Information geometry and shape analysis for radar signal processing

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Topological and Geometrical Structures of Information
31 August 2017

Main goal

Use shape analysis to improve statistical processing of radar signals

Since a locally stationary radar signal can be represented by a curve in a manifold using **information geometry**,

how can we perform **statistics on these curves**, and thereby the signals they represent ?

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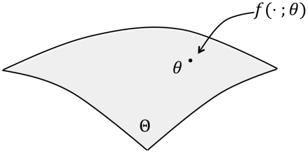
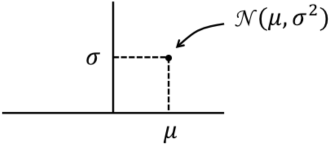
Information geometry

- Family of probability densities $\{f(\cdot,\theta),\theta\in\Theta\}$
  Each $f(\cdot,\theta)$ is represented by parameter θ in the parameter space Θ



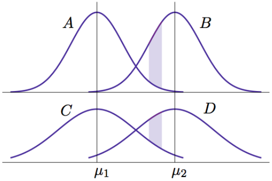
- e.g. Gaussian distribution $\mathcal{N}(\mu,\sigma^2)$ can be represented in the upper half-plane

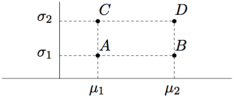


Fisher metric

Riemannian manifold structure on the space of parameters

- The Euclidean metric is not a good choice in general





[Costa, Santos, Strapasson, 2014]

$$d(A,B) > d(C,D)$$

- The Fisher metric is used instead

$$g_{ij}(\theta) = I(\theta)_{ij} = \mathbb{E}_\theta\left[\left(\frac{\partial}{\partial\theta_i}\ln f(X;\theta)\right)\left(\frac{\partial}{\partial\theta_j}\ln f(X;\theta)\right)\right] \qquad \theta = (\theta_1, \ldots, \theta_p)$$

In parametric estimation, the Fisher information $I(\theta)$

- measures the « quantity of information » contained in the data
- limits the precision with which one can estimate θ (Cramer-Rao bound)

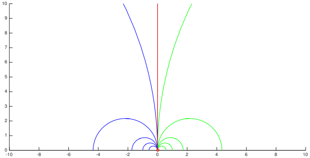
Statistical manifold

Parameter space Θ + Fisher metric = statistical manifold

e.g. for univariate Gaussian distributions $\mathcal{N}(\mu, \sigma^2)$,

Fisher geometry $\Leftrightarrow$ hyperbolic geometry.

The space of parameters (μ, σ) equipped with the Fisher metric is in bijection with the hyperbolic upper half-plane via the change of variables $(\mu, \sigma) \mapsto (\frac{\mu}{\sqrt{2}}, \sigma)$.



$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

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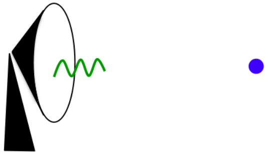
Radar setting



n pulses $\rightarrow$ n echoes $\rightarrow$ 1 vector of observations $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$

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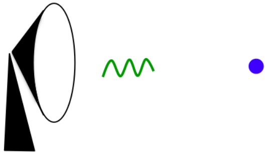

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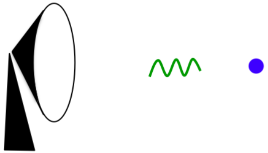

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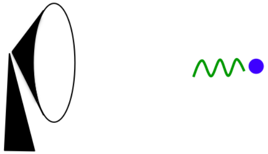

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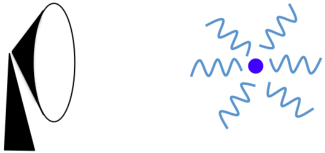

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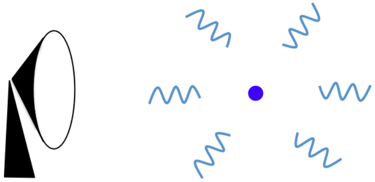

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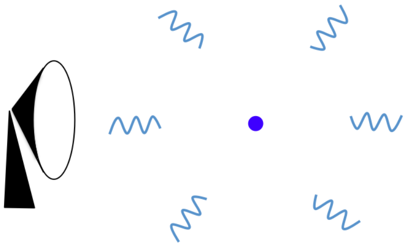

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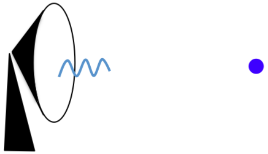

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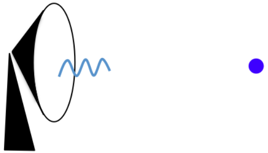

n pulses $\rightarrow$ n echoes $\rightarrow$ 1 vector of observations $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$

Stationary signal

If z is a realization of a stationary centered Gaussian vector $Z = (Z_1, \dots, Z_n)$

- Z entirely described by its covariance matrix Σ hermitian positive definite and Toeplitz

$$\Sigma = \begin{pmatrix} r_0 & r_1 & \cdots & r_{n-1} \\ \overline{r_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_1 \\ \overline{r_{n-1}} & \cdots & \overline{r_1} & r_0 \end{pmatrix}, \qquad r_k = \mathbb{E}(Z_i \overline{Z_{i+k}}) \quad k = 0, \dots, n-1.$$

- Space of parameters = $\mathcal{T}_n^+ = \{$ HPDT matrices $\}$
- Metric on $\mathcal{T}_n^+$ defined using the hessian of the entropy

$$ds^2 = -d\Sigma^* \mathrm{Hess}\, H(\Sigma) d\Sigma,$$

$$\text{with} \quad H(\Sigma) := -\mathbb{E}_\Sigma\big(\ln f(Z; \Sigma)\big) = n \ln(\pi e) + \ln(\det \Sigma).$$

- Coincides with the Fisher metric on $\mathcal{H}_n^+ = \{$ HPD matrices $\}$ but not on $\mathcal{T}_n^+$.

Stationary signal

- Equivalent coordinate system : reflection coefficients [Burg 1967]

$$\text{bijection} \quad \Phi : \quad \mathcal{T}_n^+ \to \mathbb{R}_+^* \times D^{n-1}, \qquad \text{where} \quad D = \{z \in \mathbb{C}, |z| < 1\}$$
$$\Sigma \mapsto (P_0, \mu_1, \ldots, \mu_{n-1}).$$

The coefficients $(P_0, \mu_1, \ldots, \mu_{n-1})$ are associated to $n-1$ AR models that maximize the entropy under the autocorrelation constraints given by Σ.

- The entropic metric becomes a product metric in $\mathbb{R}_+^* \times D^{n-1}$ [Barbaresco 2008]

$$ds^2 = n\left(\frac{dP_0}{P_0}\right)^2 + \sum_{k=1}^{n-1}(n-k)\frac{|d\mu_k|^2}{(1-|\mu_k|^2)^2}.$$

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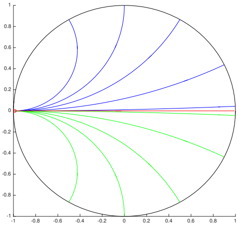
- 2 equivalent parameter spaces :
  - $\mathcal{T}_n^+$ equipped with $\quad ds^2 = -d\Sigma^* \text{Hess}\, H(\Sigma) d\Sigma$
  - The Poincaré polydisk $\quad \mathbb{R}_+^* \times \mathbb{D}^{n-1} = (\mathbb{R}_+^*, ds_0^2) \times (\mathbb{D}, ds_1^2) \times \ldots \times (\mathbb{D}, ds_{n-1}^2)$

Stationary signal

We choose the second representation :

$$\boxed{\text{1 stationary signal} \quad \longleftrightarrow \quad \text{1 point in the Poincaré polydisk} \quad \mathbb{R}_+^* \times \mathbb{D}^{n-1}}$$



$\mathbb{D}$ = Poincaré disk

Locally stationary signal

If z is a realization of a *locally* stationary centered Gaussian vector $Z = (Z_1, \dots, Z_N)$

- We decompose z in stationary portions
- For each stationary portion we estimate a covariance matrix
  $\leftrightarrow$ 1 point in the polydisk
- z is represented by a time series of covariance matrices
  $\leftrightarrow$ time series in the Poincaré polydisk.

1 locally stationary signal $\longleftrightarrow$ 1 (discrete) curve in the Poincaré polydisk

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_i \\ \vdots \\ z_{i+n} \\ \vdots \\ z_N \end{bmatrix} \rightarrow (P_0(t), \mu_1(t), \dots, \mu_{n-1}(t))$$

Motivation

Perform **statistics on locally stationary radar signals**

- Classification for target recognition
- Statistical tests for target detection

by exploiting the **shapes of the curves** that represent them in the Poincaré polydisk.

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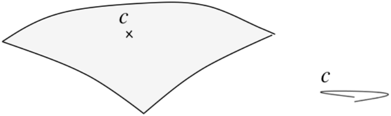

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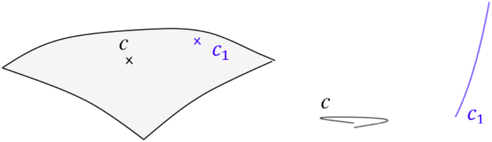

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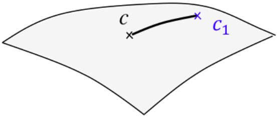

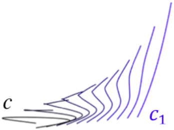

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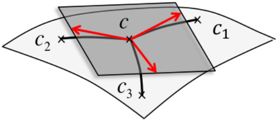

- geodesic $\leftrightarrow$ optimal deformation between 2 curves
- allows us to locally linearize around a curve (tangent space) $\rightarrow$ statistics in a flat space

Notations

$$\mathcal{M} = \{ \text{ Parameterized curves in a Rm. manifold } (M, \langle \cdot, \cdot \rangle) \text{ with velocity that never vanishes } \}$$

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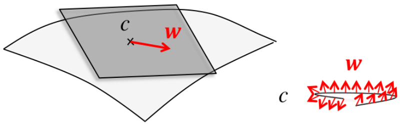

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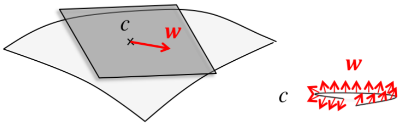

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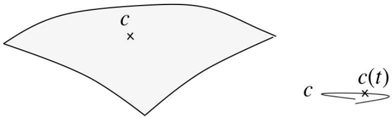

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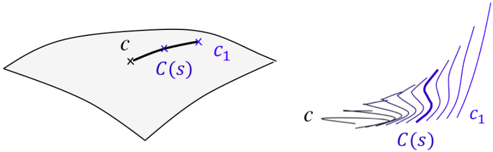

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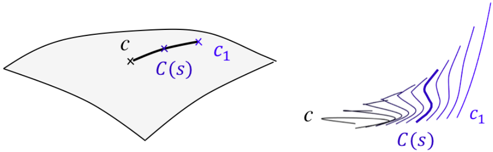

t : parameter of the curves
s : parameter of the paths of curves.

$$c_t := \frac{\partial c}{\partial t}, \quad \nabla_t c_t := \nabla_{c_t} c_t.$$

Reparameterization invariance

We equip $\mathcal{M}$ with a Riemannian metric G

$$G_c : T_c\mathcal{M} \times T_c\mathcal{M} \to \mathbb{R}, \quad (w,z) \mapsto G_c(w,z), \qquad c \in \mathcal{M}$$

It induces a distance on $\mathcal{M}$

$$\mathrm{dist}(c_0, c_1) = \inf_{c(0)=c_0, c(1)=c_1} L(c), \qquad \text{avec} \quad L(c) = \int_0^1 \|c_s(s)\|_G \, \mathrm{d}s.$$

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If G is reparameterization invariant

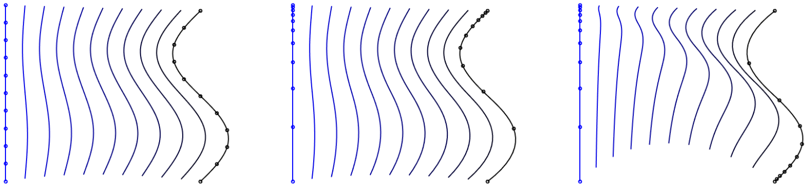
$$G_{c\circ\varphi}(w\circ\varphi, z\circ\varphi) = G_c(w, z), \qquad \forall \varphi \in \mathrm{Diff}^+([0,1]),$$

then the distance between two curves does not change if we reparameterize them the same way

$$\mathrm{dist}(c_0\circ\varphi, c_1\circ\varphi) = \mathrm{dist}(c_0, c_1), \quad \forall\varphi$$

but it does change if we reparameterize them in different ways !

