Ehrhard's inequality and hypercontractivity of Ornstein-Ulhenbeck semigroup

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Plan

Work in progress with M. Madiman, C. Roberto and P-M Samson.

I. Introduction

II. Ehrhard's inequality and its consequences

III. Proof of the main result

IV. Open questions

I. Introduction

Ornstein-Uhlenbeck semi-group

The Ornstein-Uhlenbeck semi-group is defined for all function sufficiently integrable functions $g : \mathbf{R}^n \to \mathbf{R}$ by

$$P_t g(x) = \frac{1}{(2\pi)^{n/2}} \int g(e^{-t}x + \sqrt{1-e^{-2t}}y) e^{-\frac{|y|^2}{2}}\, dy$$
$$= \mathbb{E}\left[g\left(e^{-t}x + \sqrt{1-e^{-2t}}Z\right)\right], \qquad x \in \mathbf{R}^n,$$

where $Z \sim \mathcal{N}(0, \mathrm{I}_n)$ is a standard Gaussian random vector.

P.D.E interpretation :
For any good function g, $u(t,x) = P_t g(x)$, $t \geq 0$, $x \in \mathbf{R}^n$ is solution of the equation

$$\partial_t u(t,x) = \Delta u(t,x) - x \cdot \nabla u(t,x).$$

Probabilistic interpretation :
If $(X_t^x)_{t \geq 0}$ is solution of the SDE

$$dX_t^x = \sqrt{2} dB_t - X_t^x\, dt, \qquad X_0^x = x$$

then $P_t g(x) = \mathbb{E}[g(X_t^x)]$.

Hypercontractivity of P_t

Let us denote by γ_n the standard Gaussian measure on $\mathbf{R}^n$:

$$\gamma_n(dx) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}}\, dx,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^n$.

The probability measure γ_n is invariant under the semi-group :

$$\int P_t g \, d\gamma_n = \int g \, d\gamma_n, \qquad \forall g.$$

Using Jensen inequality, it is easily seen that P_t is contractive from $\mathrm{L}_p(\gamma_n)$ to itself :

$$\|P_t g\|_{\mathrm{L_p}(\gamma_n)} \leq \|g\|_{\mathrm{L}_p(\gamma_n)}.$$

Actually a stronger property holds :

Theorem [Nelson '73]

Let $p \geq 1$; if $g \in \mathrm{L}_p(\gamma_n)$, then $P_t g \in \mathrm{L}_q(\gamma_n)$ for any $1 \leq q \leq p + (p-1)e^{2t}$ and moreover

$$\|P_t g\|_{\mathrm{L_q}(\gamma_n)} \leq \|g\|_{\mathrm{L}_p(\gamma_n)}.$$

Link with the Logarithmic Sobolev Inequality

The hypercontractivity property is a consequence of the Log-Sobolev inequality for the standard Gaussian measure

Theorem [Gross ('75), Federbusch ('69), Stam ('59)]

The standard Gaussian measure γ_n satisfies the following inequality :

$$\mathrm{Ent}_{\gamma_n}(f) \leq \frac{1}{2}\mathrm{I}_{\gamma_n}(f), \qquad \forall \text{ smooth } f \geq 0,$$

where Ent_{γ_n} denotes the Entropy functional defined by

$$\mathrm{Ent}_{\gamma_n}(f) = \int f \log f \, d\gamma_n - \int f \, d\gamma_n \log \int f \, d\gamma_n, \qquad \forall f \geq 0$$

and $\mathrm{I}_{\gamma_n}(f)$ denotes the Fisher information functional defined by

$$\mathrm{I}_{\gamma_n}(f) = \int \frac{|\nabla f|^2}{f} \, d\gamma_n.$$

Moreover hypercontravity and LSI are equivalent for a large class of Markov processes (Gross '75).

Sketch of proof of Nelson hypercontractivity theorem

Let $Lg(x) = \Delta g(x) - x \cdot \nabla g(x)$ be the generator of the semigroup P_t.

