Tangent spaces to orbit closures for representations of Dynkin quivers

Grzegorz Zwara (Toruń)

September 5, 2017

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Problem

Describe the tangent space $\mathcal{T}_{N, \overline{\mathcal{O}}_M}$.

We know that

$$\dim_k \mathcal{T}_{N,\overline{\mathcal{O}}_M} \geq \dim \overline{\mathcal{O}}_M,$$

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where

$$\boxed{[U, V] := \dim_k \mathrm{Hom}_Q(U, V)}.$$

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and we attach this integer to the corresp. mesh in A-R quiver Γ_Q:

$$\delta = \delta_{M,N} : \left\{ \text{mesh} \quad \begin{array}{ccccc} & \nearrow & w_1 & \searrow & \\ \tau L & \cdots\to & \vdots & \cdots\to & L \\ & \searrow & w_n & \nearrow & \end{array} \quad \text{in } \Gamma_Q \right\} \to \mathbb{Z}.$$

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Theorem (Bongartz)

$\Longleftarrow$ *if Q is a Dynkin or an extended Dynkin quiver.*

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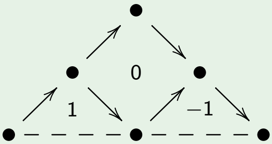

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Hence N is a degeneration of M.

Given a function

$$\delta : \{\text{mesh in } \Gamma_Q\} \to \mathbb{Z},$$

we can recover the pair (M, N) of representations with $\delta = \delta_{M,N}$ and $\mathbf{dim}\, M = \mathbf{dim}\, N$, up to a common direct summand:

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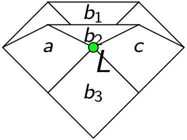

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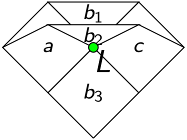

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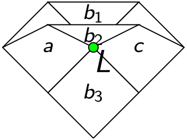

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- we define $\delta_\sigma := \delta_{W, U\oplus V}$.

Subscheme $\mathcal{C}_M$

Let $M \in \mathrm{rep}_Q(\mathbf{d})$. Given a representation L, the condition

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Reformulating Bongartz theorem:

Corollary

$(\mathcal{C}_M)_{red} = \overline{\mathcal{O}}_M$ if Q is a Dynkin or an extended Dynkin quiver.

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and consequently,

$$\left(\mathcal{T}_{N,\overline{\mathcal{O}}_M}/\mathcal{T}_{N,\mathcal{O}_N}\right) \subseteq \left(\mathcal{T}_{N,\mathcal{C}_M}/\mathcal{T}_{N,\mathcal{O}_N}\right) \subseteq \mathrm{Ext}^1_Q(N,N).$$

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Theorem (Riedtmann-Zwara)

$\mathcal{T}_{N,\mathcal{C}_M}/\mathcal{T}_{N,\mathcal{O}_N} = \mathcal{E}(N,N)$,

where $\mathcal{E}(-,-)$ is a k-subfunctor of $\mathrm{Ext}^1_Q(-,-)$, defined by

$$[\sigma]_\sim \in \mathcal{E}(V,U) \iff \mathrm{supp}(\delta_\sigma) \subseteq \mathrm{supp}(\delta_{M,N}).$$

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$W = \left[\begin{smallmatrix} N_1 & Z \\ 0 & N_2 \end{smallmatrix}\right]$ and $N = N_1 \oplus N_2 \oplus N_3$.

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$$k \to \overline{\mathcal{O}}_M, \quad t \mapsto \begin{bmatrix} N_1 & t\cdot Z & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{bmatrix}.$$

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Since N_1 and N_2 are indecomposable,

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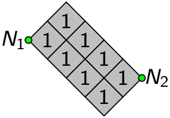

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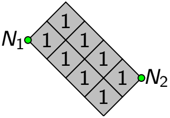

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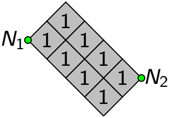

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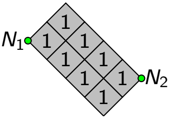

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In fact, we have more general result:

Theorem (Lakshmibai-Magyar; Riedtmann-Zwara)

If Q is a Dynkin quiver of type $\mathbb{A}$, then $\overline{\mathcal{O}}_M = \mathcal{C}_M$.