Reparameterization invariance



$\mathrm{dist}(c_0, c_1) \approx 7.17$ $\mathrm{dist}(c_0 \circ \varphi, c_1 \circ \varphi) \approx 7.16$ $\mathrm{dist}(c_0 \circ \varphi, c_1 \circ \psi) \approx 7.48$

Shape principal bundle

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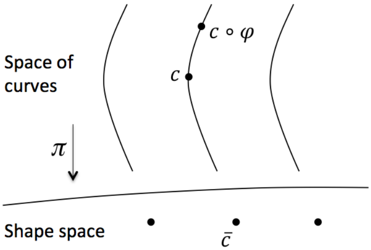
Principal bundle structure $\pi : \mathcal{M} \rightarrow \mathcal{S}$



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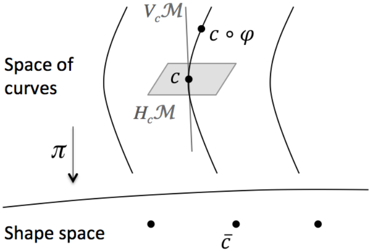
Principal bundle structure $\pi : \mathcal{M} \rightarrow \mathcal{S} \quad \Rightarrow$ Decomposition of tangent space :

$$T_c\mathcal{M} = V_c\mathcal{M} \oplus H_c\mathcal{M} \quad \text{with} \quad V_c\mathcal{M} = \ker(T_c\pi),\ H_c\mathcal{M} = (V_c\mathcal{M})^{\perp_G}$$



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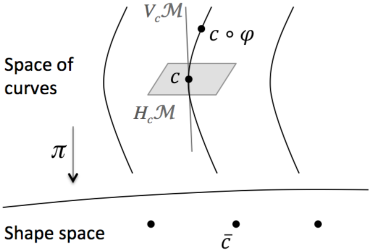

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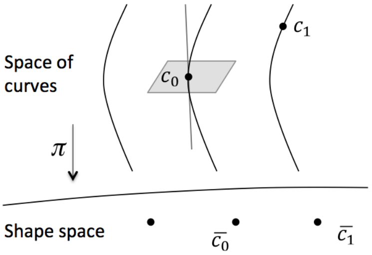
Induced distance on $\mathcal{S}$: $\quad \mathrm{dist}_{\mathcal{S}}(\bar{c}_0, \bar{c}_1) = \inf_{\varphi \in \mathrm{Diff}^+([0,1])} \mathrm{dist}(c_0, c_1 \circ \varphi).$



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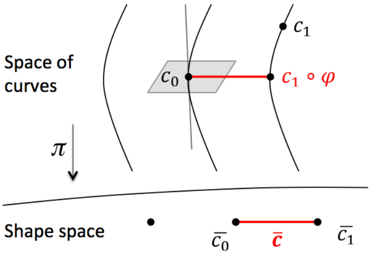
Induced distance on $\mathcal{S}$: $\quad \text{dist}_{\mathcal{S}}(\bar{c}_0, \bar{c}_1) = \inf_{\varphi \in \text{Diff}^+([0,1])} \text{dist}(c_0, c_1 \circ \varphi).$



Reparameterization invariant metrics ($M = \mathbb{R}^d$)

- The L^2 metric induces a zero distance on the shape space [Michor, Mumford, 2006]

$$G_c^0(w,z) = \int_0^1 \langle w(t), z(t) \rangle |c'(t)| \mathrm{d}t \qquad |\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$$

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$$\text{e.g.} \qquad G_c^1(w,z) = \int \langle w, z\rangle + \langle D_\ell w, D_\ell z\rangle \,\mathrm{d}\ell \qquad \begin{array}{l} D_\ell w = w'/|c'| \\ d\ell = |c'(t)|\mathrm{d}t \end{array}$$

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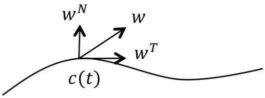
- Different weights in front of the tangential and normal components : *elastic metrics* [Mio, Srivastava, Joshi 2006]

$$G_c^{a,b}(w,z) = \int a^2 \langle D_\ell w^N, D_\ell z^N \rangle + b^2 \langle D_\ell w^T, D_\ell z^T \rangle \, \mathrm{d}\ell,$$

$D_\ell w^T = \langle D_\ell h, v \rangle v$ with $v = c'/|c'|$,
$D_\ell w^N = D\ell w - D_\ell w^T$,

a : degree "bending" of the curve,
b : degree of "stretching".



The Square Root Velocity Function

- Elastic metric for $a = 1$, $b = 1/2$ and $M = \mathbb{R}^d$

$$G_c^{1,\frac{1}{2}}(w, w) = \int \left(|D_\ell w^N|^2 + \frac{1}{4}|D_\ell w^T|^2 \right) \mathrm{d}\ell,$$

is flat [Srivastava,Klassen,Joshi,Jermyn'11], [Younes'98] :

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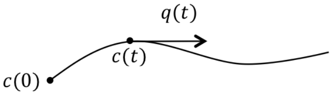

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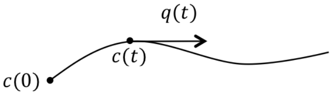

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- Extension to any metric $G^{a,b}$ with $4b^2 \geq a^2$ [Bauer, Bruveris, Marsland, Michor 2012]
- Extension to curves in a manifold using parallel transport [Zhang, Su, Klassen, Le, Srivastava 2015], [Le Brigant, Arnaudon, Barbaresco 2015]
- Extension to curves in a Lie group [Celledoni, Eslitzbichler, Schmeding 2016]

Table of contents

Our generalization of the SRV framework

$(M, \langle\cdot,\cdot\rangle, \nabla)$ Rm. manifold. We consider the metric

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$\mathrm{d}\ell = |c'|\mathrm{d}t,\ \nabla_\ell w = \frac{\nabla_t w}{|c'|},\ |\cdot| = \sqrt{\langle\cdot,\cdot\rangle},$
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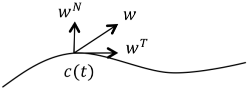

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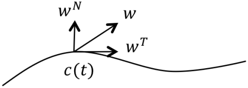

Proposition

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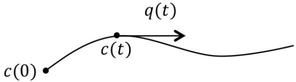

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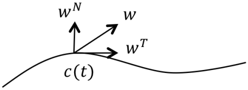

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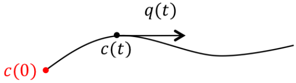

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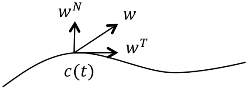

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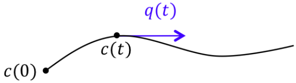


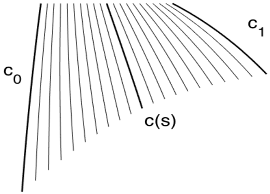
$$q(t) = \frac{c'(t)}{\sqrt{|c'(t)|}}$$

- Term of order 0 measures the difference of position between the curves
- Integral measures the difference between the velocities.

Geodesic equation

Squared norm of the speed of a path of curves $s \mapsto c(s)$:

$$\|c_s(s)\|_G^2 = G(c_s(s), c_s(s)) = |c_s(s,0)|^2 + \int_0^1 |\nabla_s q(s,t)|^2 \mathrm{d}t$$



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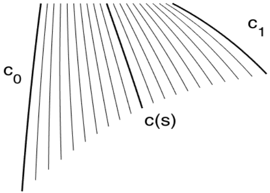
The geodesics are (locally) the length-minimizing paths and the critical points of the energy

$$L(c) = \int_0^1 \|c_s(s)\|_G \,\mathrm{d}s$$
$$E(c) = \int_0^1 \|c_s(s)\|_G^2 \,\mathrm{d}s$$



Geodesic equation

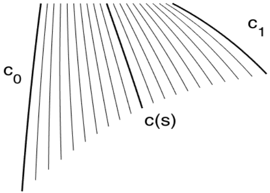
Proposition (Geodesic equation)

The shortest paths are those that verify

$$\begin{aligned} \nabla_s c_s(s,0) + r(s,0) &= 0, \quad \forall s \\ \nabla_s^2 q(s,t) + |q(s,t)| \left(r(s,t) + r(s,t)^T \right) &= 0, \quad \forall t, s \end{aligned}$$

where r depends on the curvature tensor $\mathcal{R}$ of M

$$r(s,t) = \int_t^1 \mathcal{R}(q, \nabla_s q) c_s(s,\tau)^{\tau,t} \mathrm{d}\tau.$$

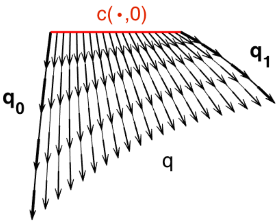


Geodesic equation

In the zero curvature case, we recover

$$\nabla_s c_s(s,0) = 0, \quad \forall s,$$
$$\nabla_s^2 q(s,t) = 0, \quad \forall t, s,$$

i.e. $c(s,\cdot)$ is a straight line between the origins and q is a linear interpolation between q_0 and q_1.

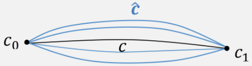


Geodesic equation

Proof

We are looking for the path $s \mapsto c(s)$ in which the derivative of the energy vanishes, i.e. s.t.

$$T_c E(w) = 0 \quad \forall w \quad \Leftrightarrow \quad \frac{d}{da}\Big|_{a=0} E(\hat{c}(a)) = 0 \text{ for any variation } \hat{c} : (-\varepsilon, \varepsilon) \to \mathcal{M},$$
$$\hat{c}(0,s,t) = c(s,t), \quad \hat{c}(a,0,t) = c_0(t), \quad \hat{c}(a,1,t) = c_1(t).$$



$$E(\hat{c}(a)) = \frac{1}{2}\int \langle \hat{c}_s(a,s,0), \hat{c}_s(a,s,0)\rangle \mathrm{d}s + \int\int \langle \nabla_s \hat{q}(s,t), \nabla_s \hat{q}(s,t)\rangle \mathrm{d}t\,\mathrm{d}s,$$
$$\frac{d}{da}E(\hat{c}(a)) = \int \langle \nabla_a \hat{c}_s(a,s,0), \hat{c}_s(a,s,0)\rangle\,\mathrm{d}s + \int\int \langle \nabla_a \nabla_s \hat{q}(a,s,t), \nabla_s \hat{q}(a,s,t)\rangle\,\mathrm{d}t\,\mathrm{d}s.$$

Geodesic equation

Proof

This can be rewritten for $a = 0$

$$\int_0^1 \langle \nabla_s c_s(s,0) + r(s,0) \, , \, \hat{c}_a(0,s,0) \, \rangle \, \mathrm{d}s$$
$$+ \int_0^1 \int_0^1 \langle \, \nabla_s \nabla_s q(s,t) + |q(s,t)| \left(r(s,t) + r(s,t)^T \right) \, , \, \nabla_a \hat{q}(0,s,t) \, \rangle \, \mathrm{d}t \, \mathrm{d}s = 0,$$

with $r(s,t) = \int_t^1 \mathcal{R}(q, \nabla_s q) c_s(s,\tau)^{\tau,t} \mathrm{d}\tau.$

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$\rightarrow$ Vanishes for any value of $\hat{c}_a(0,s,0)$ and $\nabla_a \hat{q}(0,s,t)$

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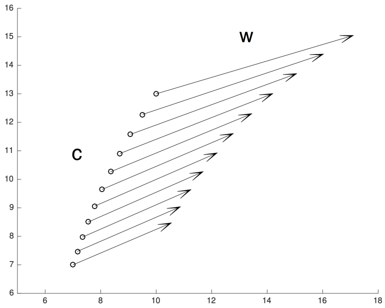
$\rightarrow$ We obtain

$$\begin{cases} \nabla_s c_s(s,0) + r(s,0) = 0 \quad \forall s, \\ \nabla_s \nabla_s q(s,t) + |q(s,t)|\left(r(s,t) + r(s,t)^T\right) = 0 \quad \forall t, s. \end{cases}$$

Constructing geodesics

Exponential map : gives the geodesic starting from c at speed w



Simulation in the hyperbolic upper half-plane

We numerically solve the geodesic equation

$$c(s+\varepsilon,t) = \exp^M_{c(s,t)}(\varepsilon c_s(s,t))$$

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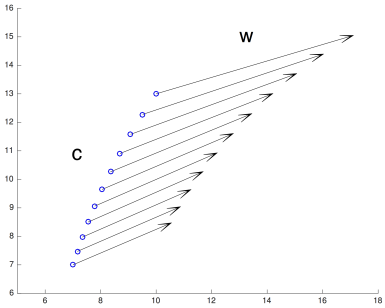

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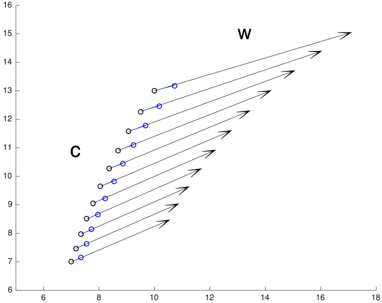

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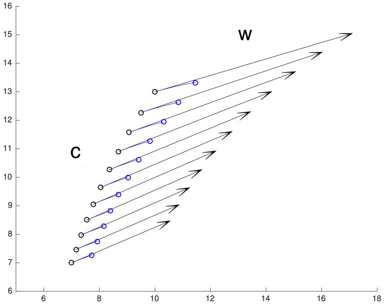

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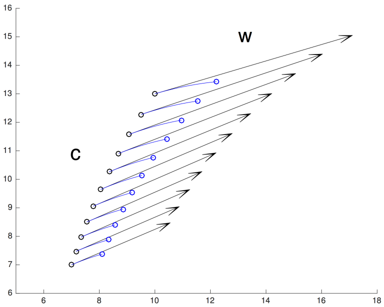

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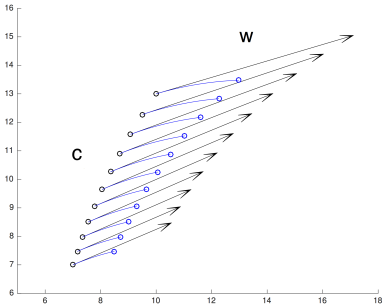

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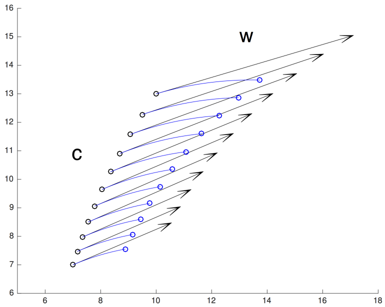

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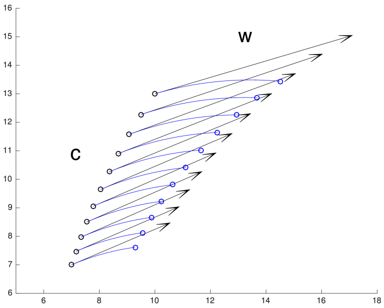

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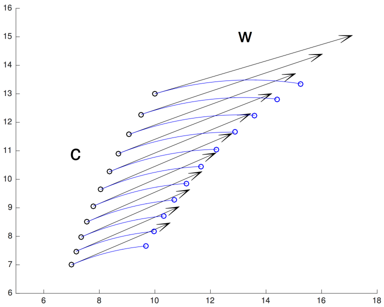

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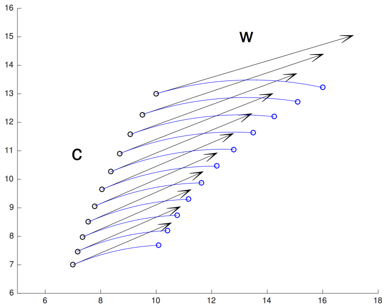

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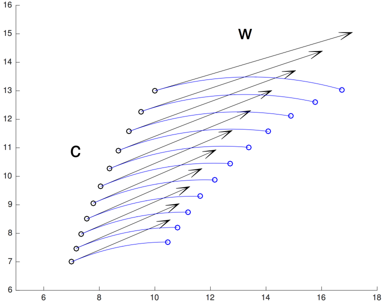

Simulation in the hyperbolic upper half-plane

We numerically solve the geodesic equation

$$c(s+\varepsilon,t) = \exp^M_{c(s,t)}\left(\varepsilon c_s(s,t)\right)$$

$$c_s(s+\varepsilon,t) = \left(c_s(s,t) + \varepsilon \nabla_s c_s(s,t)\right)^{s,s+\varepsilon}$$

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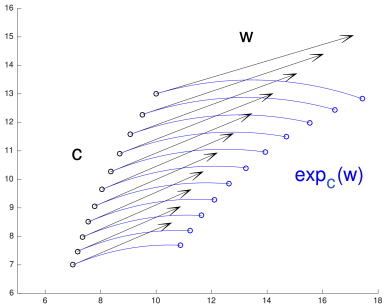