First ingredient : Using LSI and the integration by part formula

$$\int fLg\, d\gamma_n = -\int \nabla f \cdot \nabla g\, d\gamma_n$$

one sees that, for all $q > 1$,

$$\mathrm{Ent}_{\gamma_n}(f^q) \leq \frac{-q^2}{2(q-1)} \int f^{q-1} Lf\, d\gamma_n$$

Second argument :

$$\frac{d}{dt} \log \|P_t f\|_{q(t)} = C(t) \left[\mathrm{Ent}_{\gamma_n}(P_t f^{q(t)}) + \frac{q(t)^2}{q'(t)} \int (P_t f)^{q(t)-1} L P_t f\, d\gamma_n \right]$$

where $C(t) \geq 0$ and $q(t) = p + (p-1)e^{2t}$.

Third argument :

$$\frac{q(t)^2}{q'(t)} = \frac{q(t)^2}{2(q(t)-1)}.$$

Conclusion : $\varphi(t) = \|P_t f\|_{q(t)}$ is non-increasing and so $\varphi(t) \leq \varphi(0) = \|f\|_p$.

Talagrand's conjecture - continuous setting

Suppose that $g \in \mathrm{L}_1(\gamma_n)$, what can be said about $P_t g$?

Assume $g \geq 0$ and $\int g \, d\gamma_n = 1$, then by invariance $\int P_t g \, d\gamma_n = 1$ and so by Markov

$$\gamma_n(P_t g \geq u) \leq \frac{1}{u}, \qquad \forall u > 0.$$

Question : Can one improve this bound ?

More precisely, is there a function $\alpha_t :]0, +\infty[\to]0, \infty[$ with $\alpha_t(u) \to 0$ when $u \to +\infty$ such that

$$\sup_{g \geq 0, \int g \, d\gamma_n = 1} \gamma_n(P_t g \geq u) \leq \frac{\alpha_t(u)}{u}, \qquad \forall u > 0 \quad ?$$

Recent results about this question

The following result summarizes recent contributions by
(1) K. Ball, F. Barthe, W. Bednorz, K. Oleszkiewicz and P. Wolff ('13),
(2) Eldan and Lee ('14)
(3) Lehec ('16).

Theorem 1

For any $t > 0$, it holds

$$\sup_{g \geq 0, \int g \, d\gamma_n = 1} \gamma_n(P_t g \geq u) \leq \frac{\alpha_t(u)}{u}, \qquad \forall u > 1.$$

(1) (Ball et al.) $\alpha_t(u) = c(t,n) \frac{\log \log u}{\sqrt{\log(u)}}$, $c(t,n)$ depends heavily on the dimension n.

(2) (Eldan and Lee) $\alpha_t(u) = c(t) \frac{(\log \log u)^4}{\sqrt{\log(u)}}$, with $c(t)$ depending only on t.

(3) (Lehec) $\alpha_t(u) = c(t) \frac{1}{\sqrt{\log(u)}}$, with $c(t) = a \max(1; 1/t)$ and a universal constant.

Reduction to tail estimation of log semi convex functions

Recall that a smooth function $f : \mathbf{R}^n \to \mathbf{R}$ is said λ-semiconvex, $\lambda \geq 0$, if

$$\mathrm{Hess}\, f \geq -\lambda \mathrm{I}_n.$$

The following simple lemma is used in the three papers :

Lemma

Suppose that $g \geq 0$ is in $\mathrm{L}_1(\gamma_n)$, then $\log P_t g$ is $e^{-2t}/(1-e^{-2t})$-semiconvex.

Theorem 2 [Eldan-Lee/Lehec]

If $h : \mathbf{R}^n \to \mathbf{R}^+$ is such that $\log h$ is λ-semiconvex and $\int h\, d\gamma_n = 1$, then

$$\gamma_n(h \geq u) \leq c \max(1;\lambda) \frac{1}{u\sqrt{\log u}}, \qquad \forall u \geq 1,$$

c being a universal constant.

Applying Theorem 2 to $h = P_t g$ and using the lemma proves Theorem 1.

