Hochschild homology and cohomology of the super Jordan plane and more

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The super Jordan plane is the algebra

$$A = \Bbbk\langle x, y\rangle / \left(x^2, y^2x - xy^2 - xyx\right),$$

where $\Bbbk$ is an algebraically closed field and $char(\Bbbk) = 0$.

Our results

- We computed explicit bases of the Hochschild cohomology and homology spaces.
- We described completely the cup product in cohomology and the Yoneda algebra.
- We described the Lie structure of $\mathrm{H}^1(A, A)$.
- We described the Lie module structure of $\mathrm{H}^n(A, A)$ for $n > 1$.
- We computed the Yoneda algebra structure of the bosonization of the super Jordan plane and proved that it is finitely generated.

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Some motivation

Given a Hopf algebra H, the associated cohomology is

$$\mathrm{H}^\bullet(H, \Bbbk) = \mathrm{Ext}^\bullet_H(\Bbbk, \Bbbk).$$

$\mathrm{H}^\bullet(H, \Bbbk)$ is endowed with an associative and graded commutative algebra structure via the cup product.

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When is $\mathrm{H}^\bullet(H, \Bbbk)$ a finitely generated algebra?

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Some well-known results

- Finite group algebras (Golod '59, Venkov '59, Evens '61).
- Finite dimensional cocommutative Hopf algebras (Friedlander-Suslin '97).
- Finite dimensional small quantum groups (over $\mathbb{C}$), satisfying some hypotheses (Ginzburg-Kumar '93, Bendel-Nakano-Parshall-Pillen '07).

Conjecture (Etingof-Ostrik '04)

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If H is a finite dimensional Hopf algebra, then $\mathrm{H}^\bullet(H, \Bbbk)$ is finitely generated.

Theorem (Mastnak-Pevtsova-Schauenburg-Witherspoon '10)

If H is a pointed finite dimensional Hopf algebra verifying some hypotheses, then $\mathrm{H}^\bullet(H, \Bbbk)$ is finitely generated.

The proof is reduced to the case

$$\mathrm{gr}(H) \cong \mathfrak{B}(V) \# \Bbbk G$$

Recall that if H is a Hopf algebra with bijective antipode, then $\mathrm{H}^\bullet(H, \Bbbk)$ is a direct summand of $\mathrm{H}^\bullet(H, H)$.

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Nichols algebras

Definition

Let H be a Hopf algebra and let V be a $\Bbbk$-vector space. The space V is a left *Yetter-Drinfeld module over* H if V is a left H-module and H-comodule satisfying a compatibility condition.

The category of left Yetter-Drinfeld modules over H is usually denoted by ${}^H_H\mathcal{YD}$.

- If V and W belong to ${}^H_H\mathcal{YD}$, then the same is true for $V \otimes W$.
- $c_{V,W} : V \otimes W \to W \otimes V$.

These remarks lead to the definition of braided Hopf algebra.

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Example

Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}$. The tensor algebra $T(V) = \oplus_{n\geq 0} V^{\otimes n}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ with comultiplication and counit determined by

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \quad \varepsilon(v) = 0 \quad \text{for all } v \in V.$$

There is a unique maximal coideal $\mathfrak{J}(V)$ amongst the coideals of $T(V)$ contained in $\oplus_{n\geq 2} V^{\otimes n}$.

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From now on we will consider the case $H = \Bbbk G$, where $\Bbbk$ is an algebraically closed field, $\mathrm{char}(\Bbbk) = 0$ and G is an abelian group.

Problem

When is the Gelfand-Kirillov dimension of $\mathfrak{B}(V)$ finite?

The classification of pointed Hopf algebras with finite Gelfand-Kirillov dimension, abelian group of group-like elements and diagonal braiding is known if H is a domain.