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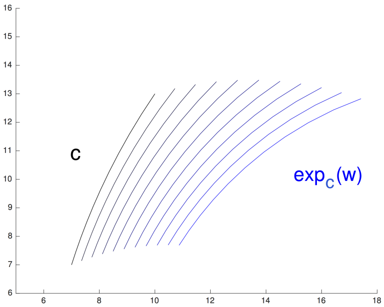

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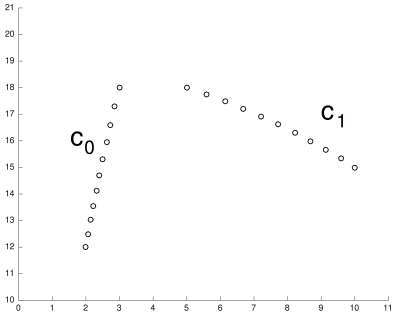

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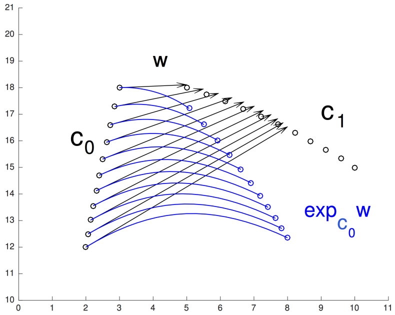

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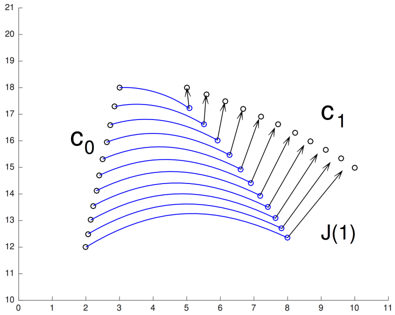

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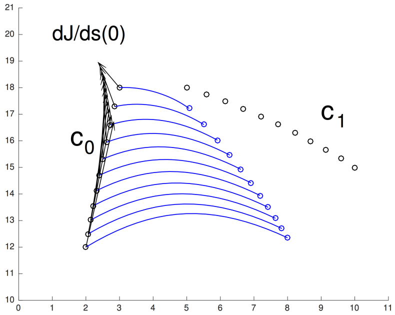

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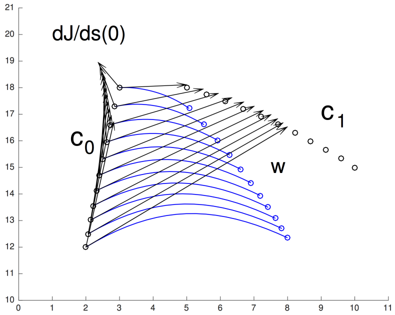

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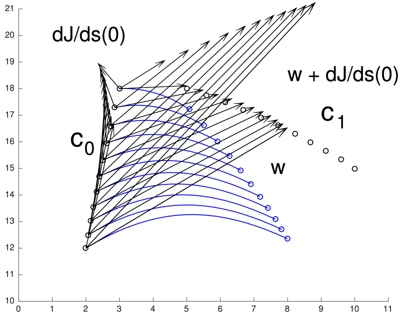

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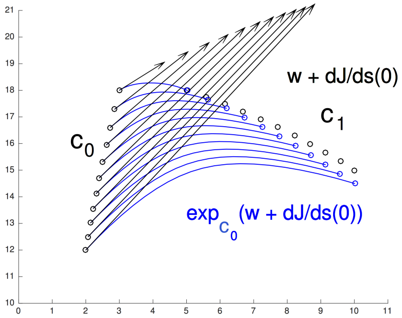

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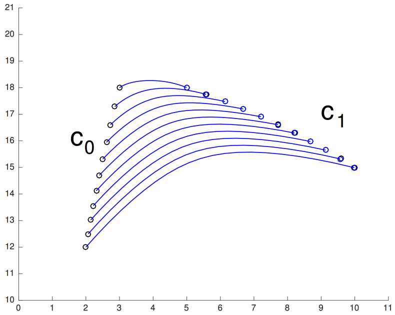

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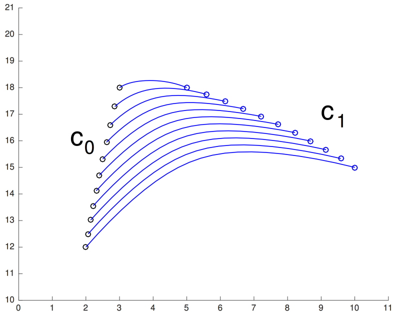

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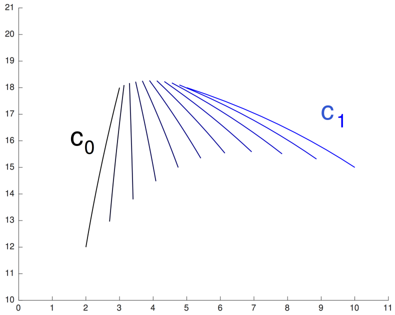

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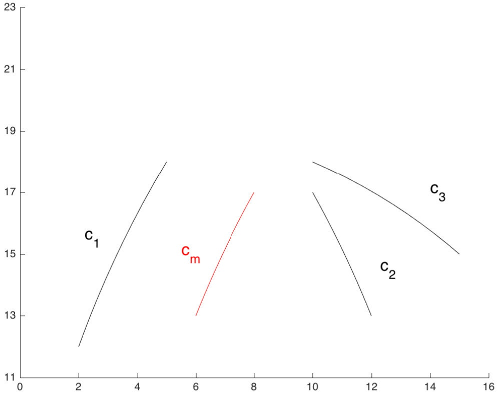
Computing a mean curve

Fréchet mean : $$c_m = \underset{c}{\operatorname{argmin}} \sum_{i=1}^{N} \operatorname{dist}^2(c_i, c)$$



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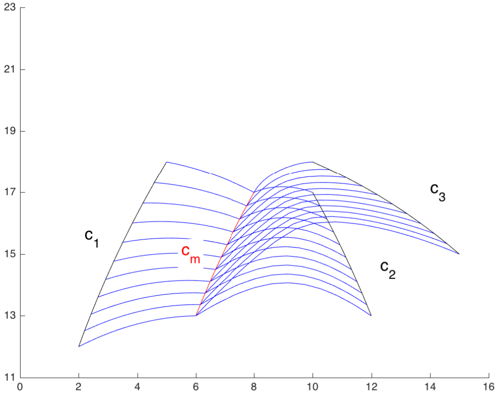

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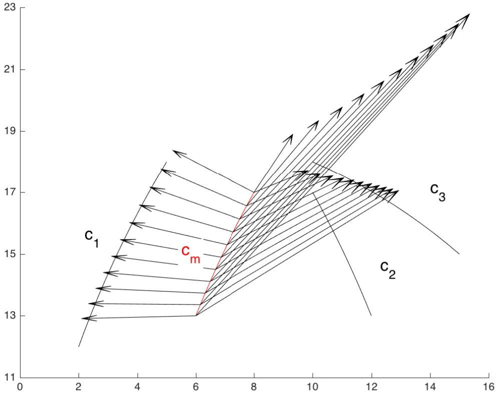

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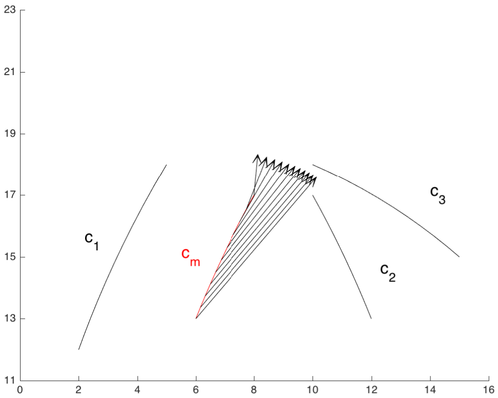

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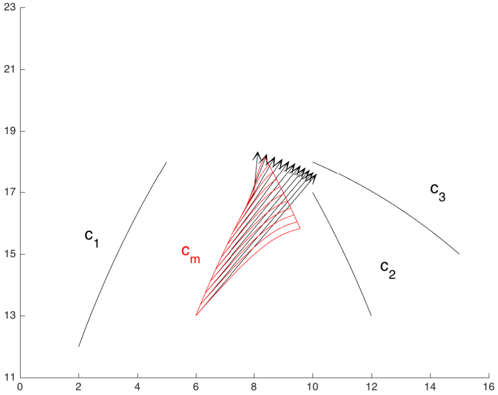

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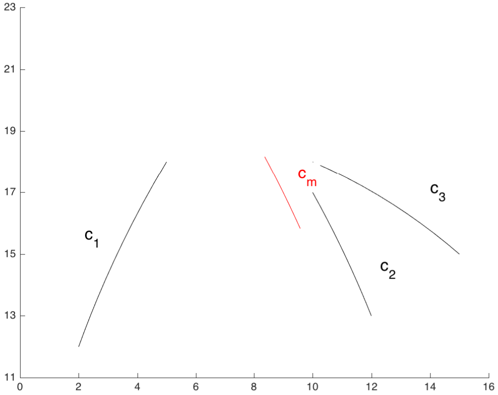

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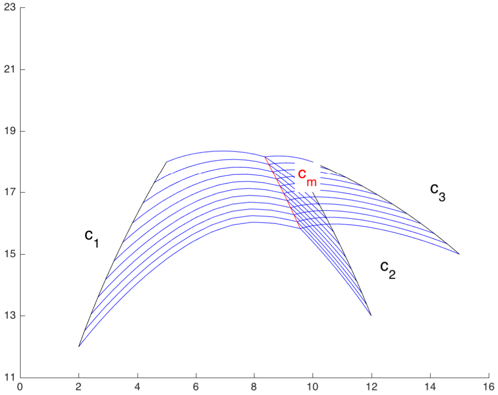

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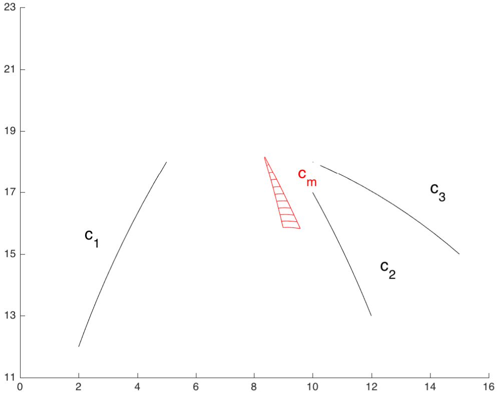

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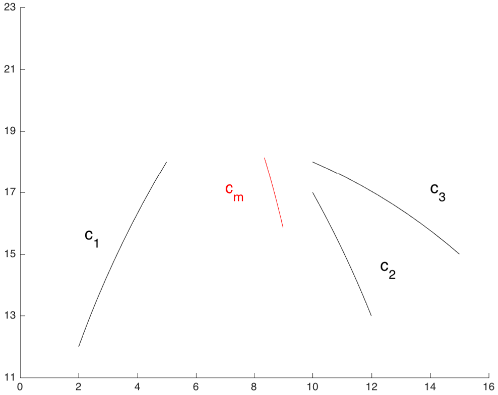

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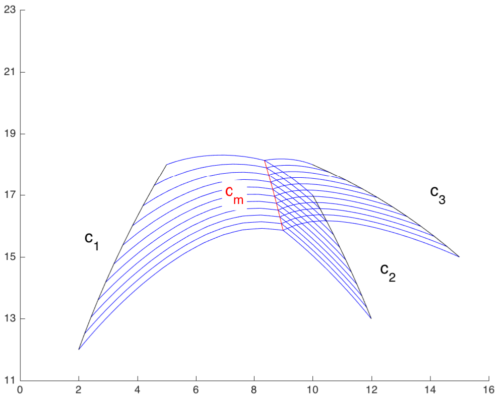

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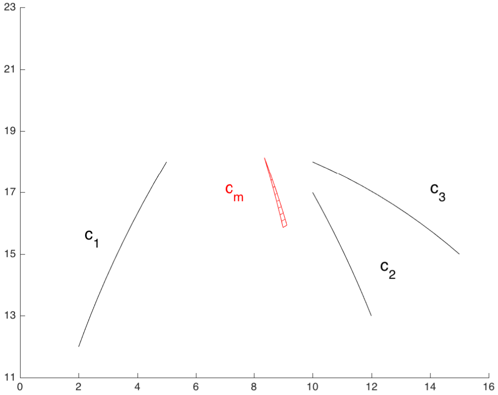

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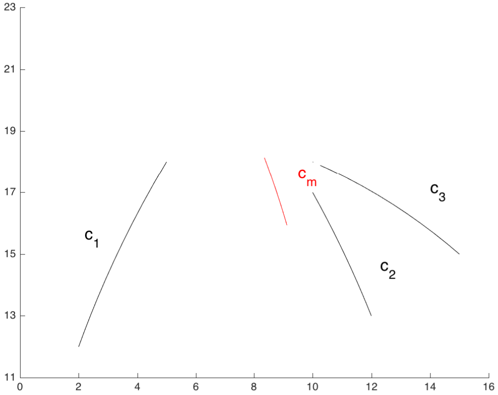

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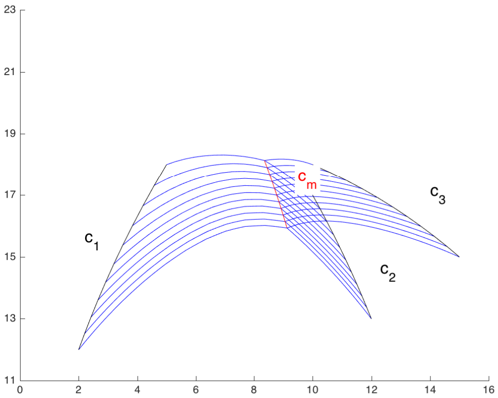

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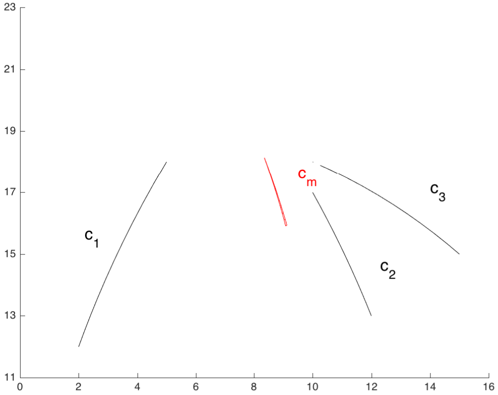