General goal : Give a new proof of Theorem 2 using functional/geometrical inequalities

- **Dimension** 1 : Elementary proof of Theorem 2 based on simple convexity inequalities.

- **Dimension** n : Proof of Theorem 2 in the log-convex case using the Gaussian Brunn-Minkowski type functional inequality of Ehrhrard.

Warm up : proof of the one dimensional case

Particular case : let $h : \mathbf{R}^n \to \mathbf{R}^+$ be such that $\log h$ is convex (for simplicity).
Write $h = e^f$ with f a convex function.
By the Fenchel-Legendre duality, it holds

$$f(x) = \sup_{y \in \mathbf{R}^n} \{x \cdot y - f^*(y)\}, \qquad \text{where} \qquad f^*(y) := \sup_{x \in \mathbf{R}^n} \{x \cdot y - f(x)\}.$$

Therefore

$$\begin{aligned} 1 = \int e^{f(x)}\, d\gamma_n(x) &= \int e^{\sup_{y \in \mathbf{R}^n} \{x \cdot y - f^*(y)\}}\, d\gamma_n(x) \\ &\geq \sup_{y \in \mathbf{R}^n} \left\{ \int e^{x \cdot y}\, d\gamma_n(x) e^{-f^*(y)} \right\} \\ &= \sup_{y \in \mathbf{R}^n} \left\{ e^{\frac{1}{2}|y|^2} e^{-f^*(y)} \right\} \end{aligned}$$

So

$$f^*(y) \geq \frac{1}{2}|y|^2$$

and so, by conjugation,

$$f(x) \leq \frac{1}{2}|x|^2$$

Warm up : proof of the one dimensional case

Therefore, for all $u > 1$,

$$\gamma_n(\{h \geq u\}) = \gamma_n(\{f \geq \log(u)\}) \leq \gamma_n\left(\left\{x : \frac{1}{2}|x|^2 \geq \log(u)\right\}\right)$$

Some simple calculations show that

$$\gamma_n\left(\left\{x : \frac{1}{2}|x|^2 \geq \log(u)\right\}\right) \leq C_n \frac{1}{u\sqrt{\log u}} \log(u)^{(n-1)/2}$$

which is interesting only for $n = 1$.

General case : If $h = e^f$ with f a λ-semiconvex function, then one can show that

$$f(x) \leq \frac{1}{2}|x|^2 + \frac{n}{2}\log(1+\lambda)$$

from which one can recover the conclusion of Theorem 2 for $n = 1$.

Conclusion : Working only with the (semi)convexity of $\log h$ is not enough. One needs to use also specific properties of the measure γ_n.

A new deviation inequality for convex functions

In what follows,

$$\Phi(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v} e^{-x^2/2}\, dx \qquad \text{and} \qquad \overline{\Phi}(v) = 1 - \Phi(v), \qquad v \in \mathbf{R}.$$

Theorem [G.-Madiman-Roberto-Samson / van Handel]

Suppose that $g : \mathbf{R}^n \to \mathbf{R}^+$ is a *log-convex* function such that $\int g\, d\gamma^n = 1$ then

$$\gamma^n(g \geq u) \leq \overline{\Phi}(\sqrt{2 \log u}), \qquad \forall u \geq 1.$$

Remarks

(1) $\overline{\Phi}(v) \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-v^2/2}}{v}$, $v > 0$. Thus $\overline{\Phi}(\sqrt{2\log u}) \leq \frac{1}{2\sqrt{\pi}} \frac{1}{u\sqrt{\log u}}$ for $u > 1$.

So one recovers Theorem 2 in the case of log-convex functions.

(2) For any $u > 1$, the function g_u defined by

$$g_u(x) = \exp\left(x_1\sqrt{2\log u} - \log u\right), \qquad x = (x_1, \ldots, x_n) \in \mathbf{R}^n,$$

achieves *equality*.

Comparison to the first approach

Let $h = e^f$ with $f : \mathbf{R}^n \to \mathbf{R}$ a convex function such that $\int h \, d\gamma_n = 1$ and $Z = (Z_1, \ldots, Z_n) \sim \mathcal{N}(0, \mathrm{I}_n)$.

