Forbidden subgraphs of some graphs representable by arcs on a circle

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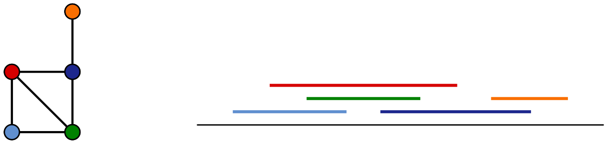
Interval graphs

Interval graphs

A graph is an interval graph if it is possible to assign an interval to each vertex in such a way that two different vertices are adjacent if and only if the corresponding intervals have nonempty intersection.



The corresponding set of intervals is called an interval model of the graph.

Hajós (1957) posed the problem of characterizing interval graphs.

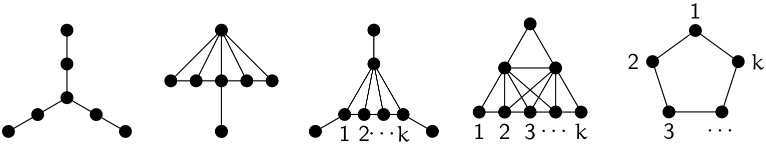
Interval graphs

There are many different characterizations of interval graphs.

The following is the minimal forbidden induced subgraph characterization of interval graphs. (An induced subgraph of a graph G is any graph obtained from G by vertex removals.)

Theorem (Lekkerkerker and Boland, 1962)

A graph is an interval graph if and only if it contains none of the following as an induced subgraph:



bipartite claw | umbrella | k-net, $k \geqslant 2$ | k-tent, $k \geqslant 3$ | C_k, $k \geqslant 4$

Recently, Lindzey and McConnell (2016) devised the first linear-time algorithm that, given any graph G which is not an interval graph, finds one of the minimal forbidden induced subgraphs of Lekkerkerker and Boland's characterization contained in G as an induced subgraph.

Characterizing and finding forbidden subgraphs

- Algorithms for recognizing interval graphs and producing an interval model if the input is an interval graph are known since long ago; e.g., Fulkerson and Gross (1965) (first polynomial-time) and Booth and Lueker (1976) (first linear-time).
- The algorithm by Lindzey and McConnell (2016) adds the possibility of producing one of Lekkerkerker and Boland's minimal forbidden induced subgraphs when the input graph is not an interval graph. (Previously, Kratsch, McConnell, Mehlhorn, and Spinrad (2006) devised an algorithm which, given any graph that is not an interval graph, produced a hole or an asteroidal triple, which form the obstruction set in a different characterization of interval graphs).
- This approach of complementing recognition algorithms by an algorithm that produces a minimal forbidden induced subgraph when the input graph is not in the class may be applied to other hereditary graph classes (i.e., graph classes closed by taking induced subgraphs).

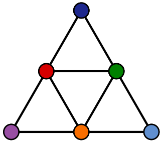
Characterizing and finding forbidden subgraphs

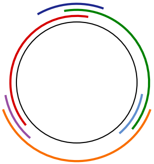
- We are interested in obtaining results analogous to those of Lekkerkerker and Boland (1962) and Lindzey and McConnell (2016), for other related hereditary graph classes.
- That is, we are interested in minimal forbidden induced subgraph characterizations and linear-time algorithms for finding one of the corresponding minimal forbidden induced subgraphs.
- In this talk, we will present some recent results in this direction regarding three different hereditary subclasses of circular-arc graphs and the main problems that remain open.

Circular-arc graphs

Circular-arc graphs

A graph is a circular-arc graph if it is possible to assign an arc of some fixed circle to each vertex in such a way that two different vertices are adjacent if and only if the corresponding arcs have nonempty intersection.





The set of circular-arcs is called a circular-arc model of the graph.

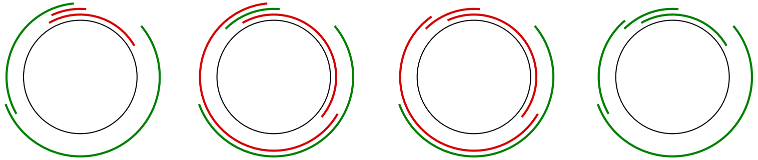
Klee (1969) was the first to pose explicitly the problem of characterizing circular-arc graphs.

Different subclasses of circular-arc graphs have been characterized.

Restricted circular-arc models and graphs

A circular-arc model is:

- proper if no arc of the model is properly contained in another arc of the model;
- normal if no two arcs together cover the whole circle;
- Helly if every set of pairwise intersecting arcs of the model has nonempty total intersection;
- proper Helly if it is simultaneously proper and Helly;
- normal Helly if it is simultaneously normal and Helly.



not proper model | not normal model | not Helly model | proper, normal and Helly model

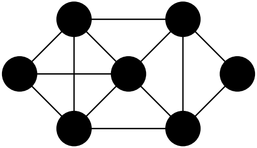
A circular-arc graph is proper, normal, Helly, proper Helly, or normal Helly if it admits at least one circular-arc model which is so.

Concave-round graphs

We will also consider the following graphs.

Concave-round graphs

A graph is concave-round if there is a circular enumeration of its vertices in such a way that the closed neighborhood of each vertex is an arc in the enumeration.



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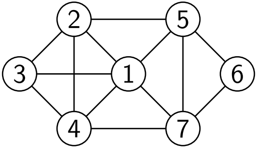

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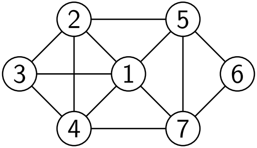
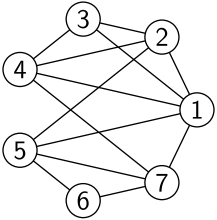


admits $1, 2, 3, 4, 5, 6, 7$ as a concave-round enumeration:



Concave-round graphs

Tucker (1971) was the first to study concave-round graphs as a special type of circular-arc graphs.

Theorem (Tucker, 1971)

Concave-round graphs form a subclass of circular-arc graphs.
Moreover, concave-round graphs form a superclass of proper circular-arc graphs.

The name 'concave-round' is due to Bang-Jensen, Huang, and Yeo (2000).

