Simple Undirected Two-Commodity Integral Flow with a Unitary Demand

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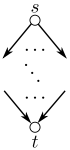
Introduction

- The classical single-commodity flow problem is well-known as MAXIMUM FLOW



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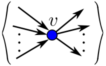

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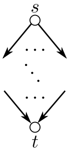

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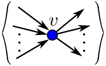

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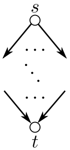

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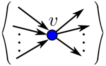

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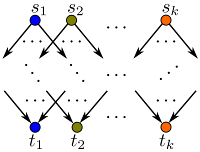

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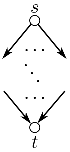

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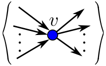

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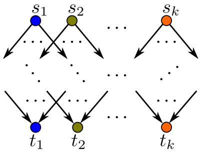

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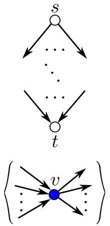

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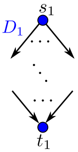

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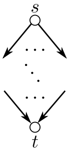

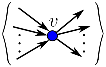

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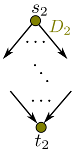

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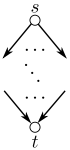

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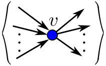

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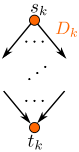

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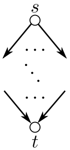

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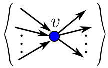

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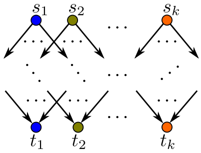

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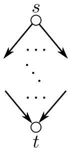

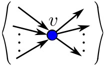

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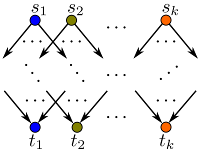

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- By using linear programming, the MULTICOMMODITY FLOW problem can be solved in polynomial-time if the flows are **real-valued** functions
- On the other hand, Karp (1975) proved that MULTICOMMODITY FLOW is a NP-complete problem if the flows must be **integral**.

Introduction: simple two-commodity integral instances

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- An instance of a network flow problem is called ***simple*** if the capacity of all edges of the input graph are **unitary**.
- A particular case of MULTICOMMODITY INTEGRAL FLOW is the SIMPLE TWO-COMMODITY INTEGRAL FLOW problem
  - Simple instances
  - Only two-commodities
  - Integral flows.

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- Nevertheless, for the **undirected case**, the hard instance constructed by them does not satisfy the condition of a demand to be unitary or even bounded by a constant.

- The main goal of our work is to **close this forty-year complexity gap**.

Simple undirected two-commodity integral flow (Simple U2CIF)

Input: An undirected graph G, two *commodities* $\{s_1, t_1\}$ and $\{s_2, t_2\}$, where s_1, t_1, s_2 and t_2 are vertices of G, and two *demands* $D_1, D_2 \in \mathbb{Z}^+$.

Question: Are there two flow functions $f_1, f_2 \colon \{\overrightarrow{uv}, \overrightarrow{vu} \mid uv \in E(G)\} \to \mathbb{Z}_0^+$ such that

1. for each $uv \in E(G)$ and $i \in \{1, 2\}$,
$$f_i(\overrightarrow{uv}) = 0 \text{ or } f_i(\overrightarrow{vu}) = 0;$$
2. for each $uv \in E(G)$, the total flow through uv does not exceed its *unitary capacity*, *i.e.*
$$\max\{f_1(\overrightarrow{uv}), f_1(\overrightarrow{vu})\} + \max\{f_2(\overrightarrow{uv}), f_2(\overrightarrow{vu})\} \leq 1;$$
3. for each $i \in \{1, 2\}$ and $v \in V \setminus \{s_i, t_i\}$, the flow function f_i is *conserved* at v, *i.e.*
$$\sum_{x \in N_G(v)} f_i(\overrightarrow{xv}) = \sum_{y \in N_G(v)} f_i(\overrightarrow{vy}); \text{ and}$$
4. for each $i \in \{1, 2\}$,
$$F_i = \sum_{v \in N_G(t_i)} f_i(\overrightarrow{vt_i}) \geq D_i?$$

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- So, let $I = (X, \mathcal{C})$ be a such instance of 3-SAT, where X is the variable set and $\mathcal{C}$ is the clause set of I.
- We construct from I an instance $g(I) = (G, \{s_1, t_1\}, \{s_2, t_2\}, D_1, D_2)$ of SIMPLE U2CIF as follows.