Theorem (Andruskiewitsch-Angiono-Heckenberger)

The Gelfand-Kirillov dimension of $\mathfrak{B}(\mathcal{V}(\varepsilon, l))$ is finite if and only if $l = 2$ y $\varepsilon = \pm 1$ (and in this case the GK-dimension is 2). Moreover,

- if $\varepsilon = 1$, then
$$\mathfrak{B}(\mathcal{V}(1,2)) = \Bbbk \langle x, y \mid yx - xy + \tfrac{1}{2}x^2 = 0 \rangle \text{ (Jordan plane).}$$
- If $\varepsilon = -1$, then
$$\mathfrak{B}(\mathcal{V}(-1,2)) = \Bbbk \langle x, y \mid x^2 = 0, y^2x - xy^2 - xyx = 0 \rangle$$
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The super Jordan plane

Going back to the algebra

$$A = \Bbbk\langle x, y\rangle / \left(x^2, y^2x - xy^2 - xyx\right).$$

- Its Gelfand-Kirillov dimension is 2.
- The set $\{x^a(yx)^b y^c : a \in \{0,1\}, b, c \in \mathbb{N}_0\}$ is a PBW basis.
- It is graded with $|x| = |y| = 1$.

First objective

Compute its Hochschild homology and cohomology.

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A projective resolution

Using the methods of Chouhy–S., '15.

Rewriting system

The ideal of relations is $(x^2, y^2x - xy^2 - xyx)$.

$$x^2 = 0,$$
$$y^2x = xy^2 + xyx.$$

Ambiguities

- x^3
- y^2x^2

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The projective resolution of A as A-bimodule that we use for the computations is the following. It is in fact minimal and free.

$$\cdots \xrightarrow{d_{n+1}} A \otimes \Bbbk\mathcal{A}_n \otimes A \xrightarrow{d_n} A \otimes \Bbbk\mathcal{A}_{n-1} \otimes A \xrightarrow{d_{n-1}} \cdots$$

$$\cdots \xrightarrow{d_2} A \otimes \Bbbk\left\{x^2, y^2x\right\} \otimes A \xrightarrow{d_1} A \otimes \Bbbk\left\{x, y\right\} \otimes A \xrightarrow{d_0} A \otimes A \longrightarrow 0$$

where $\mathcal{A}_n = \{x^{n+1}, y^2x^n\}$, for $n \geq 1$; the differentials are defined as follows:

$$\begin{aligned}
&d_0(1 \otimes v \otimes 1) = v \otimes 1 - 1 \otimes v, \\
&d_1(1 \otimes x^2 \otimes 1) = x \otimes x \otimes 1 + 1 \otimes x \otimes x, \\
&d_1(1 \otimes y^2x \otimes 1) = y^2 \otimes x \otimes 1 + y \otimes y \otimes x + 1 \otimes y \otimes yx \\
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\end{aligned}
$$

for all $n \geq 2$.

Example

$$d_2 : A \otimes \Bbbk\left\{x^3, y^2x^2\right\} \otimes A \to A \otimes \Bbbk\left\{x^2, y^2x\right\} \otimes A.$$

$$
\begin{aligned}
d_2(1 \otimes y^2x^2 \otimes 1) &= y^2 \otimes x^2 \otimes 1 - 1 \otimes y^2x \otimes x \\
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The double complex

$$\begin{tikzcd}
 & {} \arrow[d, "\partial"'] & {} \arrow[d, "\partial'"'] \\
 & A \otimes \Bbbk\{x^4\} \otimes A \arrow[d, "\delta"] & A \otimes \Bbbk\{y^2x^4\} \otimes A \arrow[l, "d"'] \arrow[d, "\delta'"] \\
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A \otimes A & A \otimes \Bbbk\{x, y\} \otimes A \arrow[l, "d_0"'] & A \otimes \Bbbk\{y^2x\} \otimes A \arrow[l, "d_1"']
\end{tikzcd}$$

Hochschild cohomology

The complex computing the Hochschild cohomology of A is

$$\cdots \xleftarrow{d^{n+1}} \mathrm{Hom}_{A^e}(A \otimes \Bbbk\left\{x^{n+1}, y^2x^n\right\} \otimes A, A) \xleftarrow{d^n} \cdots$$

Using this complex, we obtained the description of the Hochschild cohomology spaces:

$$\begin{aligned}
\mathrm{H}^0(A,A) &\cong \Bbbk, \\
\mathrm{H}^1(A,A) &\cong \langle c, s_n \mid n \geq 0 \rangle, \\
\mathrm{H}^{2p}(A,A) &\cong \left\langle t_n^{2p}, u_n^{2p} \mid n \geq 0 \right\rangle, \\
\mathrm{H}^{2p+1}(A,A) &\cong \left\langle v_n^{2p+1}, w_n^{2p+1} \mid n \geq 0 \right\rangle.
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Note that cohomology is periodic of period 2.