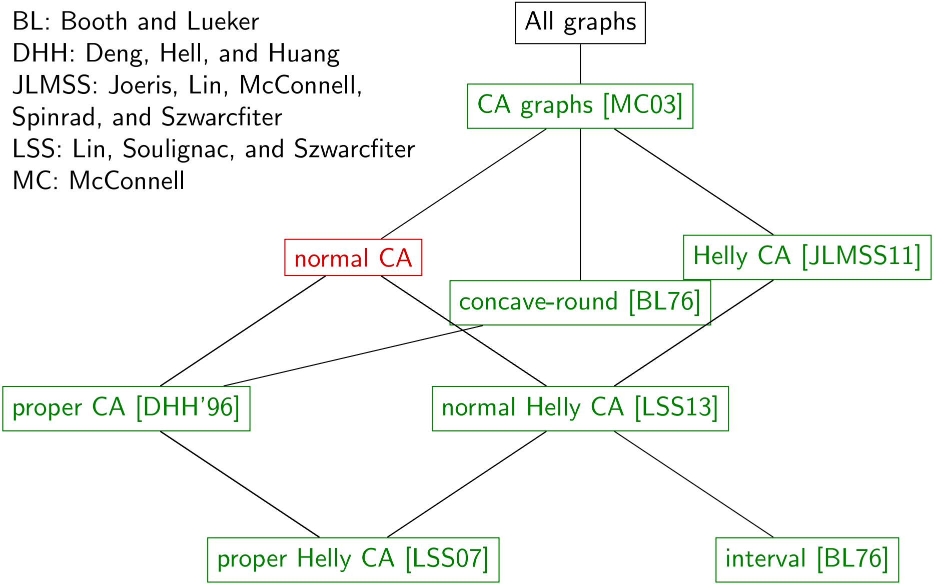
Concave-round graphs are also known as Γ circular-arc graphs or Tucker circular-arc graphs.

Circular-arc graphs: Recognition algorithms

- Linear-time recognition algorithms that also produce a corresponding circular-arc model or concave-round enumeration proving memembership to the class are known for all the subclasses of circular-arc graphs we will consider, with the only exception of normal circular-arc graphs (for which no polynomial-time recognition algorithm is known).
- The diagram that follows shows the classes we will discuss and the references to the first such linear-time algorithm in each case.

Circular-arc graphs: Recognition algorithms

BL: Booth and Lueker
DHH: Deng, Hell, and Huang
JLMSS: Joeris, Lin, McConnell, Spinrad, and Szwarcfiter
LSS: Lin, Soulignac, and Szwarcfiter
MC: McConnell



Circular-arc graphs: Forbidden subgraphs

- Our next diagram will present known results regarding forbidden induced subgraph characterizations of the same subclasses of circular-arc graphs and, on top this diagram, we will be presenting our results.

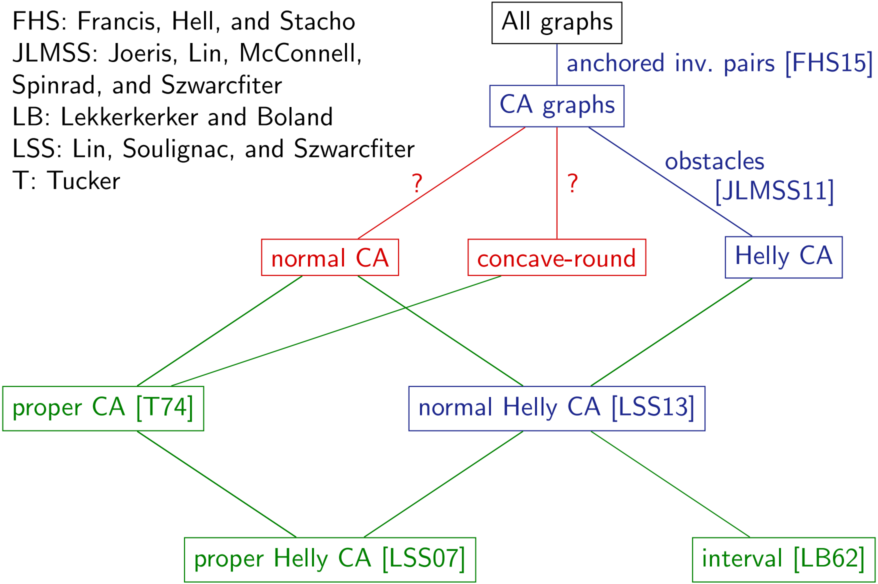
- A complete characterization of circular-arc graphs by forbidden structures, together with an $O(n^3)$-time algorithm for finding one such forbidden structure in any given graph that is not a circular-arc graph, was recently given by Francis, Hell, and Stacho (2015).

  The corresponding forbidden structures are called anchored invertible pairs which, roughly speaking, a pair of mutually avoiding walks that also avoid a fixed third vertex in a so-called circular completion of the graph.

  Although not strictly a forbidden induced subgraph characterization, it is very relevant as a forbidden structure characterization of circular-arc graphs and we include it in our chart.

Circular-arc graphs: Forbidden subgraphs/structures

FHS: Francis, Hell, and Stacho
JLMSS: Joeris, Lin, McConnell, Spinrad, and Szwarcfiter
LB: Lekkerkerker and Boland
LSS: Lin, Soulignac, and Szwarcfiter
T: Tucker