SIMPLE U2CIF: construction of the instance $g(I)$

- We define $D_1 = 1$ and $D_2 = 5m$, where $m = |\mathcal{C}|$.

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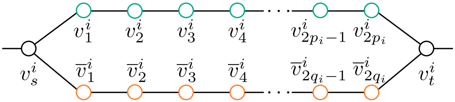
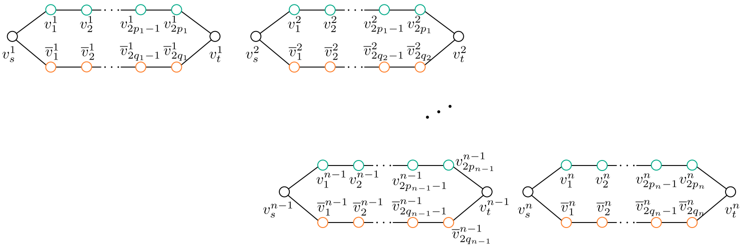
- For each variable $x_i \in X$, we create the gadget G_{x_i}:



  - p_i: the number of occurrences of the **positive literal** x_i;
  - q_i: the number of occurrences of the **negative literal** $\overline{x}_i$.

- We connect the gadgets G_{x_i} to one another in series:



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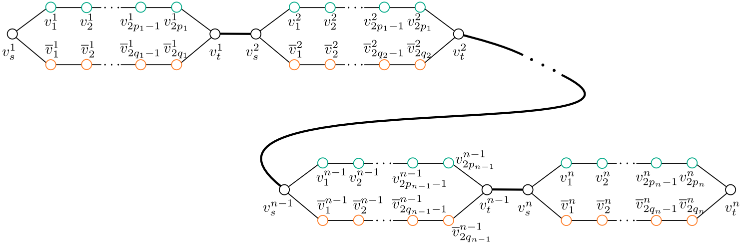

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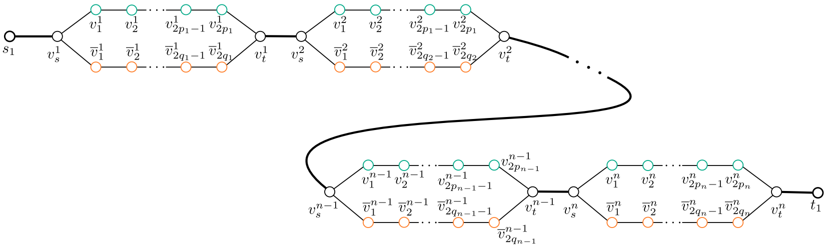

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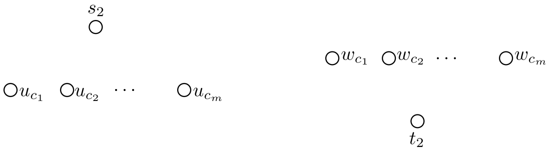

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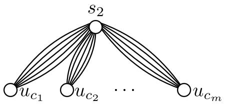

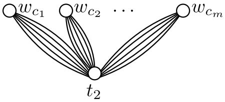

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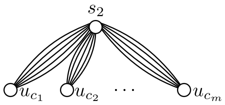

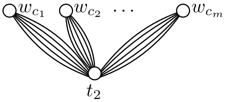

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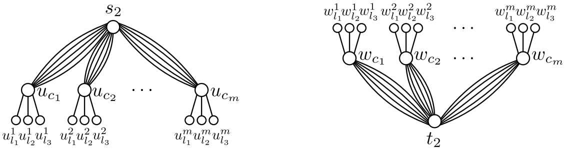

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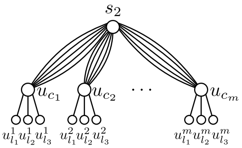

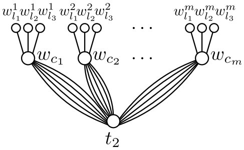

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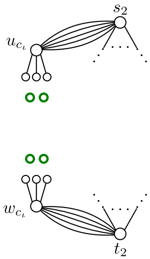

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- we create the *clause vertices* u_{c_ι} and w_{c_ι}, and we add five parallel edges between s_2 and u_{c_ι} and five parallel edges between w_{c_ι} and t_2;
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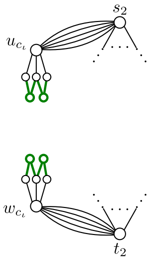