First approach :

$$f(Z) \leq \frac{|Z|^2}{2}.$$

Second approach :

$$\mathrm{Law}\left([f(Z)]_+\right) \preceq_{\mathrm{sto}} \mathrm{Law}\left(\frac{[Z_1]_+^2}{2}\right).$$

(Recall that $\mu \preceq_{\mathrm{sto}} \nu$ if $\mu(t, \infty) \leq \nu(t, \infty)$ for all $t \in \mathbf{R}$.)

Consequence

Lemma

If g is log-convex, then $P_t g$ is log-convex for all $t \geq 0$.

Corollary

If $g \geq 0$ is log-convex and $\int g \, d\gamma_n = 1$ then $\gamma_n(P_t g \geq u) \leq \overline{\Phi}(\sqrt{2 \log u})$, $u \geq 1$.

II. Ehrhard's inequality

The Borell-Ehrhard inequality

Theorem [Borell-Ehrhard]

If A, B are Borel subsets of $\mathbf{R}^n$,

$$\Phi^{-1}(\gamma_n((1-s)A+sB)) \geq (1-s)\Phi^{-1}(\gamma_n(A)) + s\Phi^{-1}(\gamma_n(B)), \qquad \forall s \in [0,1].$$

- Ehrhard ('83) proved the inequality in the case of two convex sets A, B.
- Latala ('96) then extended the result to the case where only one set is assumed to be convex.
- Finally Borell ('03) treated the general case.

Remarks :
- Ehrhard's inequality is a Gaussian improvement of Brunn-Minkowski inequality.
- There is a functional version (similar to the Prekopa-Leindler inequality).

Consequences of Ehrhard's inequality

The Ehrhard inequality gives back numerous geometric and functional inequalities for log-concave probability measures :

- Gaussian isoperimetric inequality,
  - $\rightsquigarrow$ sharp concentration of measure inequalities for γ_n
  - $\rightsquigarrow$ Log-Sobolev, Talagrand, Poincaré inequality for γ_n
- The Sudakov-Tsirelson dilation inequality for the Gaussian measure,
- The Brunn-Minkowski inequality and its equivalent functional form due to Prekopa-Leindler (for the Lebesgue measure)
  - $\rightsquigarrow$ classical isoperimetric theorem for the Euclidean space
  - $\rightsquigarrow$ many functional inequalities for general uniformly log-concave distributions
- ...

Consequences of Ehrhard's inequality

If $A \subset \mathbf{R}^n$ is a Borel set, the boundary measure of A is defined by

$$\gamma_n^+(\partial A) = \liminf_{r \to 0^+} \frac{\gamma_n(A + rB) - \gamma_n(A)}{r},$$

where B is the Euclidean unit ball.

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Gaussian isoperimetric problem : Determine the sets of a given volume that have the least boundary measure.

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Gaussian isoperimetric problem : Determine the sets of a given volume that have the least boundary measure.

Théorème [Sudakov-Tsirel'son '74, Borell '75]

Half-spaces are solution to the Gaussian isoperimetric problem.
If $A \subset \mathbf{R}^n$ is a Borel set and H is a half space such that $\gamma_n(H) = \gamma_n(A)$, then

$$\gamma_n^+(\partial A) \geq \gamma_n^+(\partial H).$$

Equivalently,

$$\gamma_n(A + rB) \geq \gamma_n(H + rB) = \Phi(\Phi^{-1}(\gamma_n(A)) + r), \qquad \forall r \geq 0,$$

where B is the Euclidean unit ball.

Consequences of Ehrhard's inequality

The isoperimetric inequality

$$\Phi^{-1}(\gamma_n(A + rB)) \geq \Phi^{-1}(\gamma_n(A)) + r, \qquad \forall r \geq 0$$

can be easily derived from Ehrhard's inequality.

Consequences of Ehrhard's inequality

The isoperimetric inequality

$$\Phi^{-1}(\gamma_n(A + rB)) \geq \Phi^{-1}(\gamma_n(A)) + r, \qquad \forall r \geq 0$$

can be easily derived from Ehrhard's inequality.