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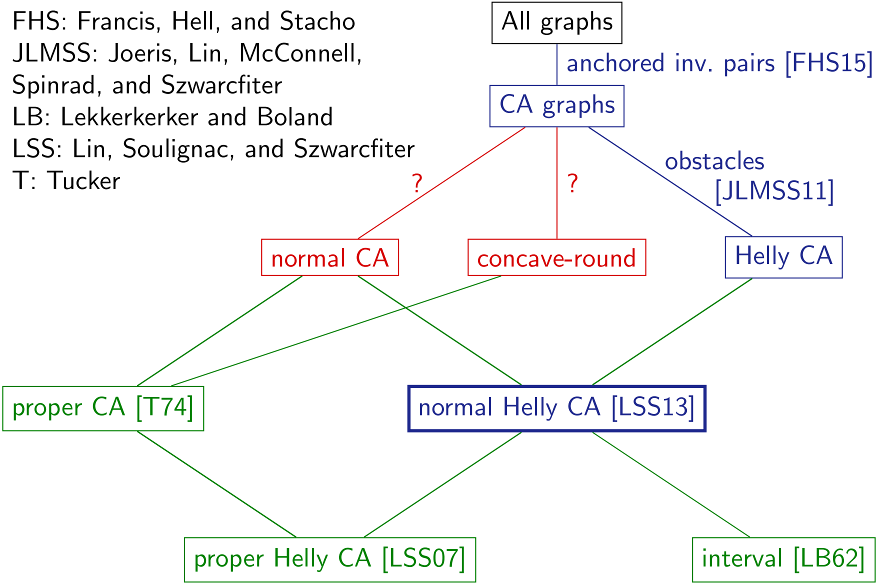
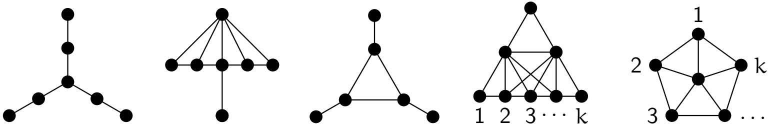


Forbidden subgraphs of normal Helly circular-arc graphs

Lin, Soulignac, and Szwarcfiter (2013) proved the following characterization of normal Helly circular-arc graphs by minimal forbidden induced subgraphs, by restricting the characterization to circular-arc graphs only.

Theorem (Lin, Soulignac, and Szwarcfiter, 2013)

Let G be a circular-arc graph. Then, G is a normal Helly circular-arc graph if and only if G contains none of the following as an induced subgraph:



bipartite claw, umbrella, net, k-tent, $k \geqslant 3$, k-wheel, $k \geqslant 4$

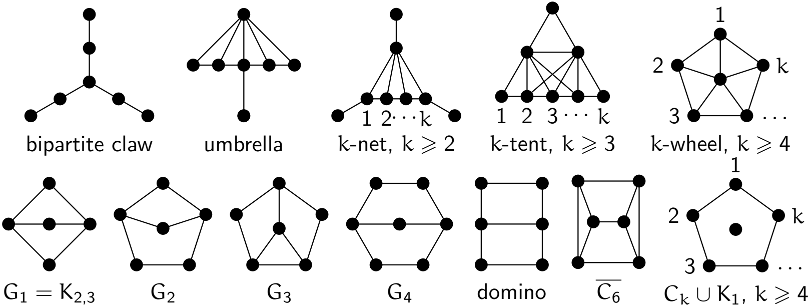
They also posed the problem of finding the complete characterization (i.e., dropping the hypothesis that G is a circular-arc graph).

Forbidden subgraphs of normal Helly circular-arc graphs

We solved this problem; i.e,. we found the complete list of minimal forbidden induced subgraphs for the class of normal Helly circular-arc graphs.

Theorem (Cao, Grippo, and S., 2017)

A graph is a normal Helly circular-arc graph if and only if it contains none of the following graphs as an induced subgraph:



Forbidden subgraphs of normal Helly circular-arc graphs

Theorem (Cao, Grippo, and S., 2017)

There is a linear-time algorithm that, given any graph G that is not a normal Helly circular-arc graph, finds an induced subgraph of G that is a minimal forbidden induced subgraph for the class of normal Helly circular-arc graphs.

The main idea behind the algorithm is that, if G is a graph that is not a normal Helly circular-arc graph, considering certain auxiliary graph $\mho(G)$, either $\mho(G)$ is non-interval or from any interval model of $\mho(G)$ we can build a normal circular-arc model of G which is not Helly. In both cases, we show how this leads to an induced subgraph of G isomorphic to one of the stated minimal forbidden induced subgraphs.

Circular-arc graphs: Forbidden subgraphs/structures

CGS: Cao, Grippo, and S.
FHS: Francis, Hell, and Stacho
JLMSS: Joeris, Lin, McConnell, Spinrad, and Szwarcfiter
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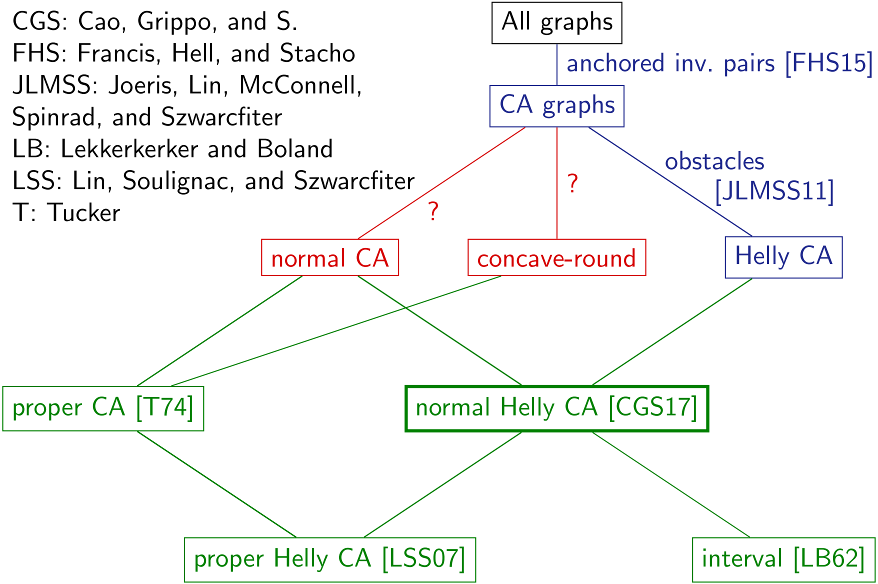