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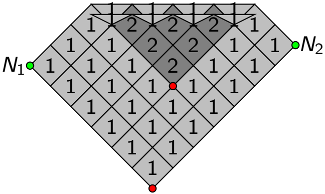
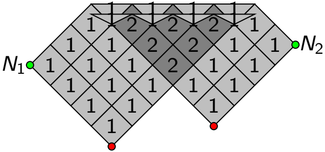
Hence $\delta_\sigma(m') = 2$, $\delta_{M,N}(m') = 1$ for some mesh m', and $\delta_\sigma =$



or



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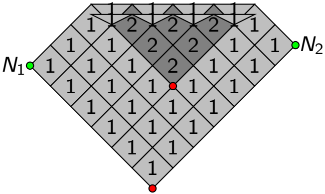

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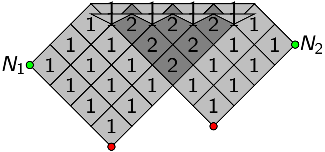
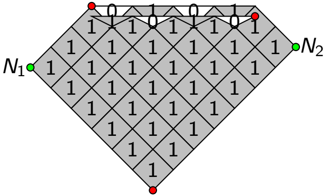
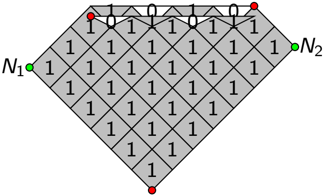


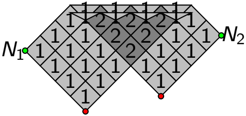
If $\dim_k \mathcal{E}(N_2, N_1) = 2$, then $[\sigma]_\sim = [\sigma']_\sim + [\sigma'']_\sim \in \left(\mathcal{T}_{N,\overline{\mathcal{O}}_M} / \mathcal{T}_{N,\mathcal{O}_N}\right)$

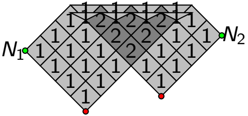
$\delta_{\sigma'}$:



$\delta_{\sigma''}$:



But if $\dim_k \mathcal{E}(N_2, N_1) = 1$ and $\delta_\sigma =$  we need a new idea.

But if $\dim_k \mathcal{E}(N_2, N_1) = 1$ and $\delta_\sigma =$ 

we need a new idea.

Lemma (2)

Let $N = N_1 \oplus N_3 \oplus N_2 \oplus N_4$ *and*

$$\eta : \quad 0 \to N_1 \xrightarrow{f} Y \to N_3 \to 0$$

be a short exact sequence such that $\delta_\eta \leq \delta_{M,N}$, *so*

$$\overline{\mathcal{O}}_N \subseteq \overline{\mathcal{O}}_{N'} \subseteq \overline{\mathcal{O}}_M,$$

where $N' = Y \oplus N_2 \oplus N_4$. *Then*

$$f \cdot [\sigma]_\sim \in \left(\mathcal{T}_{N',\overline{\mathcal{O}}_M} / \mathcal{T}_{N',\mathcal{O}_{N'}}\right) \implies [\sigma]_\sim \in \left(\mathcal{T}_{N,\overline{\mathcal{O}}_M} / \mathcal{T}_{N,\mathcal{O}_N}\right)$$

for any short exact sequence $\sigma : 0 \to N_1 \to W \to N_2 \to 0.$

Example for $\mathbb{D}_5$

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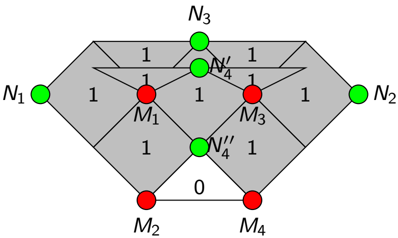
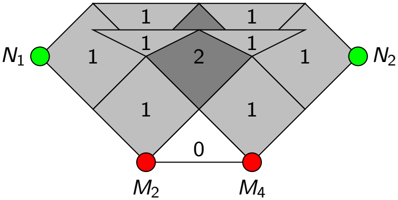
$$M = M_1 \oplus M_2 \oplus M_3 \oplus M_4, \qquad N = N_1 \oplus N_3 \oplus N_2 \oplus N_4' \oplus N_4''$$

$$\sigma : \quad 0 \to N_1 \to M_2 \oplus M_4 \to N_2 \to 0,$$

$\delta_{M,N}$:



δ_σ :



Example for $\mathbb{D}_5$

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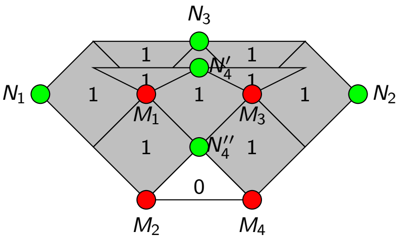
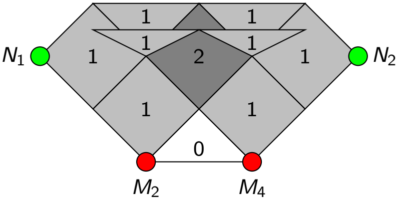
$\delta_{M,N}$:



δ_σ :

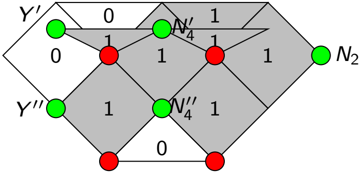


Let $Y' := \tau N_4'$, $Y'' := \tau N_4''$ and

$$\eta : 0 \to N_1 \xrightarrow{f = \left(\begin{smallmatrix} f' \\ f'' \end{smallmatrix}\right)} Y' \oplus Y'' \to N_3 \to 0.$$

$N' = Y' \oplus Y'' \oplus N_2 \oplus N'_4 \oplus N''_5$ and

$\delta_{M,N'} =$



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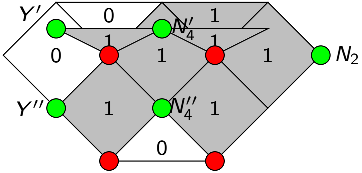
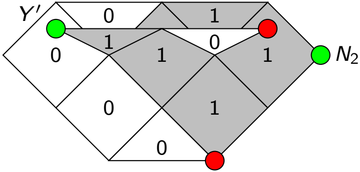
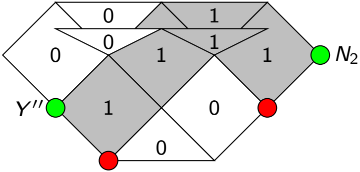
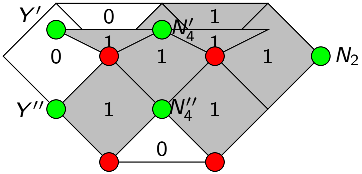
$\delta_{M,N'} =$ 

Pushouts of σ via f' and f'':





$N' = Y' \oplus Y'' \oplus N_2 \oplus N_4' \oplus N_5''$ and

$\delta_{M,N'} =$ 

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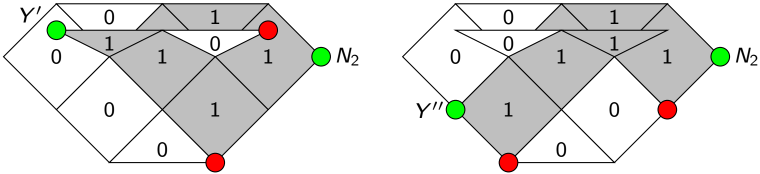

Since $\delta_{f'.\sigma} \leq \delta_{M,N'}$ and $\delta_{f''.\sigma} \leq \delta_{M,N'}$,

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By Lemma (2), $[\sigma]_\sim \in \left(\mathcal{T}_{N,\overline{\mathcal{O}}_M} / \mathcal{T}_{N,\mathcal{O}_N}\right)$. □

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General case.

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Thus $f \cdot [\sigma]_\sim \in \left(\mathcal{T}_{N',\overline{\mathcal{O}}_M} / \mathcal{T}_{N',\mathcal{O}_{N'}}\right)$.
By Lemma (2), $[\sigma]_\sim \in \left(\mathcal{T}_{N,\overline{\mathcal{O}}_M} / \mathcal{T}_{N,\mathcal{O}_N}\right)$. $\square$

General case.
Given a Dynkin quiver Q, we use Lemmas (1), (2), the fact that $\mathcal{T}_{N,\overline{\mathcal{O}}_M}$ is a vector space, and the induction on codimension

$$\dim \mathcal{O}_M - \dim \mathcal{O}_N = [N, N] - [M, M].$$