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Description of $\mathrm{H}^\bullet(A, A)$ as a graded algebra

Problem

Our aim: to describe the product in $\mathrm{H}^\bullet(A, A)$.

One possible way is using the bar resolution. For this, we constructed comparison maps between this resolution and ours, using again the rewriting rules.

$$\begin{tikzcd} \arrow[r, "d_{n+1}"] & A \otimes \Bbbk\left\{x^{n+1}, y^2x^n\right\} \otimes A \arrow[r, "d_n"] \arrow[d, shift right, "f_{n+1}"'] & A \otimes \Bbbk\left\{x^n, y^2x^{n-1}\right\} \otimes A \arrow[r, "d_{n-1}"] \arrow[d, shift right, "f_n"'] & {} \\ \arrow[r, "b_{n+1}"] & A \otimes A^{\otimes n+1} \otimes A \arrow[r, "b_n"] \arrow[u, shift right, "g_{n+1}"'] & A \otimes A^{\otimes n} \otimes A \arrow[r, "b_{n-1}"] \arrow[u, shift right, "g_n"'] & {} \end{tikzcd}$$

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- for $n = 1$,

$$f_1 : A \otimes \Bbbk\{x, y\} \otimes A \to A \otimes A \otimes A,$$
$$f_1(1 \otimes v \otimes 1) = 1 \otimes v \otimes 1 \quad \text{for all } v \in \Bbbk\{x, y\},$$

- for $n \geq 2$,

$$f_n : A \otimes \Bbbk\left\{x^n, y^2x^{n-1}\right\} \otimes A \to A \otimes A^{\otimes n} \otimes A,$$
$$f_n(1 \otimes x^n \otimes 1) = 1 \otimes x^{\otimes n} \otimes 1,$$
$$f_n(1 \otimes y^2x^{n-1} \otimes 1) = y \otimes y \otimes x^{\otimes n-1} \otimes 1 + 1 \otimes y \otimes yx \otimes x^{\otimes n-2} \otimes 1$$
$$- x \otimes y \otimes y \otimes x^{\otimes n-2} \otimes 1 - 1 \otimes x \otimes y^2 \otimes x^{\otimes n-2} \otimes 1$$
$$- x \otimes y \otimes x^{\otimes n-1} \otimes 1 - 1 \otimes x \otimes yx \otimes x^{\otimes n-2} \otimes 1$$
$$+ \cdots .$$

We do not have an explicit expression for the maps

$$g_n : A \otimes A^{\otimes n} \otimes A \to A \otimes \Bbbk \left\{x^n, y^2x^{n-1}\right\} \otimes A.$$

Example

$$g_2 : A \otimes A^{\otimes 2} \otimes A \to A \otimes \Bbbk \left\{x^2, y^2x\right\} \otimes A$$

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Proposition

The algebra $\mathrm{H}^\bullet(A, A)$ is not finitely generated.

Recall that $\mathrm{H}^0(A, A) \cong \Bbbk$. Given $\lambda \in \mathrm{H}^0(A, A)$ and $\varphi \in \mathrm{H}^\bullet(A, A)$,

$$\varphi \smile \lambda = \lambda \smile \varphi = \lambda\varphi.$$

Recalling the notation

$$\begin{aligned}
\mathrm{H}^1(A, A) &\cong \langle c, s_n \mid n \geq 0 \rangle, \\
\mathrm{H}^{2p}(A, A) &\cong \left\langle t_n^{2p}, u_n^{2p} \mid n \geq 0 \right\rangle, \\
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Table of products of the generators