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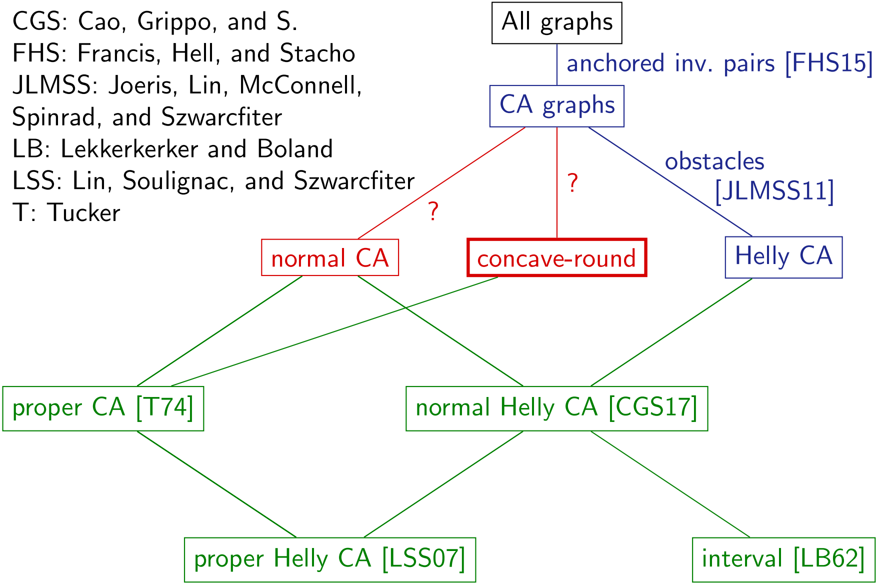


Concave-round graphs and the circular-ones property

Circular-ones property (Tucker, 1970)

- A matrix has the circular-ones property for rows if there is a circular ordering of its columns in such a way that that the ones in each row are consecutive in this circular ordering.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

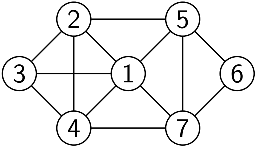
- The circular-ones property for columns is defined analogously.

Concave-round graphs and the circular-ones property

An augmented adjacency matrix $M(G)$ of a graph G arises by putting 1's all along the diagonal of an adjacency matrix of G.

Remark

A graph G is concave-round if and only if the augmented adjacency matrix $M(G)$ has the circular-ones property for rows and columns.



$$M(G) = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

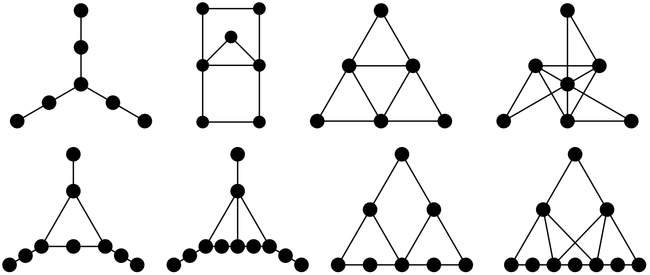
Forbidden subgraphs of concave-round graphs

Bang-Jensen, Huang, and Yeo (2000) posed the problem of characterizing concave-round graphs by forbidden induced subgraphs.

The result below gives the solution to the problem.

Theorem (S., 2016[+][a])

A graph is concave-round if and only if G contains none of the following as an induced subgraph: $C_k \cup K_1$ for any $k \geqslant 4$, $\overline{C_{2k+1} \cup K_1}$ for any $k \geqslant 1$, $\overline{C_{2k}}$ for any $k \geqslant 3$, plus



Forbidden subgraphs of concave-round graphs

Theorem (S., 2016$^+$[a])

There is a linear-time algorithm that, given a graph G that is not concave-round, finds an induced subgraph of G which is a minimal forbidden induced subgraph for the class of concave-round graphs.

The proof is in two steps.

- We first identify the minimal forbidden submatrices for the circular-ones property for rows and columns and show that one such submatrix (if present) can be found in linear time.
- Then, we apply this result to the augmented adjacency matrix and exploit the fact that the class of proper circular-arc graphs is a subclass of the class of concave-round graphs.

We also observed, by combining results from Tucker (1971, 1974), Golumbic (1980), Müller (1997), and Hell and Huang (2004), that concave-round graphs are normal circular-arc graphs.

Circular-arc graphs: Forbidden subgraphs/structures

CGS: Cao, Grippo, and S.
FHS: Francis, Hell, and Stacho
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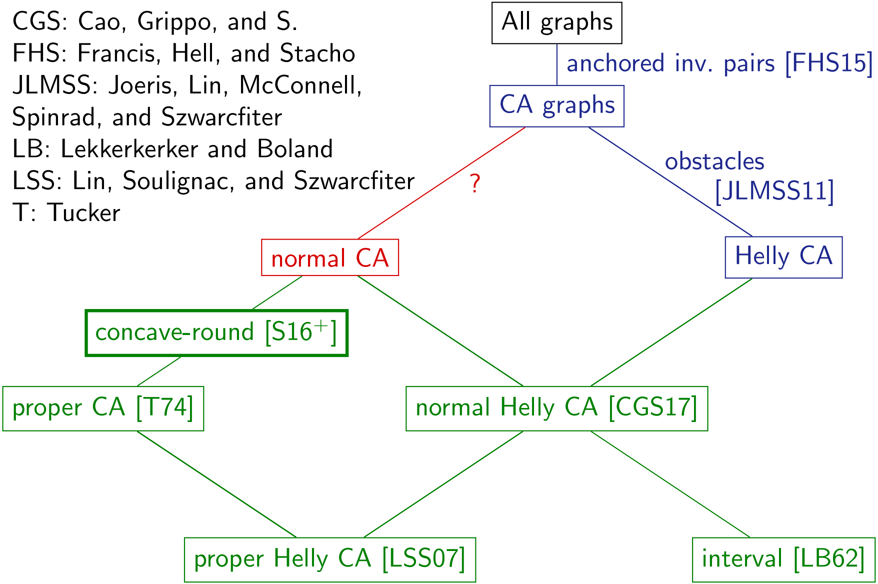