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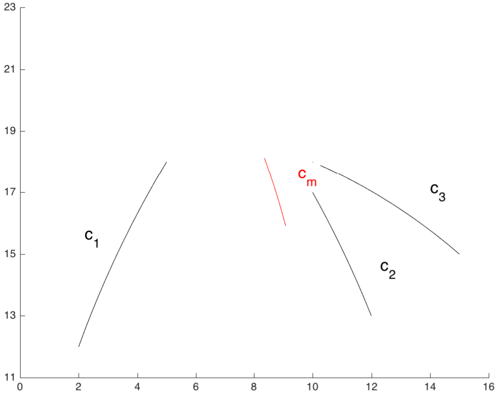

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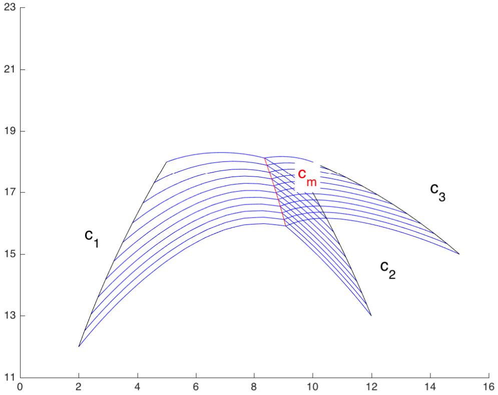

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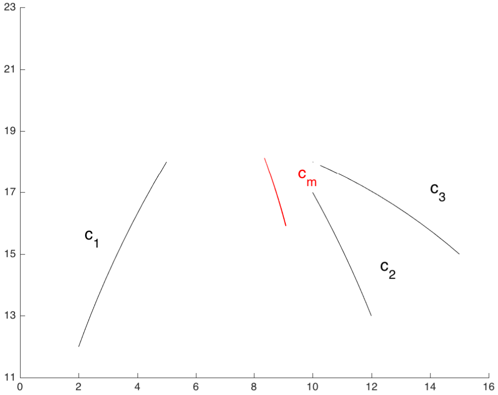


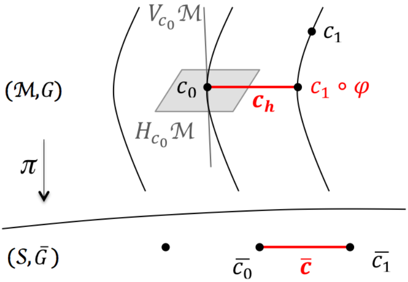
Table of contents

Optimal matching between two curves

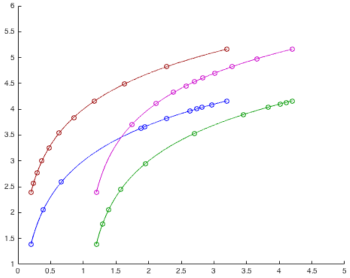
The geodesics of $\mathcal{S}$ are projections of the horizontal geodesics of $\mathcal{M}$.

Fix c_0, and search for $c_1 \circ \varphi$ that minimizes the distance between the two fibers.

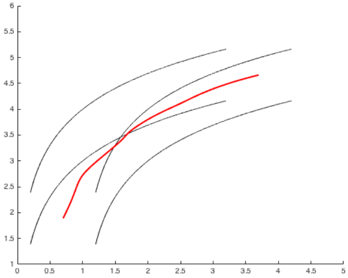
$(c_0, c_1 \circ \varphi)$ gives an optimal matching between the two shapes $\bar{c}_0$ and $\bar{c}_1$.



Motivation for optimal matching



Several curves with same shape but different parameterizations



Pointwise mean

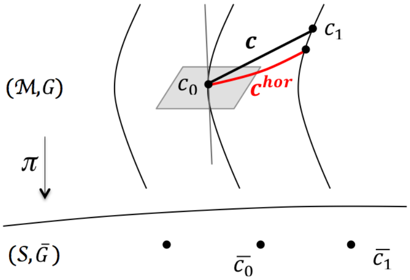
$\rightarrow$ it is interesting to **redistribute** the points on the different curves.

Optimal matching algorithm

We decompose any path of curves $s \mapsto c(s) \in \mathcal{M}$ into

$$c(s) = c^{hor}(s) \circ \varphi(s)$$

where c^{hor} horizontal path and φ path in $\mathrm{Diff}^+([0,1])$.

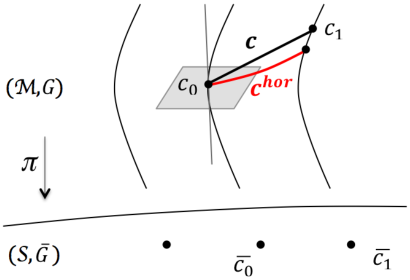


Optimal matching algorithm

Proposition

The horizontal part of a path of curves is at most as long as the path itself

$$L(c^{hor}) \leq L(c).$$



Optimal matching algorithm

Algorithm : set $\hat{c}_1 = c_1$ and iterate

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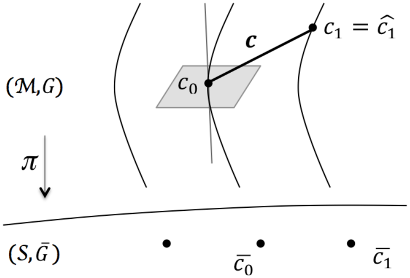

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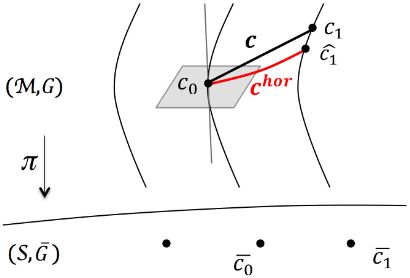

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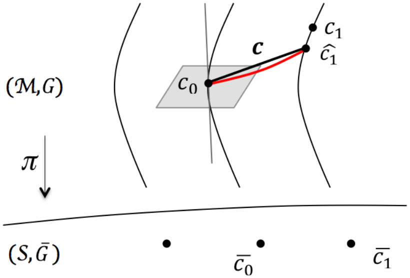

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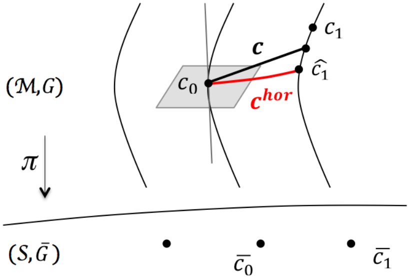

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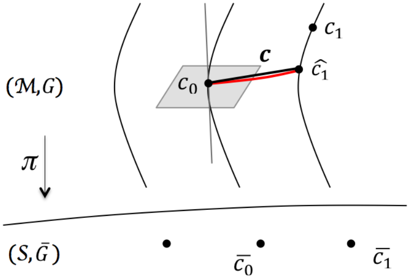

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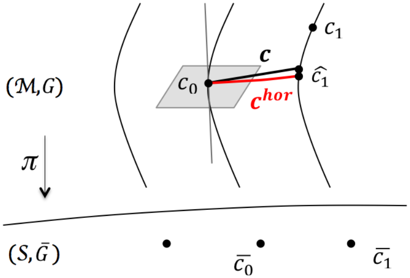

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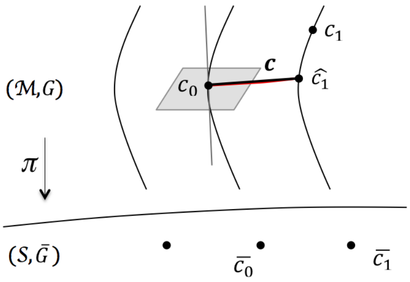

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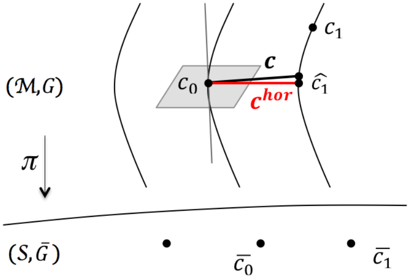

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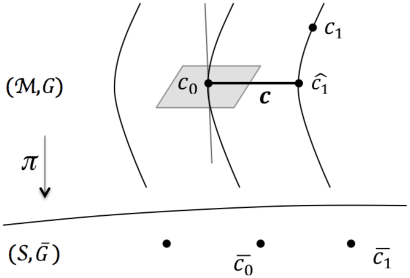

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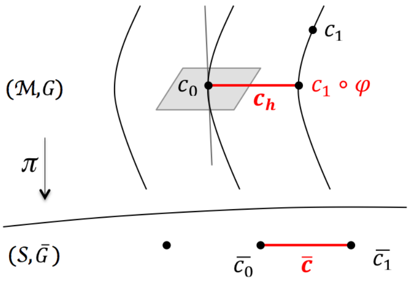
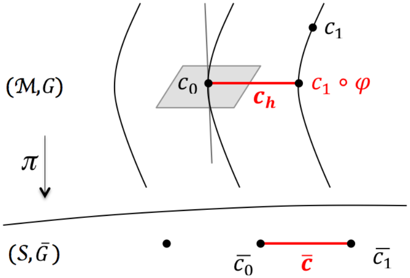


Optimal matching algorithm

How can we compute the horizontal part of a path ?



Horizontal part of a tangent vector

Vertical space : $\mathrm{Ver}_c = \{mv = mc'/|c'|, \quad m \in C^\infty([0,1],\mathbb{R}),\ m(0) = m(1) = 0\}.$

Horizontal space : $\mathrm{Hor}_c = (\mathrm{Ver}_c)^{\perp_G}.$

Proposition (Horizontal vector and horizontal part of a vector)

A vector $h \in T_c\mathcal{M}$ tangent in $c \in \mathcal{M}$ is horizontal for the elastic metric $G^{a,b}$ iff

$$\left((a/b)^2 - 1\right)\langle \nabla_t h, \nabla_t v\rangle - \langle \nabla_t^2 h, v\rangle + |c'|^{-1}\langle \nabla_t c', v\rangle\langle \nabla_t h, v\rangle = 0.$$

$$a = 2b = 1: \qquad 3\langle \nabla_t h, \nabla_t v\rangle - \langle \nabla_t^2 h, v\rangle + |c'|^{-1}\langle \nabla_t c', v\rangle\langle \nabla_t h, v\rangle = 0.$$

The vertical and horizontal parts of a vector $w \in T_c\mathcal{M}$ are given by

$$w^{ver} = mv, \qquad w^{hor} = w - mv,$$

where $m \in C^\infty([0,1],\mathbb{R})$ is solution of

$$\begin{cases} m'' - \langle \nabla_t c'/|c'|, v\rangle m' - 4|\nabla_t v|^2 m = \langle \nabla_t^2 w, v\rangle - 3\langle \nabla_t w, \nabla_t v\rangle - \langle \nabla_t c'/|c'|, v\rangle\langle \nabla_t w, v\rangle. \\ m(0) = m(1) = 0. \end{cases}$$

Horizontal part of a path of curves

Proposition (Horizontal part of a path of curves)

Let $s \mapsto c(s)$ be a path in $\mathcal{M}$. Its horizontal part is given by

$$c^{hor}(s,t) = c(s, \varphi(s)^{-1}(t)),$$

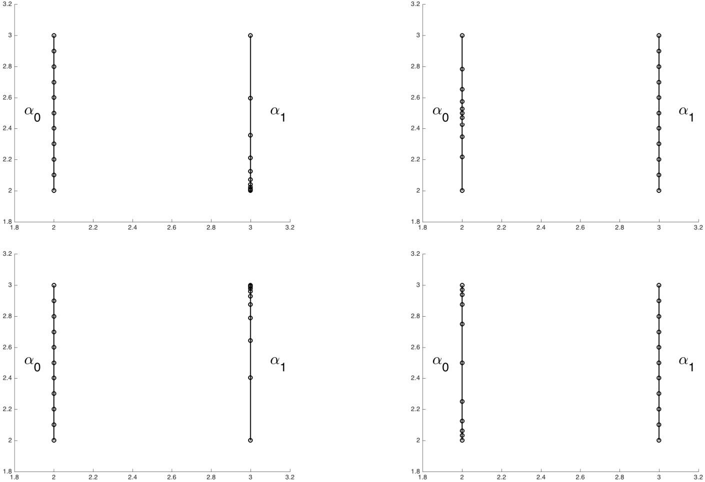
where $s \mapsto \varphi(s)$ is solution of

$$\begin{cases} \varphi_s(s,t) = m(s,t)/|c_t(s,t)| \cdot \varphi_t(s,t), \\ \varphi(0,\cdot) = \mathrm{Id} \end{cases}$$

and where $m(s) : [0,1] \to \mathbb{R}$, $t \mapsto m(s,t)$ is solution for all s of

$$\begin{cases} m_{tt} - \langle \nabla_t c_t/|c_t|, v\rangle m_t - 4|\nabla_t v|^2 m = \langle \nabla_t^2 c_s, v\rangle - 3\langle \nabla_t c_s, \nabla_t v\rangle - \langle \nabla_t c_t/|c_t|, v\rangle\langle \nabla_t c_s, v\rangle \\ m(s,0) = m(s,1) = 0. \end{cases}$$

Example : optimal matching in $\mathbb{H}^2$



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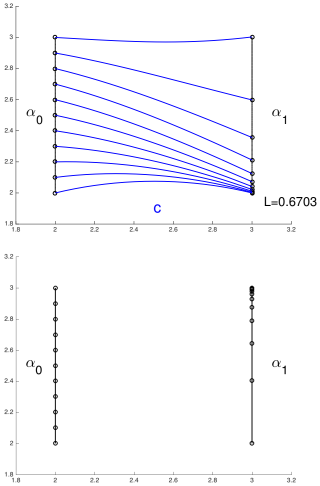

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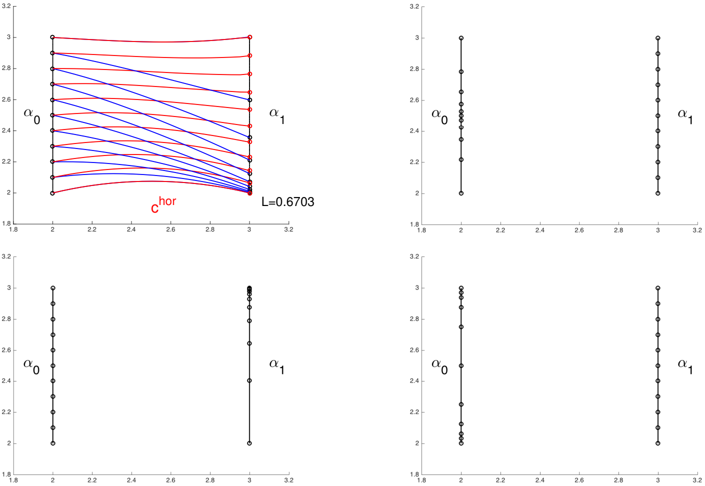