SIMPLE U2CIF: construction of the instance $g(I)$

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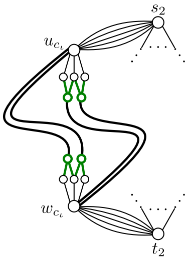

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- And, we add the edges $u^\iota_{l_\kappa} \overline{v}^i_{2j-1}$ and $\overline{v}^i_{2j} w^\iota_{l_\kappa}$ if the κ-th literal in C_ι corresponds to the *j*-th occurrence of the **negative** literal $\boldsymbol{\overline{x}_i}$.

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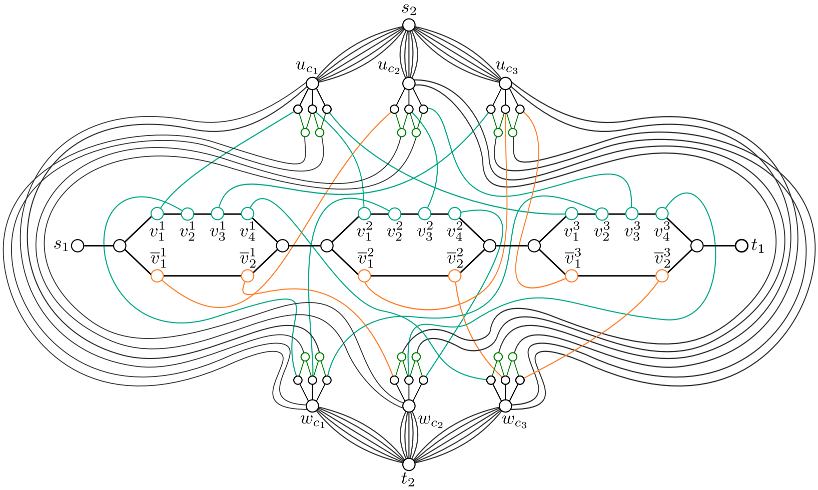

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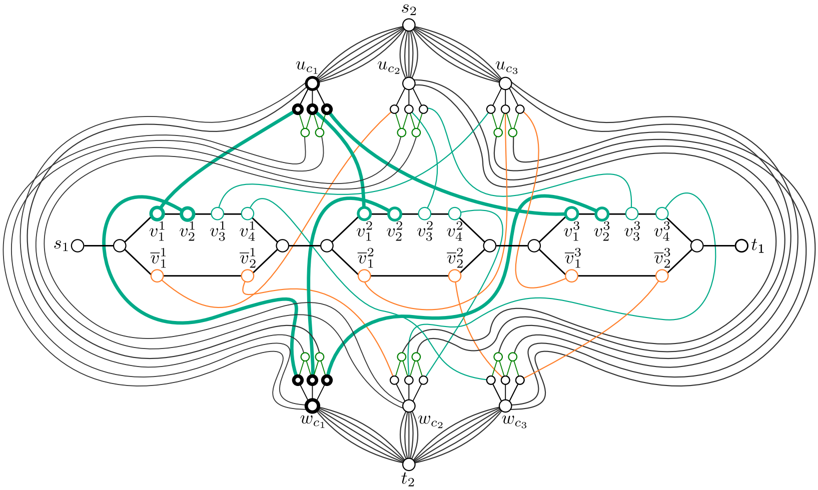

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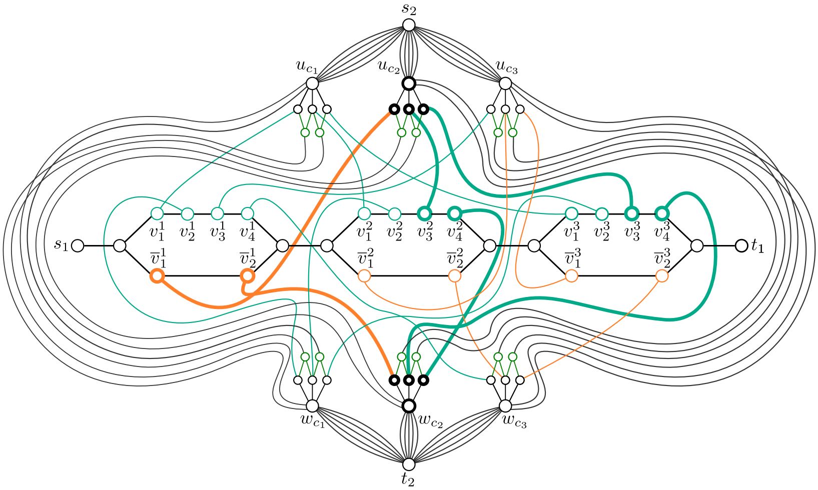