Namely,

$$\begin{aligned} \Phi^{-1}(\gamma_n(A+rB)) &= \Phi^{-1}\left(\gamma_n\left((1-\lambda)\frac{A}{1-\lambda} + \lambda\frac{rB}{\lambda}\right)\right) \\ &\geq (1-\lambda)\Phi^{-1}\left(\gamma_n\left(\frac{A}{1-\lambda}\right)\right) + \lambda\Phi^{-1}\left(\gamma_n\left(\frac{rB}{\lambda}\right)\right). \end{aligned}$$

When $\lambda \to 0$, a direct calculation on the χ^2 distribution shows that

$$\lambda\Phi^{-1}\left(\gamma_n\left(\frac{rB}{\lambda}\right)\right) \to r.$$

A specific application : Ehrhard's lemma

Lemma [Ehrhard]

Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is a convex function, then the function

$$\mathbf{R} \to \mathbf{R} : t \mapsto \Phi^{-1} \circ \gamma^n(f \leq t)$$

is concave.

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is concave.

Proof. Let $t_1, t_2 \in \mathbf{R}$ and $s \in [0, 1]$; by convexity

$$f((1-s)x_1 + sx_2) \leq (1-s)f(x_1) + sf(x_2)$$

and therefore

$$(1-s)\{f \leq t_1\} + s\{f \leq t_2\} \subset \{f \leq (1-s)t_1 + st_2\}.$$

So

$$\Phi^{-1} \circ \gamma^n ((1-s)\{f \leq t_1\} + s\{f \leq t_2\}) \leq \Phi^{-1} \circ \gamma^n (\{f \leq (1-s)t_1 + st_2\})$$

Applying Ehrhard's inequality yields the claim.

A specific application : Ehrhard's lemma

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function and denote by μ_f the image of γ^n under the map f, *i.e*

$$\mu_f(A) = \gamma^n(f^{-1}(A)), \qquad \forall A \subset \mathbf{R}.$$

Let F_f be the cumulative distribution function of μ_f defined by

$$F_f(u) = \mu_f(]-\infty, u]), \qquad u \in \mathbf{R}$$

and denote by F_f^{-1} its generalized inverse :

$$F_f^{-1}(t) = \inf\{u \in \mathbf{R} : F_f(u) \geq t\}, \qquad t \in (0,1).$$

It is well known that $T_f = F_f^{-1} \circ \Phi$ is a transport map between γ_1 and μ_f :

$$\int \varphi(y)\, \mu_f(dy) = \int \varphi(T_f(x))\, \gamma_1(dx), \qquad \forall \varphi : \mathbf{R} \to \mathbf{R}.$$

Equivalent form of Ehrhard's lemma :

Lemma

If $f : \mathbf{R}^n \to \mathbf{R}$ is *convex*, the transport map T_f sending γ_1 on μ_f is also convex.

III. Proof of the main result

Main result

Theorem [G.-Madiman-Roberto-Samson / van Handel]

Suppose that $g : \mathbf{R}^n \to \mathbf{R}^+$ is a *log-convex* function such that $\int g \, d\gamma_n = 1$ then

$$\gamma^n(g \geq u) \leq \overline{\Phi}(\sqrt{2 \log u}), \qquad \forall u \geq 1.$$

Letting $g = e^f$ with f convex, the preceding result restates as follows :

Theorem

Suppose that $f : \mathbf{R}^n \to \mathbf{R}^+$ is a convex function such that $\int e^f \, d\gamma^n = 1$ then

$$\gamma^n(f \geq r) \leq \overline{\Phi}(\sqrt{2r}), \qquad \forall r \geq 0.$$

Proof (1/2)

Let f be a convex function on $\mathbf{R}^n$.

$$1 = \int_{\mathbf{R}^n} e^f \, d\gamma_n = \int_{\mathbf{R}} e^y \, \mu_f(dy) = \int_{\mathbf{R}} e^{T_f(x)} \, \gamma_1(dx).$$

Proof (1/2)

Let f be a convex function on $\mathbf{R}^n$.

$$1 = \int_{\mathbf{R}^n} e^f \, d\gamma_n = \int_{\mathbf{R}} e^y \, \mu_f(dy) = \int_{\mathbf{R}} e^{T_f(x)} \, \gamma_1(dx).$$

Claim. $T_f(x) \leq x^2/2$ for all $x \in \mathbf{R}$.