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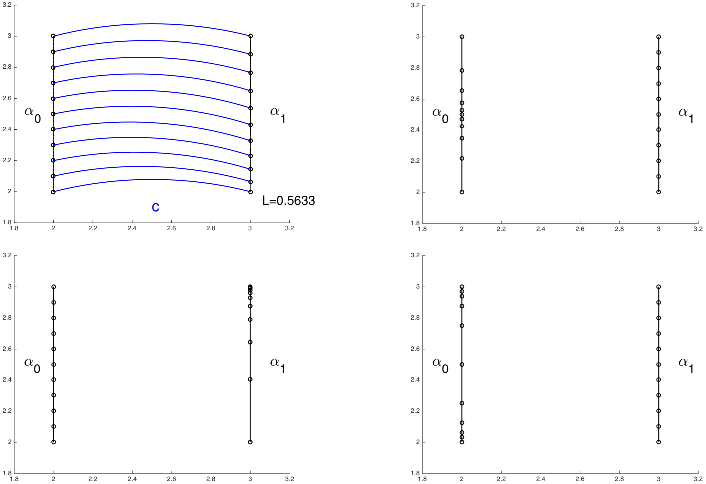

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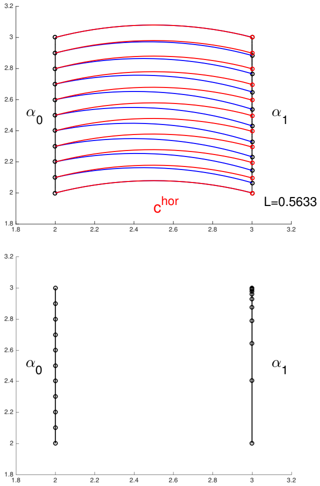

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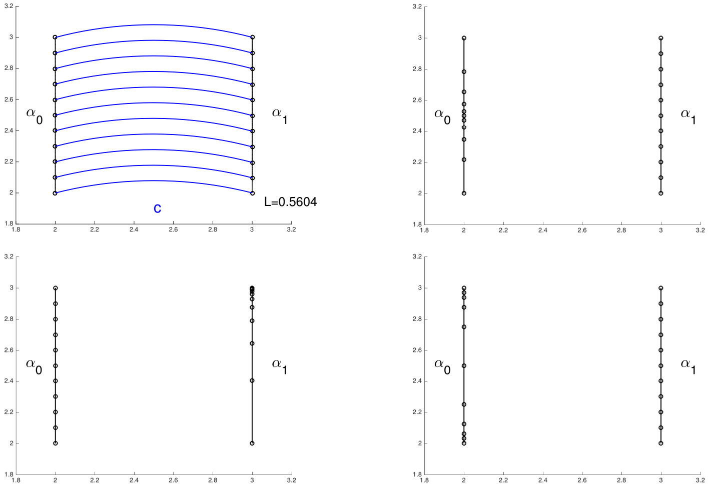

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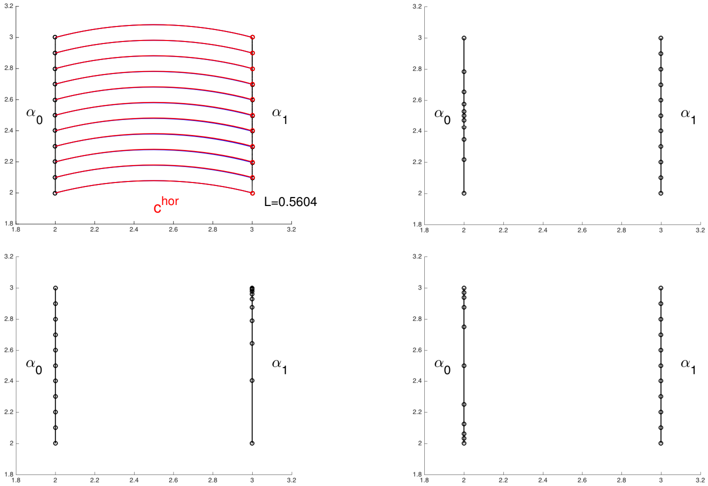

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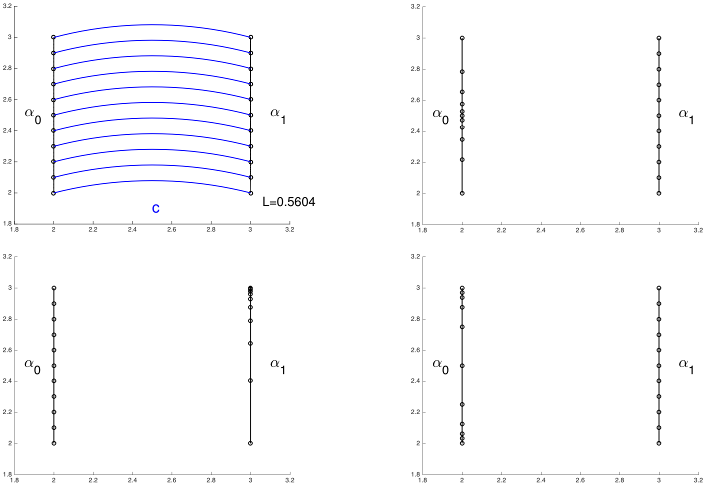

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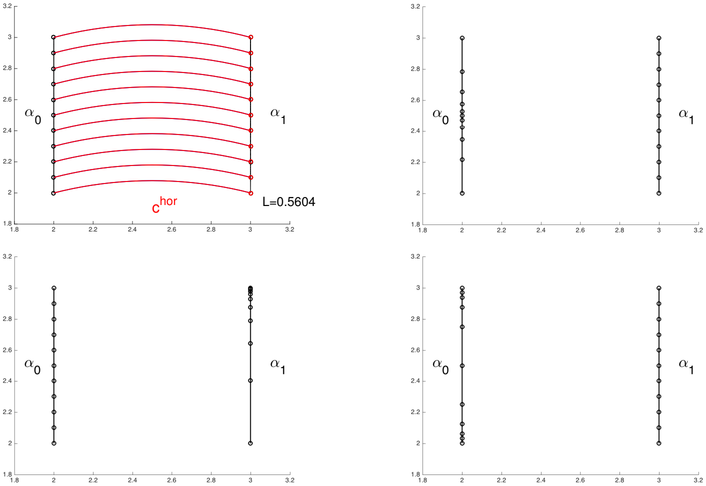

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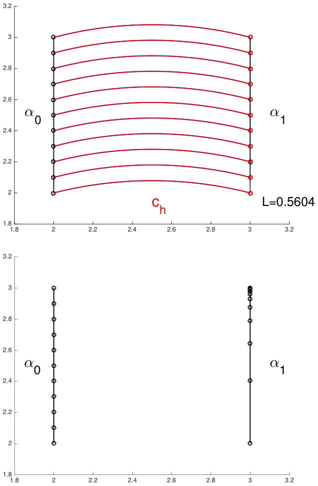

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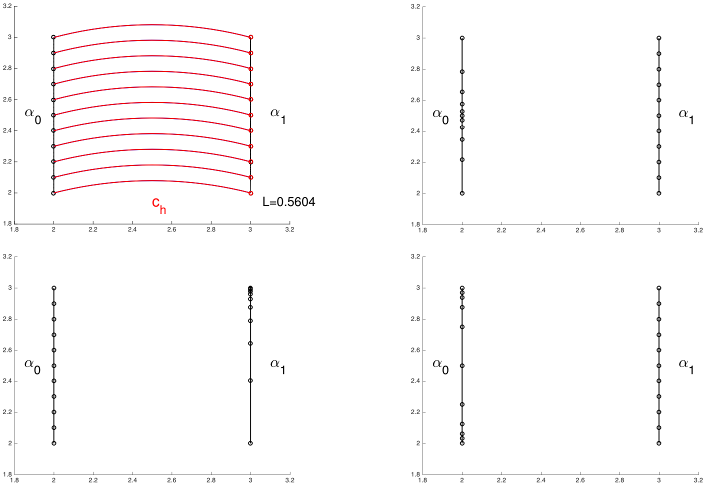

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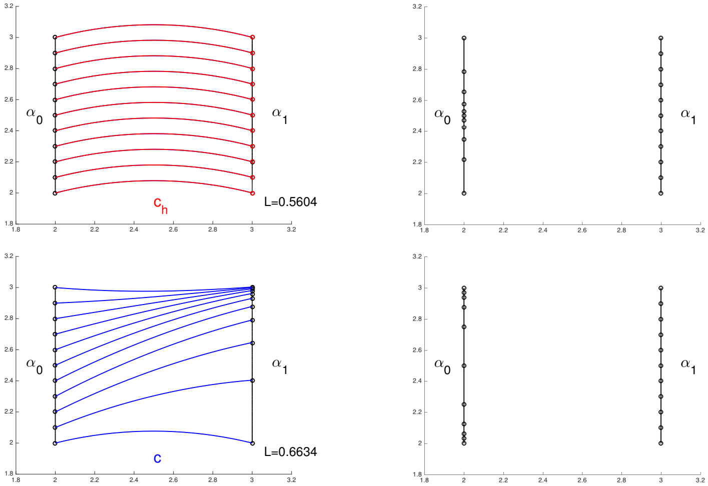

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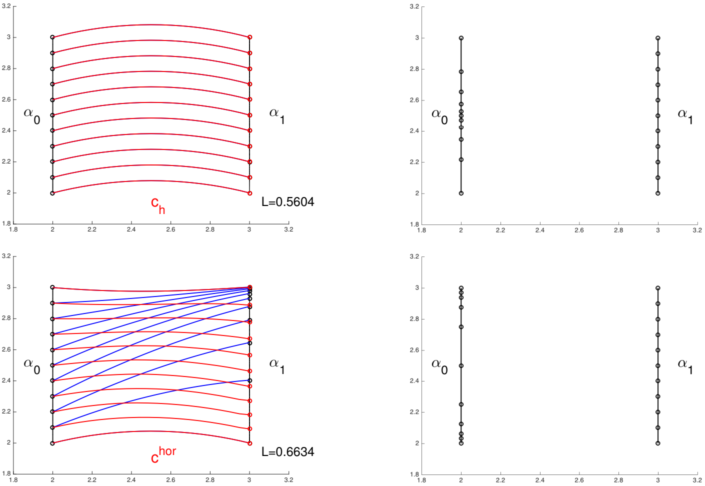

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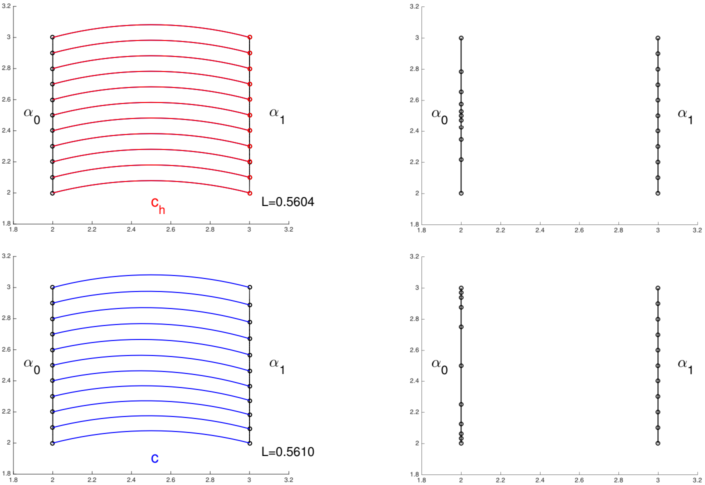

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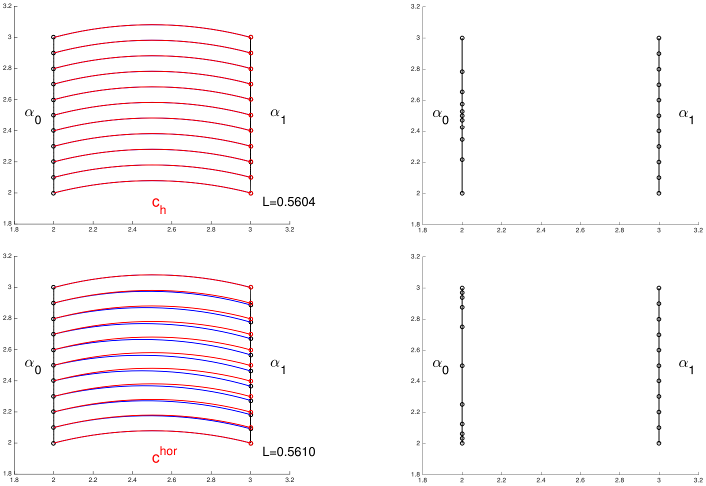

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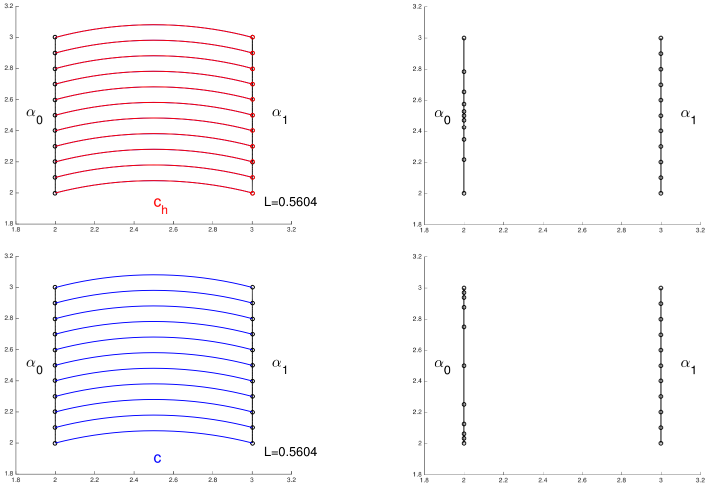

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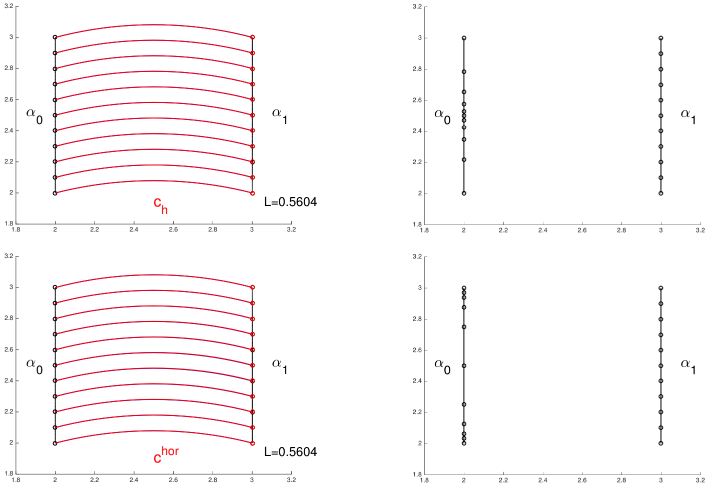