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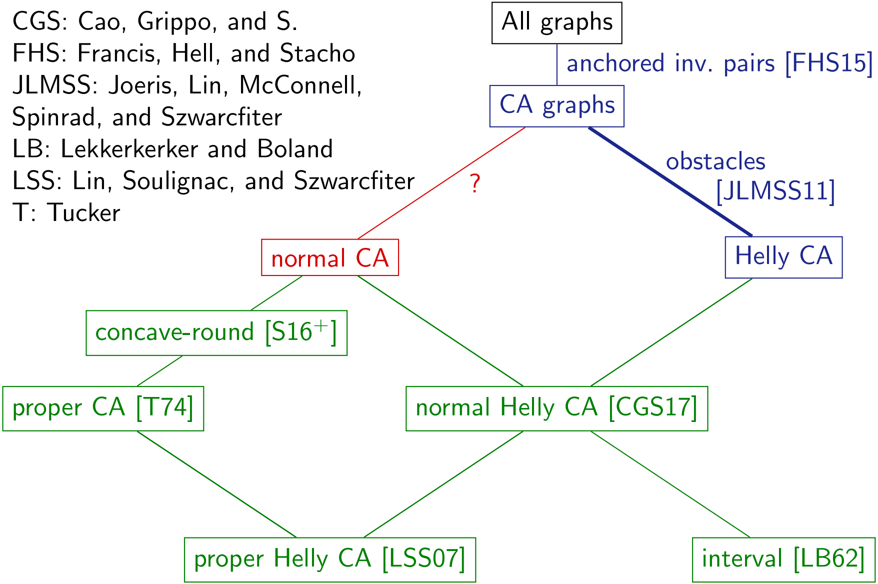


Obstacles and Helly circular-arc graphs

Theorem (Joeris, Lin, McConnell, Spinrad, and Szwarcfiter, 2011)

Let G be a circular-arc graph. Then, G is a Helly circular-arc graph if and only if G contains no induced obstacle.

This theorem gives a characterization of Helly circular-arc graphs by forbidden induced subgraphs restricted to circular-arc graphs.

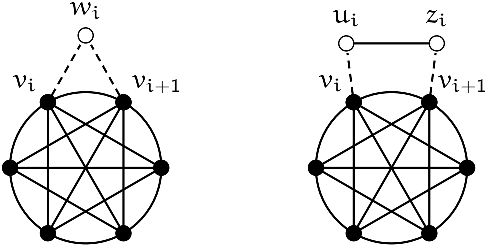
Nevertheless, this characterization is not by minimal forbidden induced circular-arc subgraphs because:

- There are obstacles that contain other obstacles with fewer vertices as induced subgraphs.
- There are obstacles which are not circular-arc graphs.

Obstacle enumeration and witnesses

Obstacle enumeration

An obstacle enumeration of a graph is circular enumeration $v_1, v_2, \ldots, v_k$ of the vertices for some clique Q such that $k \geqslant 3$ and, for each two consecutive vertices v_i and v_{i+1} in the circular enumeration (where v_{k+1} stands for v_1), one of the following situations holds:



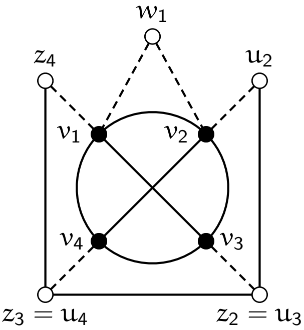
Black vertices are those of the clique Q and white vertices are called the witnesses of the circular enumeration.

Notice that adjacencies between witnesses corresponding to different pairs of consecutive vertices of the enumeration are arbitrary.

Obstacles

Obstacles
(Joeris, Lin, McConnell, Spinrad, and Szwarcfiter, 2011)

An obstacle is a graph G admitting an obstacle enumeration such that the vertices of G are those of the obstacle enumeration and its witnesses.



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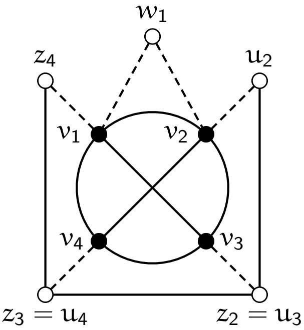

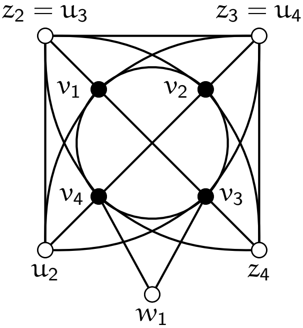

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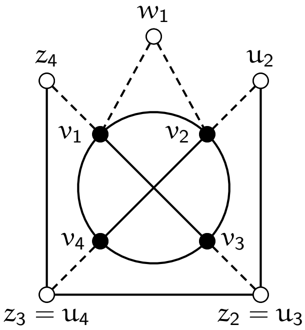

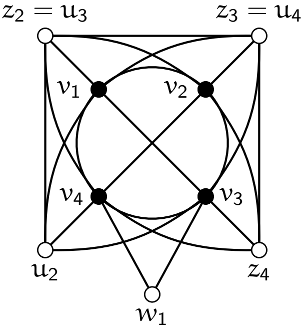

Except for those edges required by the definition of obstacle enumeration (u_2z_2, u_3z_3, and u_4z_4) adjacencies among white vertices are arbitrary.

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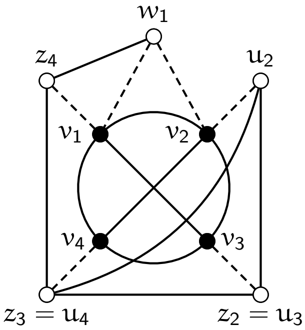

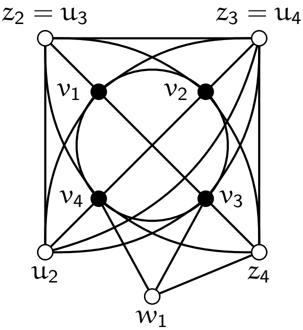

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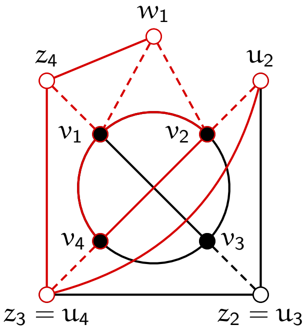
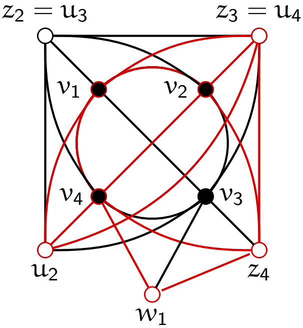
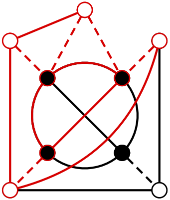


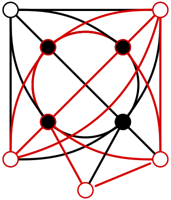


Except for those edges required by the definition of obstacle enumeration (u_2z_2, u_3z_3, and u_4z_4) adjacencies among white vertices are arbitrary.