This follows from the convexity of T_f and our first naive approach using convex duality (in dimension 1) !

Proof (2/2)

One just proved that $T_f(x) \leq x^2/2$ for all $x \in \mathbf{R}$.

Since $T_f(x) = F_f^{-1} \circ \Phi(x)$, one gets

$$\Phi(x) \leq F_f(x^2/2) = \gamma_n(f \leq x^2/2)$$

and so

$$\Phi(\sqrt{2r}) \leq \gamma_n(f \leq r), \qquad \forall r \geq 0.$$

Another recent application of Ehrhard inequality

Theorem [Paouris-Valettas '16]

Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is a convex function belonging to $\mathrm{L}_2(\gamma_n)$, then

$$\gamma_n\left(f \leq \int f\, d\gamma_n - r\right) \leq \exp\left(-c\frac{r^2}{\mathrm{Var}_{\gamma_n}(f)}\right), \qquad \forall r \geq 0.$$

This type of deviation inequalities is usually true for Lipschitz functions and does not contain variance terms in general.

Yet another applications : the median of convex functions

Lemma [Kwapien '94]

Suppose that $f \in \mathrm{L}_1(\gamma_n)$ is a convex function. Then

$$\mathrm{med}_{\gamma_n}(f) \leq \int f \, d\gamma_n.$$

Yet another applications : the median of convex functions

Lemma [Kwapien '94]

Suppose that $f \in \mathrm{L}_1(\gamma_n)$ is a convex function. Then

$$\mathrm{med}_{\gamma_n}(f) \leq \int f \, d\gamma_n.$$

Proof (slightly different from Kwapien's original proof)

$$\int_{\mathbf{R}^n} f(z)\,\gamma_n(dz) = \int_{\mathbf{R}} y\,\mu_f(dy) = \int_R T_f(x)\,\gamma_1(dx) \geq T_f\left(\int_R x\,\gamma_1(dx)\right) = T_f(0).$$

But

$$T_f(0) = F_f^{-1} \circ \Phi(0) = F_f^{-1}(1/2) = \mathrm{med}_{\gamma_n}(f).$$

IV. Open questions

Open questions

Can one recover the semi-convex case using Ehrhard inequality ?

We proved

$$f \text{ convex } \Rightarrow T_f \text{ convex}$$

so it is tempting to formulate the following conjecture

Conjecture 1

If $f : \mathbf{R}^n \to \mathbf{R}$ is λ-semiconvex, then the map T_f transporting γ_1 on $\mu_f = \gamma_n \circ f^{-1}$ is $\kappa(\lambda)$-semiconvex.

Conjecture 1 with $\kappa(\lambda) = \lambda$ would imply

Conjecture 2

If $f : \mathbf{R}^n \to \mathbf{R}$ is λ-semiconvex and such that $\int e^f \, d\gamma_n = 1$ then

$$\gamma_n(f \geq r) \leq \overline{\Phi}(\sqrt{2r - \log(1+\lambda)}), \qquad \forall r \geq \frac{1}{2}\log(1+\lambda).$$

Conjecture 2 seems plausible since the result by Lehec implies it with some additional universal constant.

About Conjecture 1

Unfortunately, Conjecture 1 is false. . .

Other questions

- What about the original question by Talagrand on the discrete cube ?

- Is there a discrete Ehrhard inequality on the discrete cube ?

- Is there a link between the refined hypercontractivity result and the estimation of the deficit in the Gaussian log-Sobolev inequality ?

- In the same spirit, can one derive from Ehrhard's inequality improvements of the log-Sobolev or Talagrand inequalities (perhaps in restriction to convex functions) ?

Thank you for your attention !