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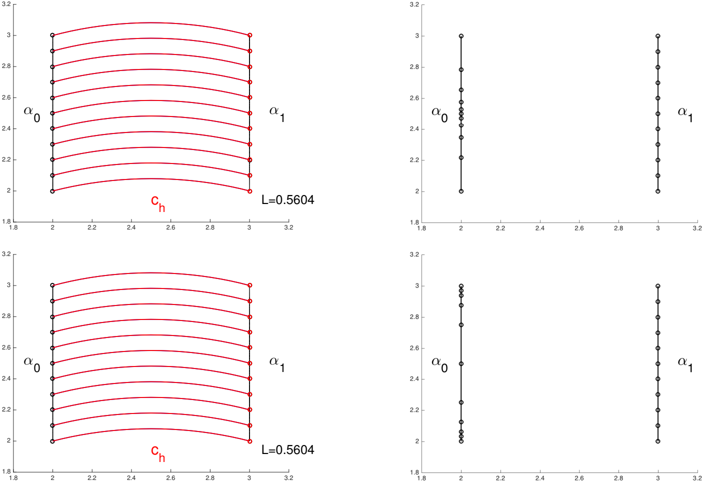

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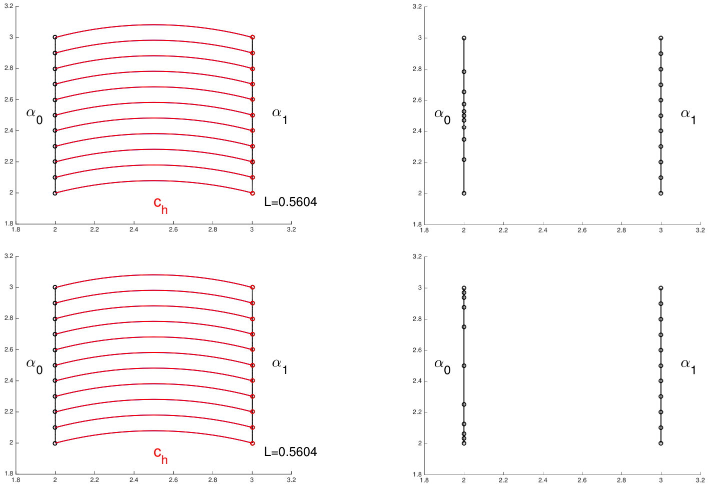

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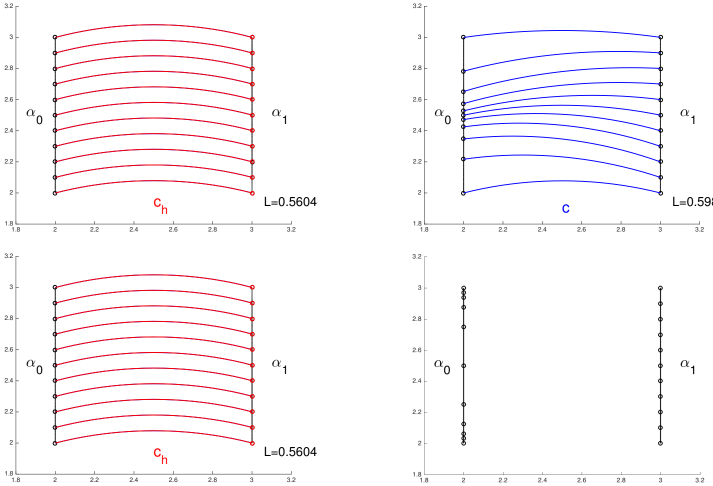

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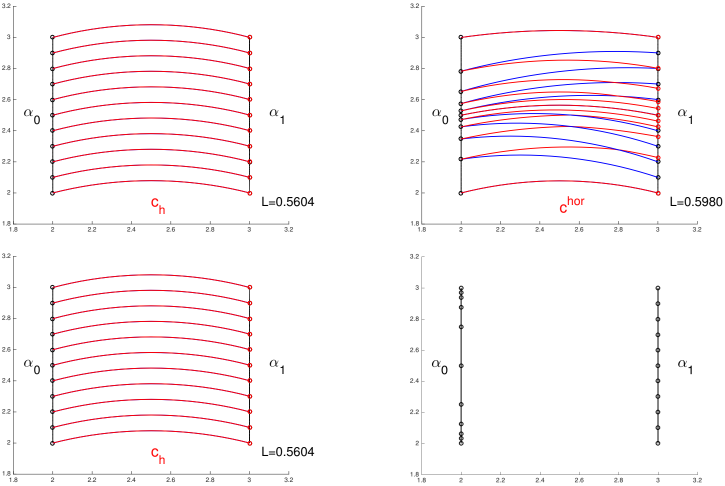

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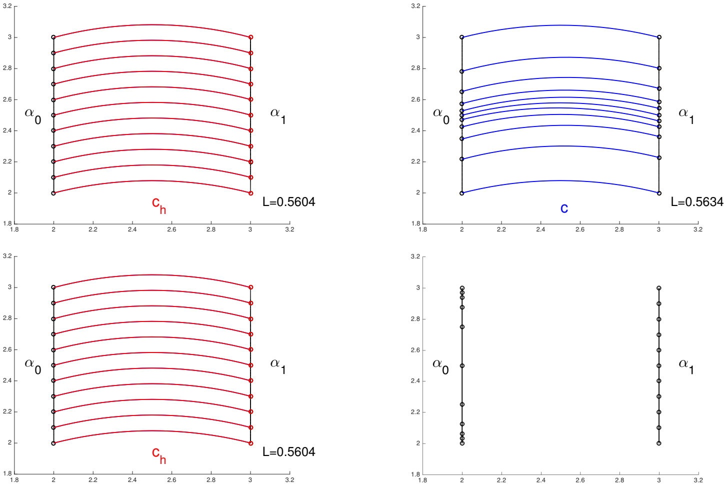

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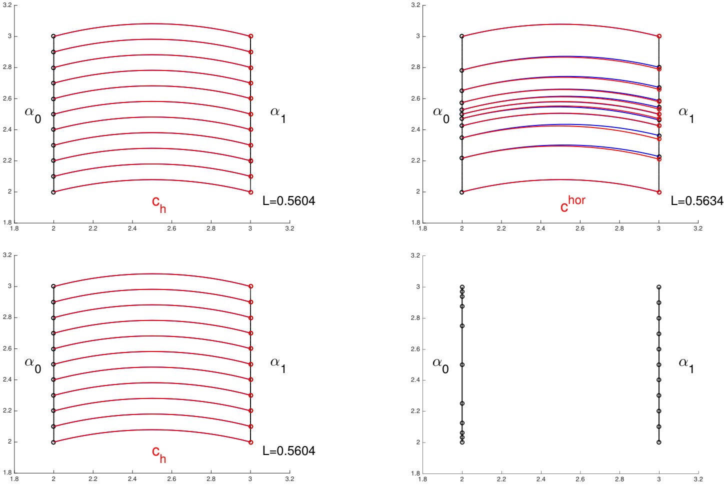

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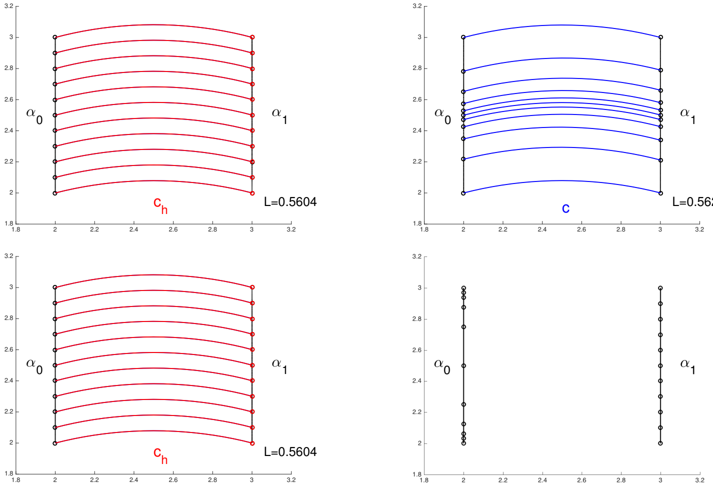

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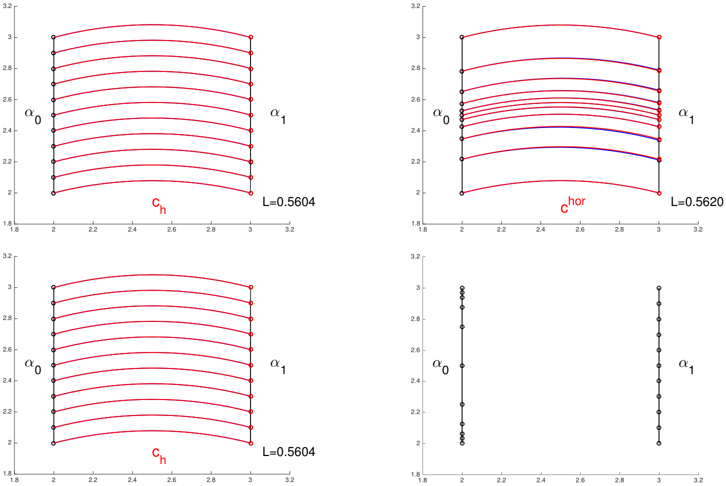

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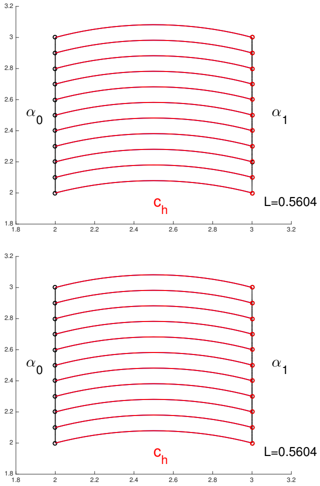

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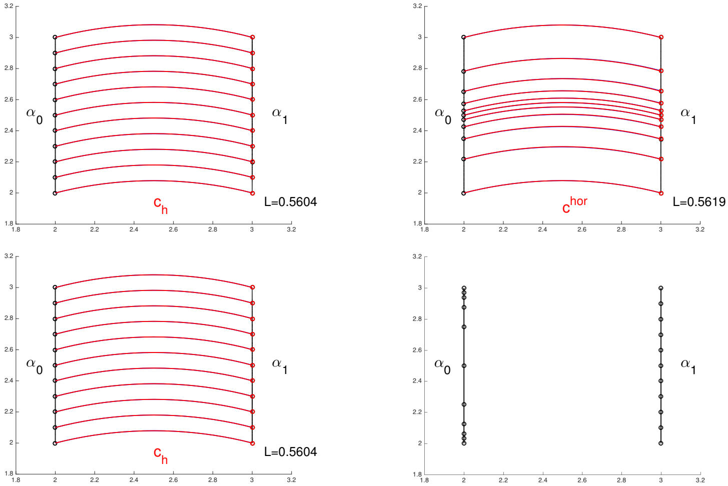

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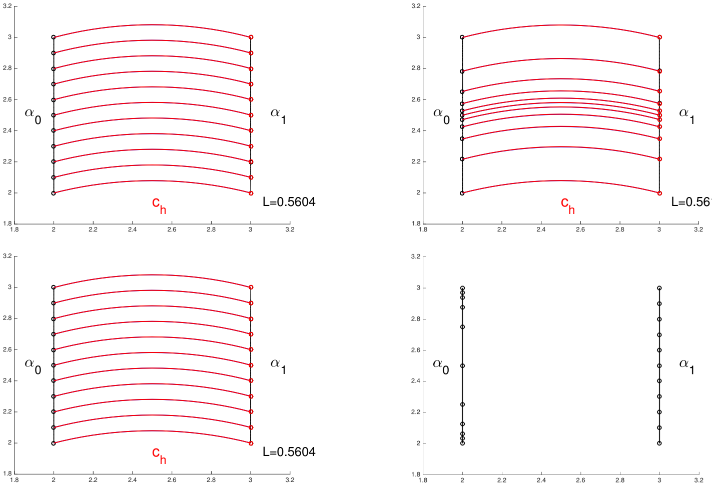

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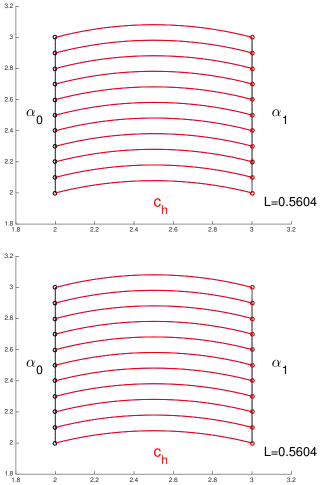

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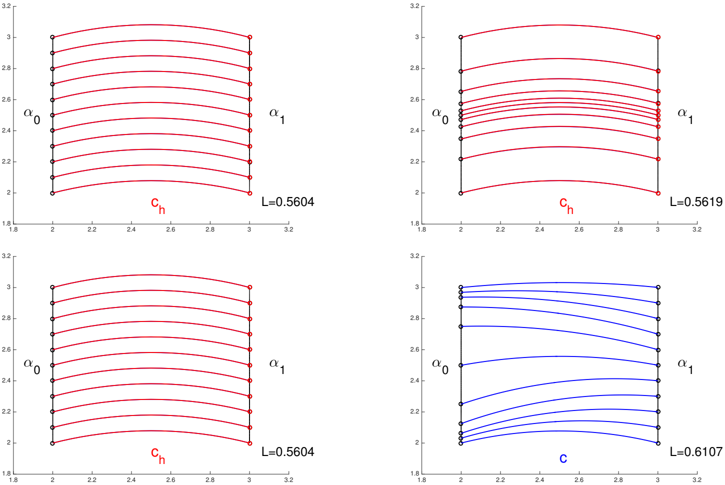