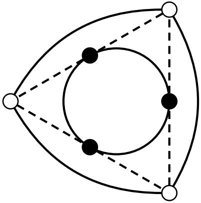
Obstacles may not be minimal or circular-arc

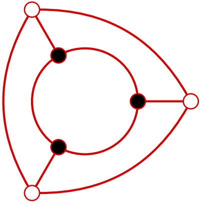
- Not only there are obstacles which contain other obstacles as induced subgraphs:





- As already noticed by Joeris, Lin, McConnell, Spinrad, and Szwarcfiter (2011), obstacles may not be circular-arc graphs.





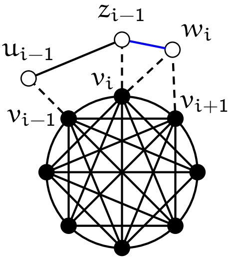
$\overline{C_6}$ is not circular-arc

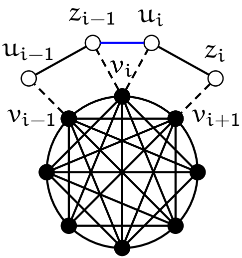
Valid edges

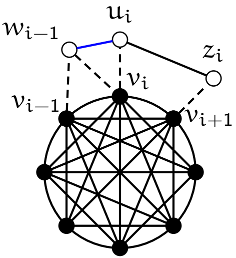
In order to overcome these drawbacks, we introduce essential obstacles, which are defined in terms of valid edges.

Valid edges

An edge joining two witnesses is valid if it is of one of the following types:







Essential enumerations and essential obstacles

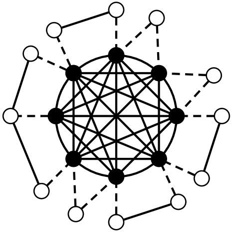
An obstacle enumeration $\mathcal{Q}$ is essential if every edge joining two of its witnesses is valid.

An obstacle is essential if it admits an essential obstacle enumeration.

Essential enumerations and essential obstacles

An obstacle enumeration Ω is essential if every edge joining two of its witnesses is valid.

An obstacle is essential if it admits an essential obstacle enumeration.



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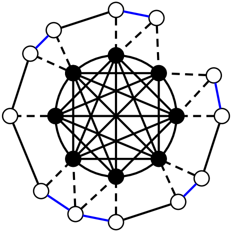

- Each of the blue edges may or may not be present.
- No other edges between white vertices are allowed.

Essential enumerations and essential obstacles

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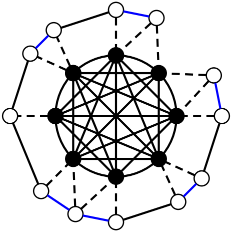


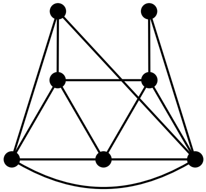
- Each of the blue edges may or may not be present.
- No other edges between white vertices are allowed.

Lemma (S., 2016^{+}[b])

Each essential obstacle is a circular-arc graph and a minimal forbidden induced subgraph for the class of Helly circular-arc graphs.

Non-essential obstacle enumerations

The following example shows a minimal forbidden induced subgraph for the class of Helly circular-arc graphs which is also a circular-arc graph:



Non-essential obstacle enumerations

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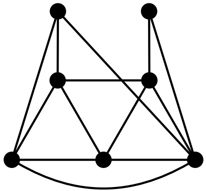

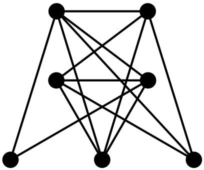

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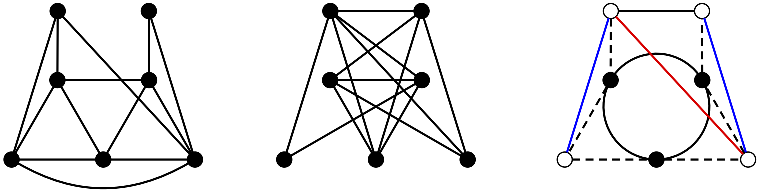

Non-essential obstacle enumerations

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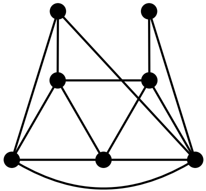

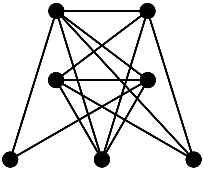

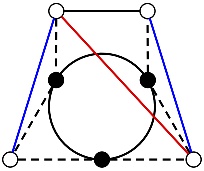

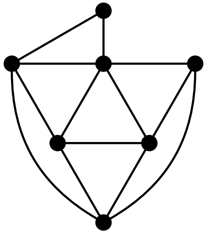

Non-essential obstacle enumerations

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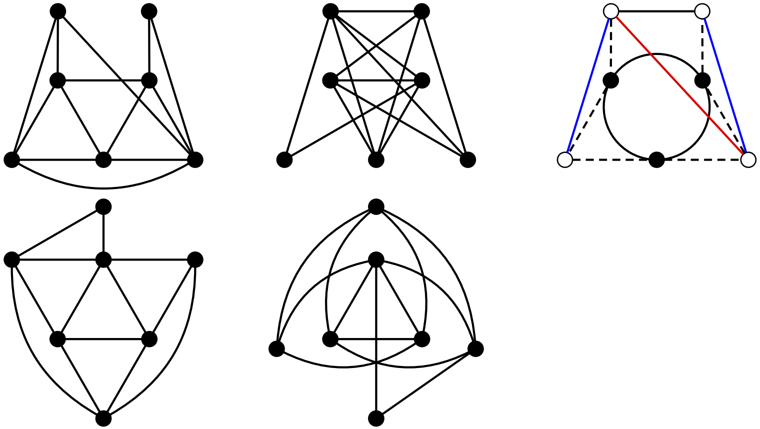