$c \smile \varphi = 0$ for all $\varphi \in \mathrm{H}^\bullet(A, A)$

	s_n	t_n^{2q}	u_n^{2q}	v_n^{2q+1}	w_n^{2q+1}
s_m	$4(n-m)t^2_{n+m+1}$	0	$2v^{2q+1}_{n+m+1}$ $+(2n+1)w^{2q+1}_{n+m}$	$-(2n+1)t^{2p+2}_{n+m}$	$2t^{2p+2}_{n+m+1}$
t_m^{2p}	0	0	t^{2p+2q}_{m+n}	0	0
u_m^{2p}	$2v^{2p+1}_{n+m+1}$ $+(2n+1)w^{2p+1}_{n+m}$	t^{2p+2q}_{n+m}	u^{2p+2q}_{m+n}	$v^{2p+2q+1}_{n+m}$	$w^{2p+2q+1}_{n+m}$
v_m^{2p+1}	$(2n+1)t^{2p+2}_{n+m}$	0	$v^{2p+2q+1}_{m+n}$	0	$t^{2p+2q+2}_{m+n}$
w_m^{2p+1}	$-2t^{2p+2}_{n+m+1}$	0	$w^{2p+2q+1}_{m+n}$	$-t^{2p+2q+2}_{m+n}$	0

$\mathrm{H}^\bullet(A, A)$ is generated as an algebra by the elements 1, c, t_0^2, s_n, u_n^2 y v_n^3 with $n \geq 0$.

Description of $\mathrm{H}^1(A, A)$ as a Lie algebra

Recall that $\mathrm{H}^1(A, A) \cong \mathrm{Der}_{\Bbbk}(A, A)/\operatorname{Inn}_{\Bbbk}(A, A)$. The structure of Lie algebra is given by

$$[\overline{f}, \overline{h}] = \overline{f} \circ \overline{h} - \overline{h} \circ \overline{f}$$

for all $\overline{f}, \overline{h} \in \mathrm{H}^1(A, A)$.

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- $[c, s_n] = 0$
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Virasoro algebra

The Virasoro algebra, denoted by Vir, is the Lie algebra with basis $\{L_n, c \mid n \in \mathbb{Z}\}$ defined as follows: for all $n, m \in \mathbb{Z}$,

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c,$$
$$[L_m, c] = 0.$$

The triangular decomposition of the Virasoro algebra is:

$$\mathrm{Vir} \cong \mathrm{Vir}^+ \oplus \mathfrak{h} \oplus \mathrm{Vir}^- .$$

where

$$\mathrm{Vir}^+ = \bigoplus_{n=1}^{\infty} \Bbbk L_n \qquad \mathfrak{h} = \Bbbk c \oplus \Bbbk L_0 \qquad \mathrm{Vir}^- = \bigoplus_{n=1}^{\infty} \Bbbk L_{-n}.$$

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In order to identify $\mathrm{H}^1(A, A)$ with a Lie subalgebra of the Virasoro algebra, we set

$$L'_m = 2^{-m-1} s_m$$

for all $m \geq 0$. The bracket is thus

$$[L'_m, L'_n] = (m-n)L'_{m+n} \text{ and } [L'_m, c] = 0.$$

Theorem

There exists a Lie algebra isomorphism

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Remarks and questions

- Localizations? Drinfeld doubles?
- Any representation M of our Nichols algebra provides by induction of representations, a module over $\mathcal{U}(\mathrm{Vir})$.
- Given $n \in \mathbb{N}$, the vector space $\mathrm{H}^n(A, A)$ is a representation of $\mathrm{H}^1(A, A)$.
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Action of $\mathrm{H}^1(A, A)$ on $\mathrm{H}^n(A, A)$

Theorem

- The action of c on all the generators is trivial.
- $\left[s_m, t_n^{2p}\right] = 2\Big(n - (2p-1)m - p\Big)t_{n+m}^{2p}$,
- $\left[s_m, u_n^{2p}\right] = 2\Big((n - 2pm - p)u_{n+m}^{2p} + pm(2m+1)t_{n+m}^{2p}\Big)$.
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- $\left[s_m, w_n^{2p+1}\right] = \Big(n - 2pm - p\Big)w_{n+m}^{2p+1}$.