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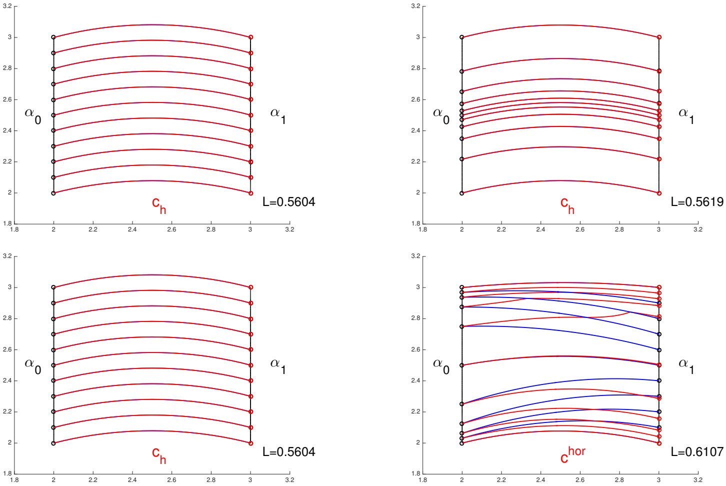

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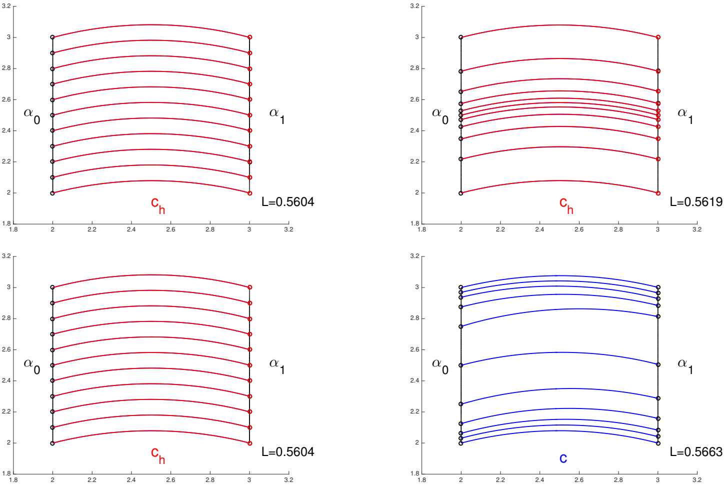

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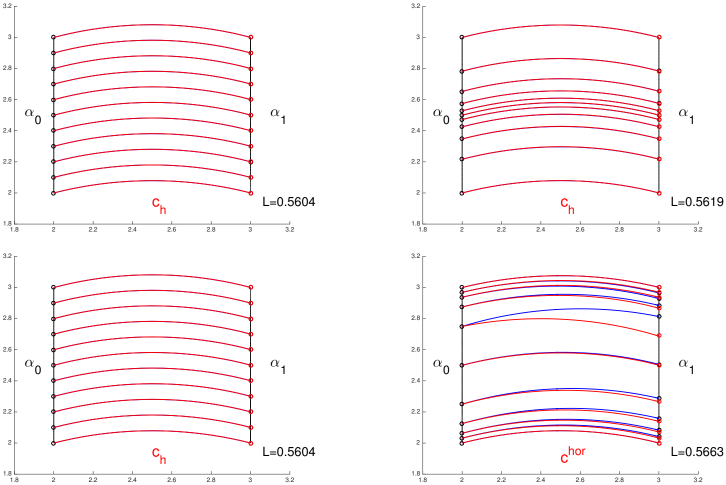

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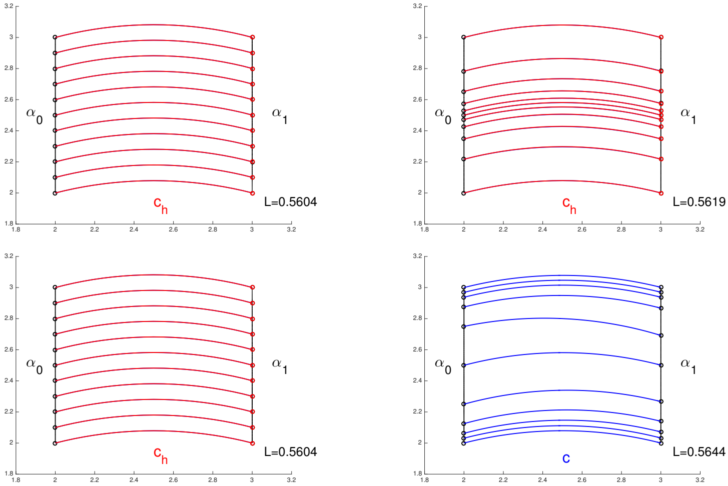

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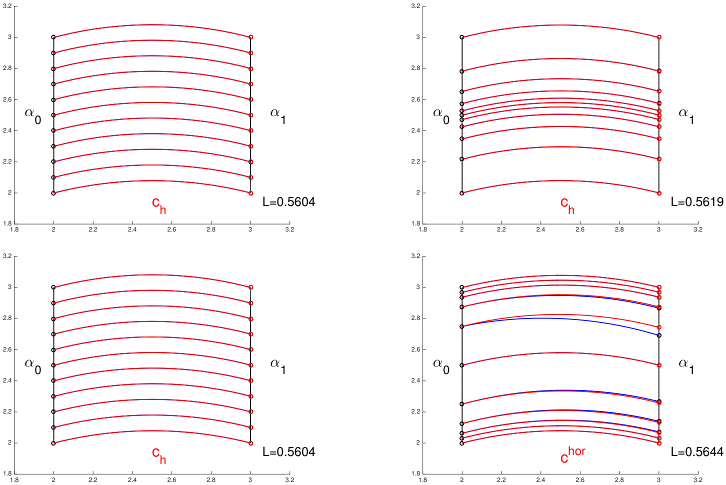

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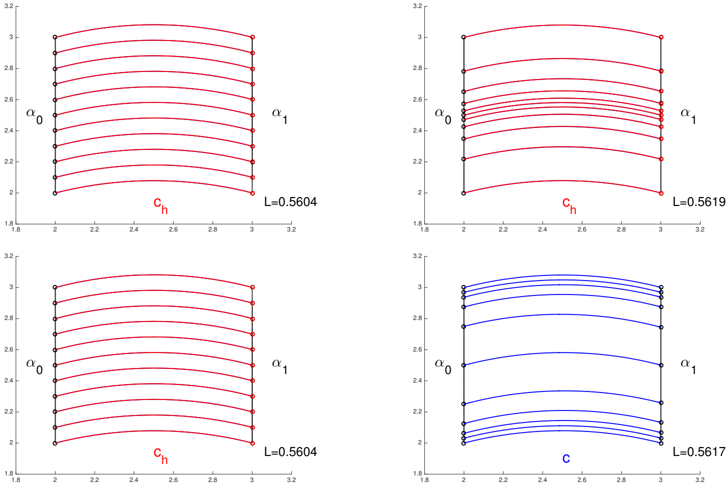

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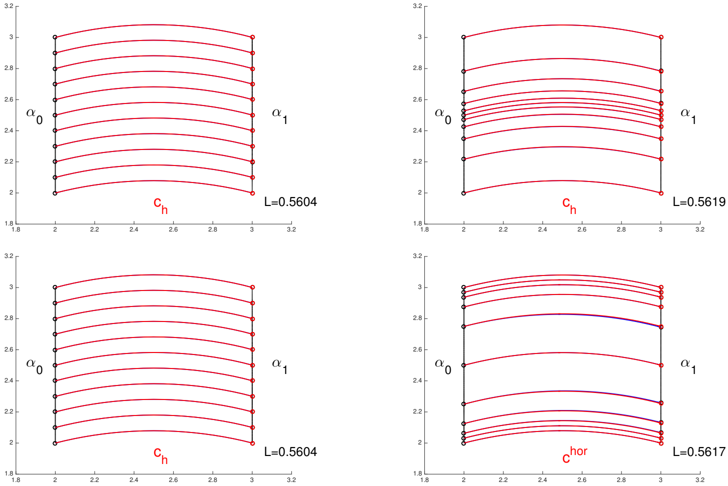

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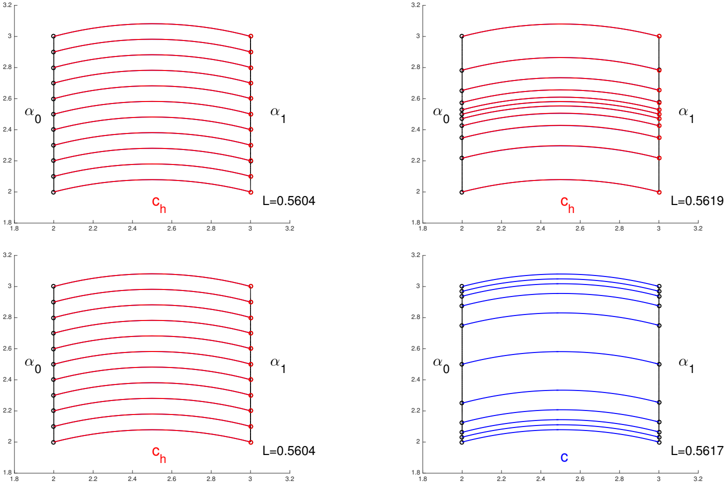

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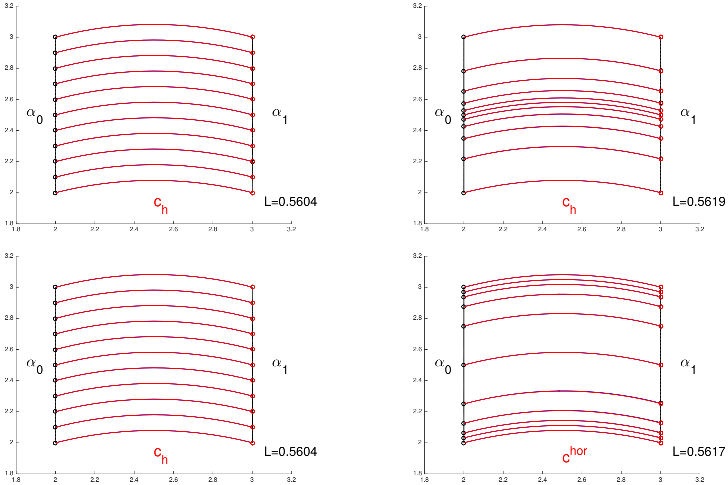

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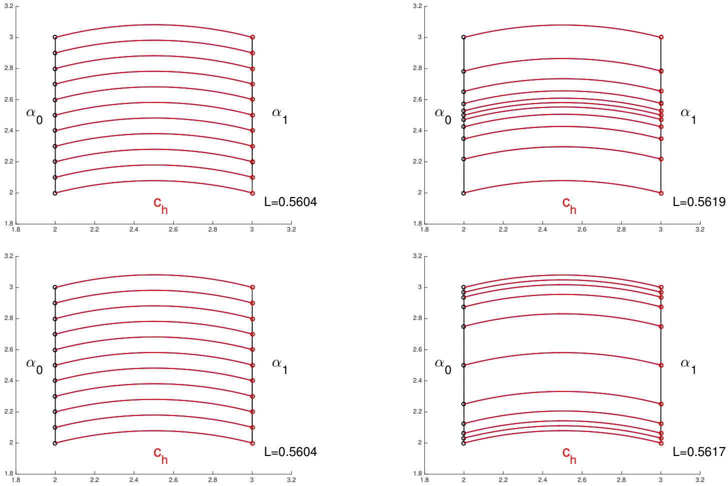

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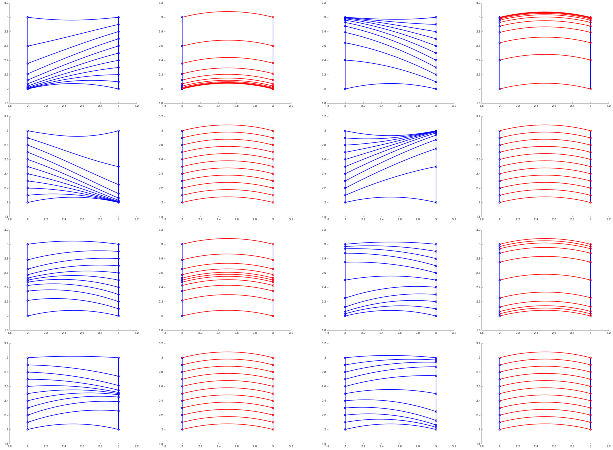
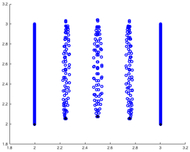


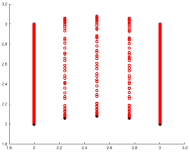
Geodesics between different pairs of parameterizations of two segments of $\mathbb{H}^2$ (blue) and the corresponding horizontal geodesics (red)

Example : optimal matching in $\mathbb{H}^2$

Superposition of horizontal geodesics $\rightarrow$ geodesic between the shapes



Initial geodesics



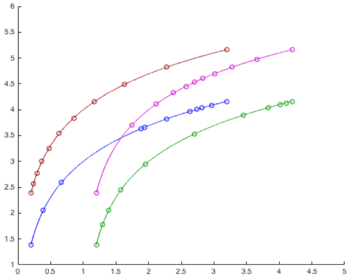
Horizontal geodesics

Common lengths of the horizontal geodesics $\rightarrow$ distance between the shapes $(d \approx 0.56)$

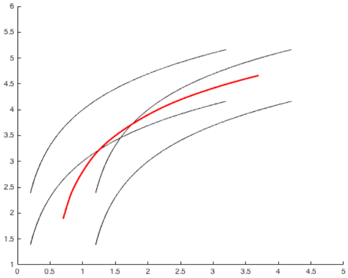
0.6287	0.5611	0.6249	0.5633
0.7161	0.5601	0.7051	0.5601
0.5798	0.5608	0.6106	0.5615
0.6213	0.5601	0.6104	0.5601

Length of the initial geodesics (blue)
and the corresponding horizontal geodesics (red)

Back to the first example



Several curves with same shape but different parameterizations



Mean for our metric after optimal matching

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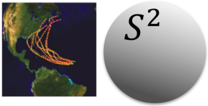
The applications

- In practice, the applications give series of points
  Space of "discrete curves" = M^{n+1}
- Hyp : M has constant sectional curvature K.

$K = +1$

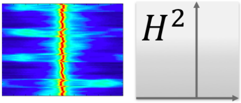


Hurricane trajectories*

$K = 0$

Car trajectories**

$K = -1$



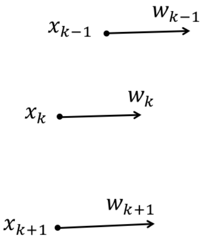
Radar signals

*[Su, Kurtek, Klassen, Srivastava '14]
**[Zhang, Su, Klassen, Le, Srivastava '15]

Riemannian structure on M^{n+1}

"Discrete curve" $\quad \alpha = (x_0, \ldots, x_n) \in M^{n+1}$,

Tangent vector $\quad w = (w_0, \ldots, w_n), \quad w_k \in T_{x_k} M$.