Non-essential obstacle enumerations

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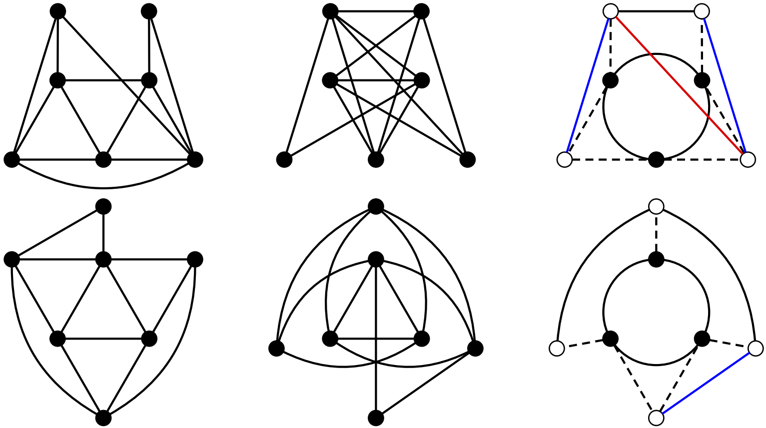
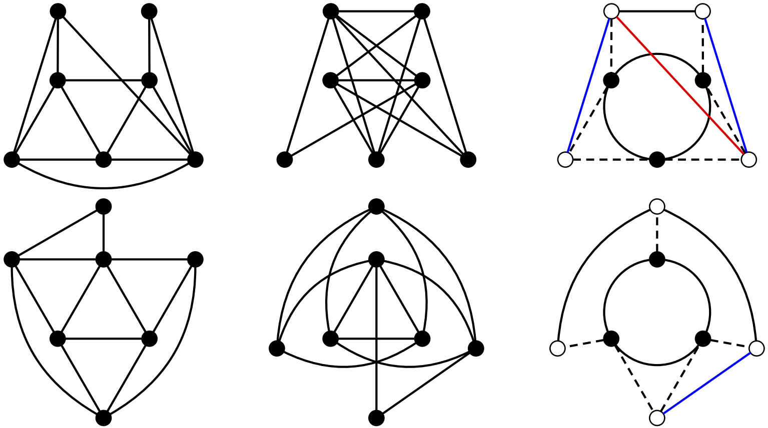


Non-essential obstacle enumerations

The following example shows a minimal forbidden induced subgraph for the class of Helly circular-arc graphs which is also a circular-arc graph:



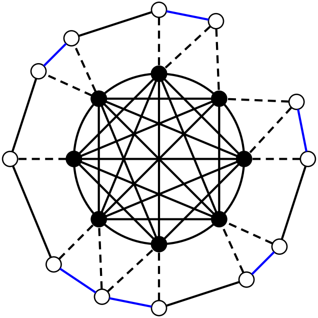
Lemma (S., 2016⁺[b])

Each minimal forbidden induced subgraph for the class of Helly circular-arc graphs that is a circular-arc graph is an essential obstacle.

Essential obstacles

Theorem (S., 2016$^+$[b])

Essential obstacles are precisely the minimal forbidden induced circular-arc subgraphs for the class of Helly circular-arc graphs.



Corollary

The number of minimal forbidden induced subgraphs for the class of Helly circular-arc having at most N vertices grows exponentially with N.

Essential obstacles

Joeris et al. (2011) gave an algorithm for finding an induced obstacle in any circular-arc graph which is not a Helly circular-arc graph.

Theorem (Joeris, Lin, McConnell, Spinrad, and Szwarcfiter, 2011)

Given a circular-arc graph G which is not a Helly circular-arc graph, it is possible find in linear time an obstacle induced in G.
Moreover, if a circular-arc model of G is given as input, the time bound reduces to $O(n)$.

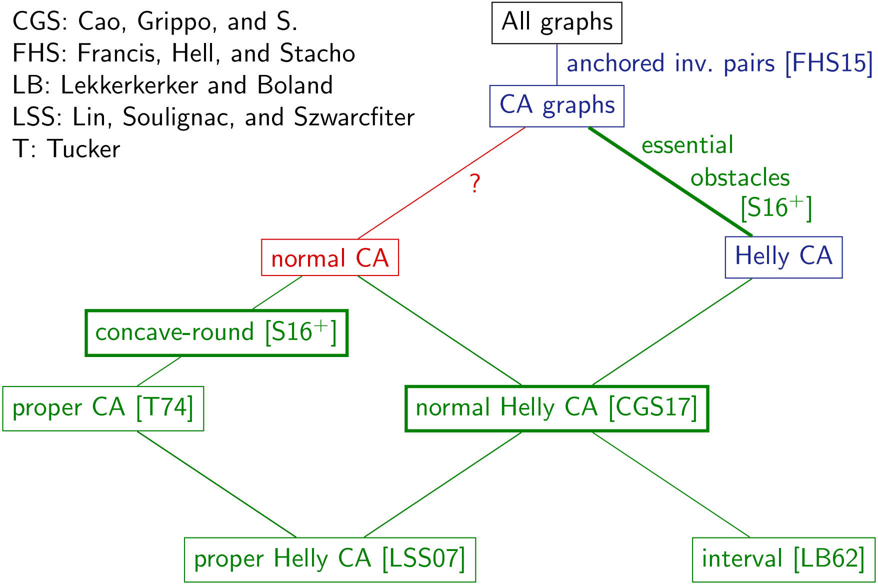
We managed to modify their algorithm so as to produce an essential obstacle instead, within the same time bound.

Theorem (S., 2016$^+$[b])

Given any circular-arc graph G which is not a Helly circular-arc graph, it is possible to find in linear time an essential obstacle induced in G.
Moreover, if a circular-arc model of G is given as input, the time bound reduces to $O(n)$.