The proof uses Suárez-Álvarez method to compute the brackets.

Looking at an intermediate series module $V_{a,b}$ as a module over $Vir^+ \oplus \mathfrak{h}$ -that we denote $V_{a,b}^+$- and rescaling, we can see that the space generated by (for example) the t_n^{2p}'s for a fixed p is isomorphic to a submodule of the intermediate series module $V_{-(2p-1),-p}^+$.

These modules are indecomposable.

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The Yoneda algebra

Proposition

- The space $\mathrm{H}^0(A,\Bbbk)$ is isomorphic to $\Bbbk$ with basis $\{e : 1 \otimes 1 \to 1\}$.
- $\mathrm{H}^1(A,\Bbbk)$ is 2-dimensional with basis $\left\{\eta^1, \omega^1\right\}$ defined by

$$\eta^1(1 \otimes x \otimes 1) = 1, \quad \eta^1(1 \otimes y \otimes 1) = 0,$$
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The Hilbert series of $\oplus_{i \geq 0} \mathrm{H}^i(A,\Bbbk)$ is $h(t) = 1 + 2\sum_{i \geq 1} t^i = \frac{1+t}{1-t}$.

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The algebra $H^\bullet(A, \Bbbk)$ is generated by $\{e, \eta^1, \omega^1, \omega^2\}$. Moreover, $H^\bullet(A, \Bbbk)$ is isomorphic to the graded algebra

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Notice that $(H^\bullet(A, \Bbbk), \smile)$ is not a graded commutative algebra and in particular, $(H^\bullet(A, \Bbbk), \smile)$ is not a subalgebra of $(H^\bullet(A, A), \smile)$.
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The Yoneda algebra of $A\#\Bbbk\mathbb{Z}$

A is a $\Bbbk\mathbb{Z}$-module algebra, where the action of $\Bbbk\mathbb{Z}$ on A corresponds to the braiding c of $V(-1,2)$.
Using Grothendieck's spectral sequence for the derived functors of the composition of two functors , we get:

$$E_2^{p,q} = \mathrm{H}^p\left(\mathbb{Z}, \mathrm{H}^q(A,\Bbbk)\right) \Rightarrow \mathrm{H}^{p+q}(A\#\Bbbk\mathbb{Z},\Bbbk).$$

We have a first quadrant spectral sequence with only two non trivial rows, and the differential $d_2 : E_2^{p,q} \to E_2^{p-1,q+2}$ can only be non trivial when $p=1$. Moreover, there is a five term exact sequence

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Due to the shape of the spectral sequence, it will collapse at $E_3^{\bullet,\bullet}$.
Moreover, $E_2^{0,1}$ is 0, and since both $E_2^{1,0}$ and $E_2^{0,2}$ are one dimensional, d_2 is either zero or an isomorphism, and this will depend on whether $\mathrm{H}^1(A\#\Bbbk\mathbb{Z},\Bbbk)$ is zero or not.

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This last space is isomorphic to $\mathrm{Ext}^1_{A\#\Bbbk\mathbb{Z}}(\Bbbk,\Bbbk)$, that is, the space of classes of isomorphisms of 1-extensions of $\Bbbk$ by $\Bbbk$.

We found nontrivial extensions, so

$$\mathrm{H}^1(A\#\Bbbk\mathbb{Z},\Bbbk) \cong \Bbbk.$$

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The Yoneda algebra of $A\#\Bbbk\mathbb{Z}$

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Theorem

The Yoneda algebra $\mathrm{E}(A\#\Bbbk\mathbb{Z}) = \oplus_{i\geq 0}\mathrm{H}^i(A\#\Bbbk\mathbb{Z},\Bbbk)$ is the $\Bbbk$-algebra generated by $\overline{e},\eta^2,\omega^3$, where $\deg(\overline{e}) = 1$, $\deg(\eta^2) = 2$, $\deg(\omega^3) = 3$. It is graded commutative, summarizing

$$\mathrm{E}(A\#\Bbbk\mathbb{Z}) \cong \Bbbk\left[\eta^2\right] \otimes \Lambda(\omega^3,\overline{e}).$$

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