Riemannian structure on M^{n+1}

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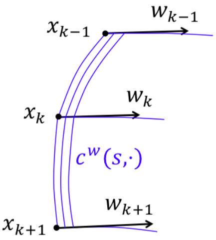


Discrete metric on M^{n+1} :

$$G^n_\alpha(w,w) = |w_0|^2 + \frac{1}{n}\sum_{k=0}^{n-1} |\nabla_s q^w(0, \tfrac{k}{n})|^2,$$

$s \mapsto c^w(s,\cdot)$ path of piecewise-geodesic curves

and q^w the SRV of c^w.

Convergence of the discrete model to the continuous model

Définition

We say that $\alpha = (x_0, \ldots, x_n) \in M^{n+1}$ is *the discretization of size n* of $c \in \mathcal{M}$ when

$$c(\tfrac{k}{n}) = x_k \quad \text{for all } k = 0, \ldots, n.$$

A path $s \mapsto \alpha(s)$ of discrete curves is *the discretization of size n* of a path of curves $s \mapsto c(s)$ when $\alpha(s)$ is the discretization of $c(s)$ for all s.

We show the convergence of the energies when $n \to \infty$

$$E^n(\alpha) = \frac{1}{2}\int_0^1 \left(|x_0'(s)|^2 + \frac{1}{n}\sum_{k=0}^{n-1} |\nabla_s q_k(s)|^2 \right) \mathrm{d}s$$

$$E(c) = \frac{1}{2}\int_0^1 \left(|c_s(s,0)|^2 + \int_0^1 |\nabla_s q(s,t)|^2 \mathrm{d}t \right) \mathrm{d}s$$

Convergence of the discrete model to the continuous model

Theorem (Convergence of the discrete model to the continuous model)

Let $s \mapsto c(s)$ be a C^1 path of C^2 curves whose speed in t never vanishes, identifiable to an element $(s,t) \mapsto c(s,t)$ of $C^{1,2}([0,1] \times [0,1], M)$ such that $c_t \neq 0$.

Let $s \mapsto \alpha(s) = (x_0(s), \ldots, x_n(s))$ be the discretization of size n of c.

Then there exists a constant $\lambda > 0$ that does not depend on c and such that for n big enough,

$$|E(c) - E^n(\alpha)| \leq \frac{\lambda}{n} (\inf |c_t|)^{-1} |c_s|^2_{2,\infty} (1 + |c_t|_{1,\infty})^3,$$

with

$$|c_t|_{1,\infty} := |c_t|_\infty + |\nabla_t c_t|_\infty,$$
$$|c_s|_{2,\infty} := |c_s|_\infty + |\nabla_t c_s|_\infty + |\nabla_t^2 c_s|_\infty,$$

and $|w|_\infty := \sup_{s,t \in [0,1]} |w(s,t)|$.

Discrete geodesic equation

We find the geodesic equation by a method analogous to that of the continuous case.
The coefficients depend on the curvature K of M.

Proposition (Discrete geodesic equation)

The path $s \mapsto \alpha(s) = (x_0(s), \ldots, x_n(s)) \in M^{n+1}$ is a geodesic for G^n iff its SRV coordinates $s \mapsto \big(x_0(s), (q_k(s))_k\big)$ verify

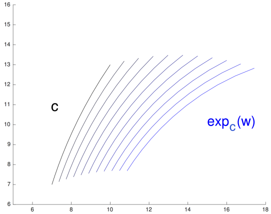
$$\begin{aligned} &\nabla_s {x_0}'(s) = -r_0(s) + o(1), \\ &\nabla_s^2 q_k(s) = -|q_k(s)|(r_k(s) + r_k(s)^T) + o(1), \quad k = 0, \ldots, n-1, \end{aligned}$$

for all $s \in [0,1]$, with

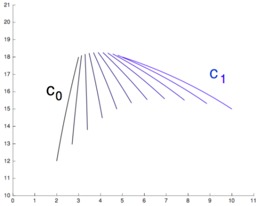
$$\begin{aligned} &r_k(s) := \frac{1}{n} \sum_{\ell=k+1}^{n-1} P_c^{\frac{l}{n}, \frac{k}{n}} \big(\mathcal{R}(q, \nabla_s q) c_s(s, \tfrac{\ell}{n})\big) \underset{n\to\infty}{\longrightarrow} r(s, \tfrac{k}{n}), \quad k = 1, \ldots, n-2, \\ &r_{n-1}(s) := 0. \end{aligned}$$

Implementation

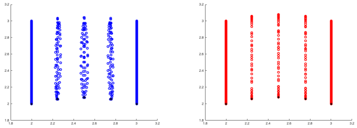
This allowed us to implement :



1. the exponential map



2. the geodesic shooting algorithm



3. the optimal matching algorithm

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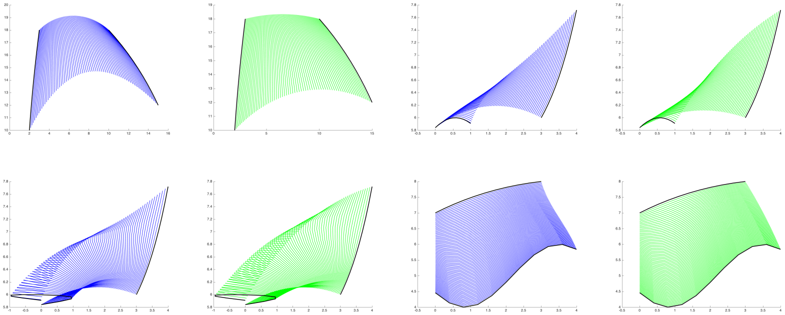
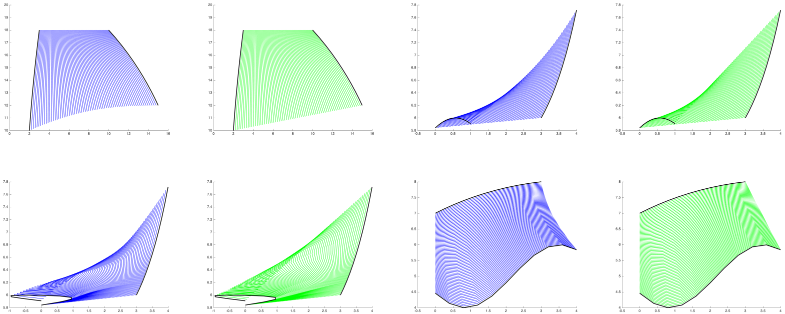
Geodesic shooting in $\mathbb{H}^2$



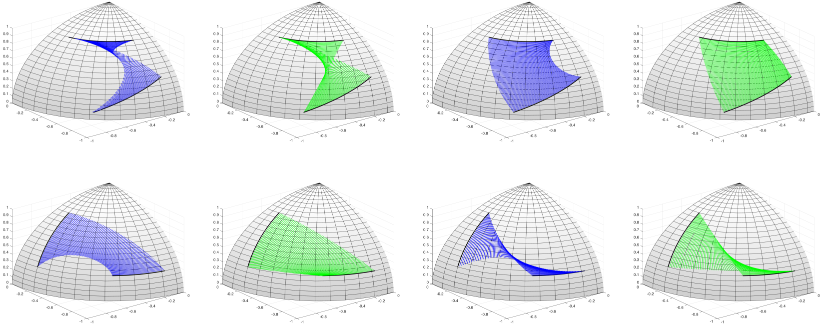
Geodesics between parameterized curves in $\mathbb{H}^2$ for metric G (blue) and the L^2 metric (green)

Geodesic shooting in $\mathbb{R}^2$



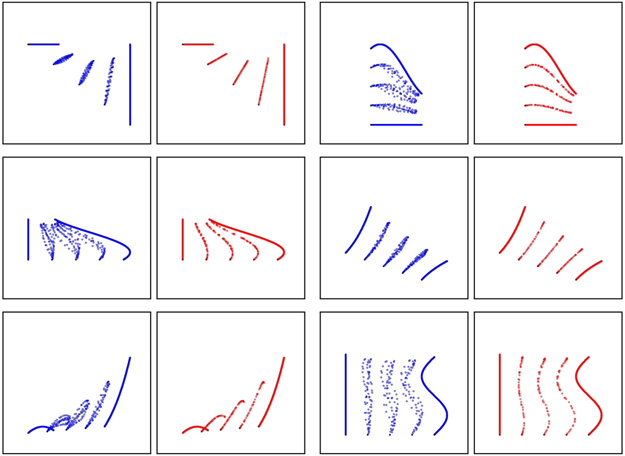
Geodesics between parameterized curves in $\mathbb{R}^2$ for metric G (blue) and the L^2 metric (green)

Geodesic shooting in $\mathbb{S}^2$



Geodesics between parameterized curves in $\mathbb{S}^2$ for metric G (blue) and the L^2 metric (green)

Optimal matching in $\mathbb{R}^2$



Superposition of geodesics between different parameterizations of the same curves (blue) and of the associated horizontal geodesics (red)

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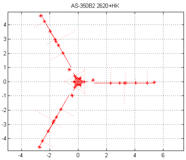
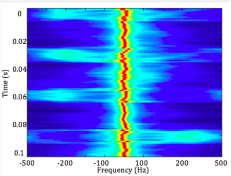
Computing a mean helicopter signature

- Data : m vectors $X^k = (X_1^k, \ldots, X_N^k)$ of N radar observations obtained using a simulator of helicopter signatures





- Each observation vector X^k corresponds to a slightly different rotation speed of the blades
- We want to create a mean signature that takes these variations into account.

Computing a mean helicopter signature

- For each X^k, we estimate the evolution of the reflection coefficients

$$\left(P^k(t),\, \mu_1^k(t), \dots,\, \mu_{n-1}^k(t)\right) \in \mathbb{R}_+^* \times \mathbb{D}^{n-1}$$

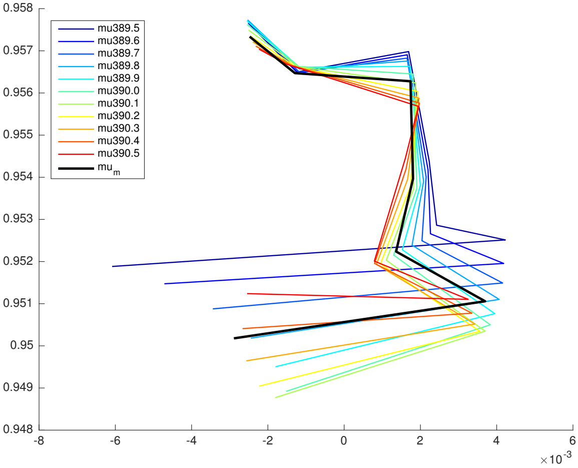
  n = size of the gliding window = size of the stationary portions.

- Since we have a product metric on $\mathbb{R}_+^* \times \mathbb{D}^{n-1}$, to compare X^k and X^ℓ we simply pairwise compare the curves μ_i^k and μ_i^ℓ for each $i = 1, \dots, n-1$.

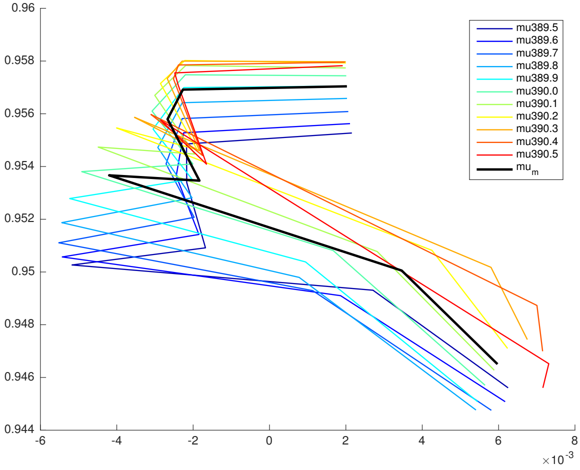
- They are curves in the Poincaré disk.

$$z = \begin{bmatrix} z_1 \\ \vdots \\ \boxed{\begin{matrix} z_i \\ \vdots \\ z_{i+n} \end{matrix}} \\ \vdots \\ z_N \end{bmatrix} \longrightarrow (P_0(t), \mu_1(t), \dots, \mu_{n-1}(t))$$

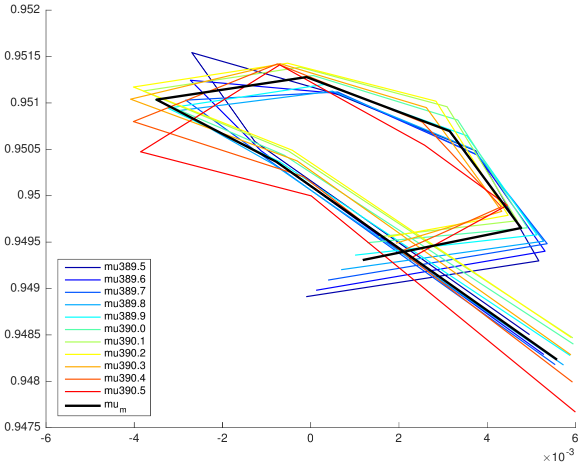
Computing a mean helicopter signature



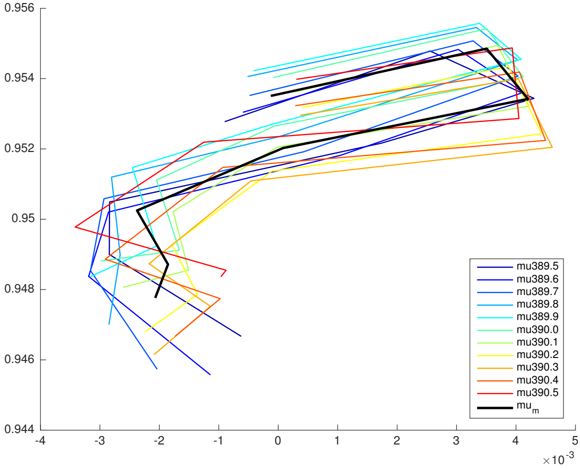
Computing a mean helicopter signature



Computing a mean helicopter signature



Computing a mean helicopter signature



What still needs to be done

- These mean curves are computed without optimal matching :
  - interpolate between points (splines in the hyperbolic space),
  - compute the mean between the shapes of the interpolations (with optimal matching).
- Compare the results obtained by considering the signals as locally stationary (curves) to those obtained by considering them as stationary (points).
- In the locally stationary case, compare the efficiency of different metrics between curves for target detection or recognition.
- More generally : exploit the Riemannian setting to perform statistics on sets of curves.

Thank you for your attention !