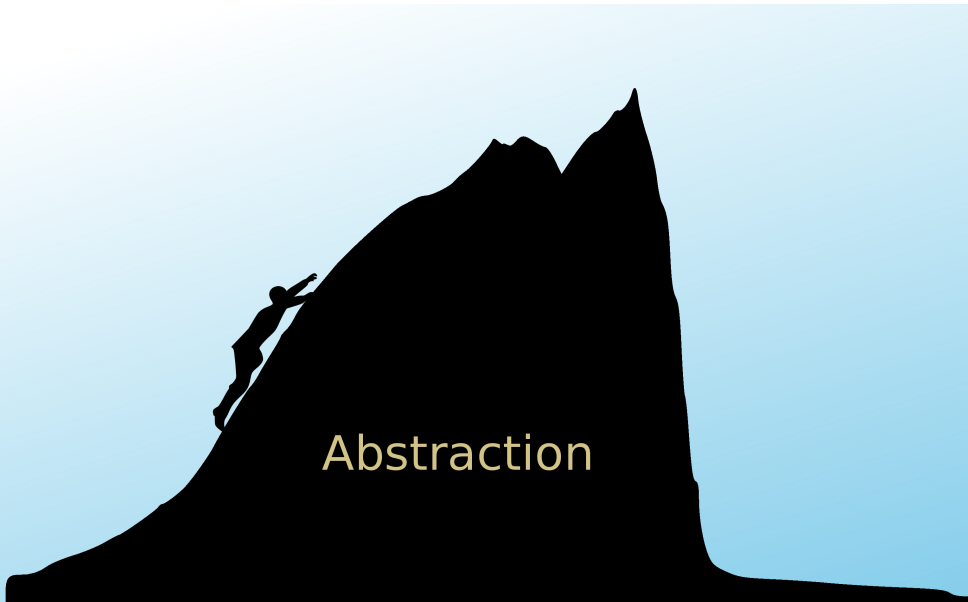


Abstraction

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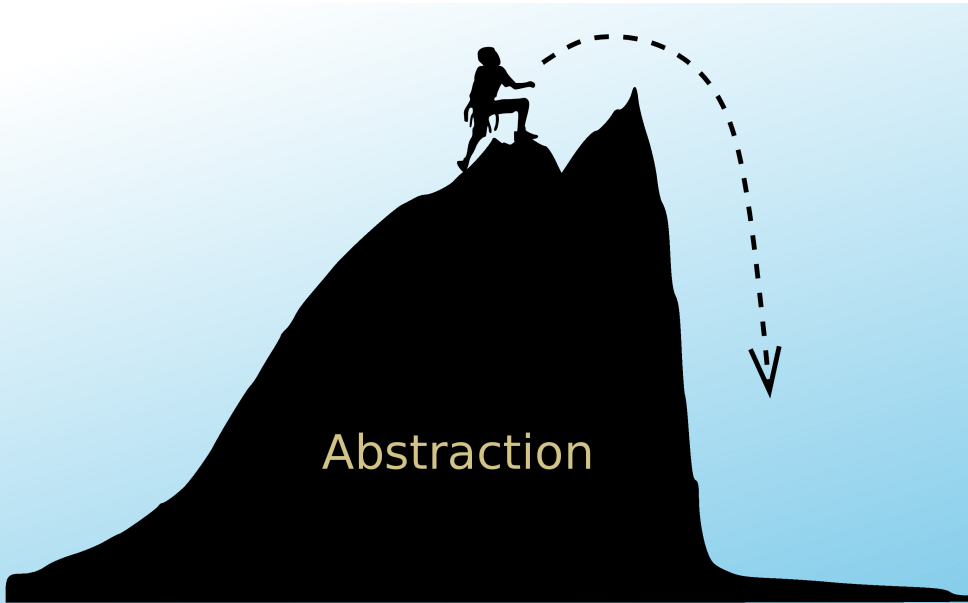
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4. Low-tech corollary

(with John Baez and Tobias Fritz)

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So, an internal Δ -algebra in $(\mathbb{R}, +, 0)$ is a functor **FinProb** $\longrightarrow (\mathbb{R}, +, 0)$ satisfying certain axioms.

Where are we now?

Where are we now?

Abstraction

A black silhouette of a person climbing a jagged mountain peak. The person is positioned near the top right of the mountain, using a long pole or rope for support. The mountain itself is a large, dark shape with a prominent peak. The background is a light blue gradient.

Where are we now?



Where are we now?

Abstraction

A conceptual illustration featuring a large, jagged black silhouette of a mountain against a light blue background. The word "Abstraction" is written in a yellow, sans-serif font across the lower-middle section of the mountain. To the right of the mountain, a small black silhouette of a person is shown in mid-air, having fallen off the edge of the mountain. The person's arms and legs are outstretched, suggesting a fall. The overall image serves as a metaphor for the dangers of becoming too abstract.

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Abstraction



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Then there is some constant $c \in \mathbb{R}$ such that

$$L\left((X, \mathbf{p}) \xrightarrow{f} (Y, \mathbf{q})\right) = c \cdot (H(\mathbf{p}) - H(\mathbf{q}))$$

for all f .

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This gives a new and entirely explicit characterization of Shannon entropy.