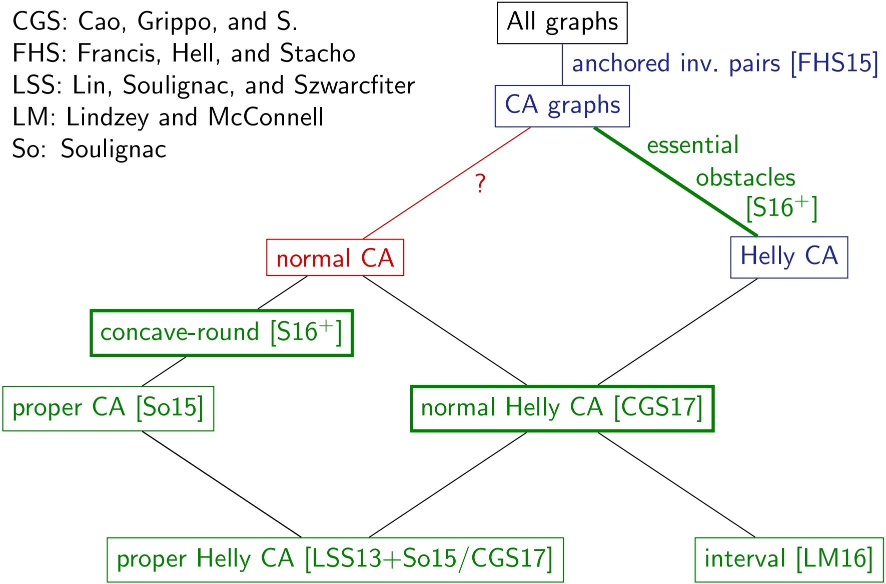
Circular-arc graphs: Forbidden subgraphs/structures

CGS: Cao, Grippo, and S.
FHS: Francis, Hell, and Stacho
LB: Lekkerkerker and Boland
LSS: Lin, Soulignac, and Szwarcfiter
T: Tucker



... and finding forbidden subgraphs/structures

CGS: Cao, Grippo, and S.
FHS: Francis, Hell, and Stacho
LSS: Lin, Soulignac, and Szwarcfiter
LM: Lindzey and McConnell
So: Soulignac



Main open problems

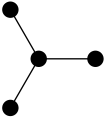
- Characterize circular-arc graphs by forbidden (sub)structures that can be found and authenticated in less than $O(n^3)$ time. The $O(n^3)$-time bound is matched by the algorithm by Francis, Hell, and Stacho (2015).
- Characterize normal circular-arc graphs by forbidden structures (at least within circular-arc graphs) and devise a polynomial-time algorithm for finding one of the corresponding forbidden substructures (if present).

  Hell and Huang (2004) proved that normal circular-arc graphs which are co-bipartite are precisely the complement of interval bigraphs, for which the forbidden induced subgraphs are only partially known (A. K. Das, S. Das, and Sen, 2016). More partial characterizations of normal circular-arc graphs were found by Bonomo, Durán, Grippo, and S. (2009).
- Characterize Helly circular-arc graphs by forbidden induced subgraphs not restricted to circular-arc graphs and design a polynomial-time algorithm for finding one of the corresponding forbidden induced subgraphs.

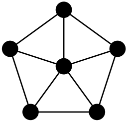
{claw, 5-wheel}-free Helly circular-arc graphs

As a partial solution to the last problem, we found:

- the characterization by minimal forbidden induced subgraphs of Helly circular-arc graphs restricted to {claw, 5-wheel}-free graphs, and
- a linear-time algorithm that, given any graph G that is not a Helly circular-arc graph, finds an induced subgraph of G which is isomorphic to claw, 5-wheel, or a minimal forbidden induced subgraph for the class of Helly circular-arc graphs.



claw



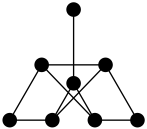
5-wheel

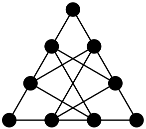
Notice that no forbidden induced subgraph characterization for circular-arc graphs restricted to {claw, 5-wheel}-free graphs is known. It is known for the following more restricted classes: complements of bipartite graphs (Trotter and Moore, 1976) and claw-free chordal graphs (Bonomo, Durán, Grippo, and S., 2009).

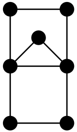
$\{$claw, 5-wheel$\}$-free Helly circular-arc graphs

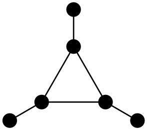
Theorem (S., 2016^+[b])

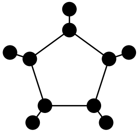
Let G be $\{$claw, 5-wheel$\}$-free graph. Then, G is a Helly circular-arc graph if and only if G contains no induced $\overline{3K_2}$, $\overline{P_7}$, $\overline{F_1}$, $\overline{F_2}$, $\overline{H_3}$, net, $\overline{2P_4}$, $\overline{F_8}$, $\overline{C_6}$, tent $\cup K_1$, or $C_k \cup K_1$ for any $k \geqslant 4$.

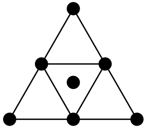

F_1


F_2


H_3


net


F_8

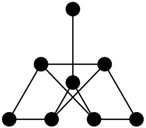

tent $\cup K_1$

For the proof, we determine explicitly all claw-free essential obstacles and exploit our characterization for concave-round graphs (S., 2016^+[a]).

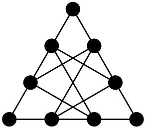
$\{$claw, 5-wheel$\}$-free Helly circular-arc graphs

Theorem (S., 2016^{+}[b])

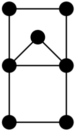
There is a linear-time algorithm that, given any graph G that is not a Helly circular-arc graph, finds an induced subgraph of G isomorphic to claw, 5-wheel, or one of the following minimal forbidden induced subgraphs for the class of Helly circular-arc graphs: $\overline{3K_2}$, $\overline{P_7}$, $\overline{F_1}$, $\overline{F_2}$, $\overline{H_3}$, net, $\overline{2P_4}$, $\overline{F_8}$, $\overline{C_6}$, tent $\cup K_1$, or $C_k \cup K_1$ for any $k \geqslant 4$.



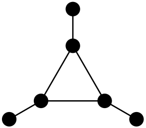
F_1



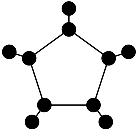
F_2



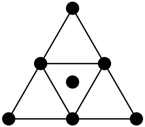
H_3



net



F_8



tent $\cup K_1$

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Thank you very much for your attention!

 

Happy 150 Years!