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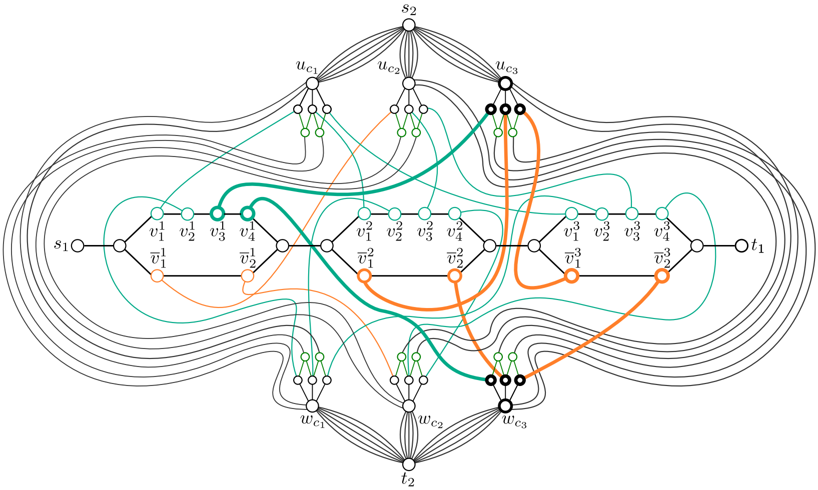

SIMPLE U2CIF: relation between the $g(I)$ and I instances

Lemma

If $g(I)$ is a YES *instance of* SIMPLE U2CIF*, then the first commodity flow uses only edges whose endpoints belong to* $\{s_1, t_1\} \cup V(G_{x_1}) \cup \cdots \cup V(G_{x_n})$.

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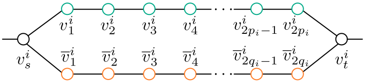

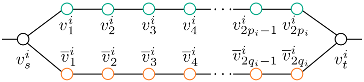

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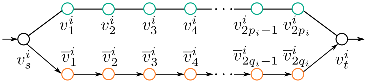

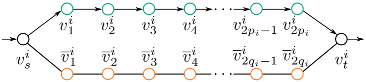

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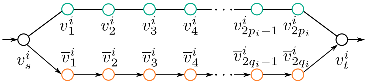

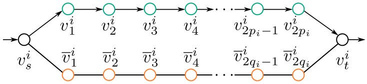

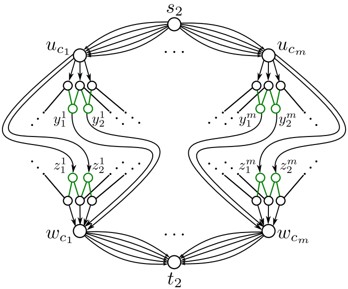

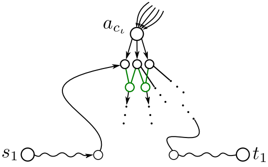

$a_{c_\iota} \in \{u_{c_\iota}, w_{c_\iota}\}$

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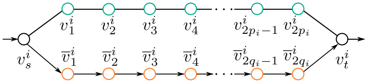

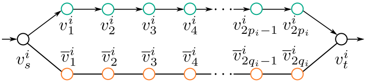

Lemma

$g(I)$ is Yes instance of SIMPLE U2CIF *if and only if I is a* YES *instance of 3-SAT.*

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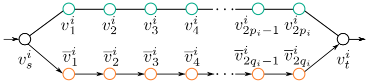
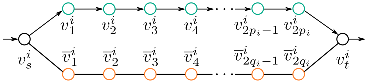


$\alpha(x_i) =$ *true*.



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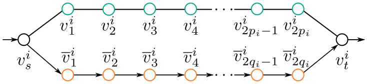

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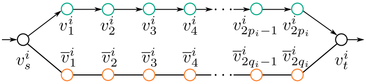

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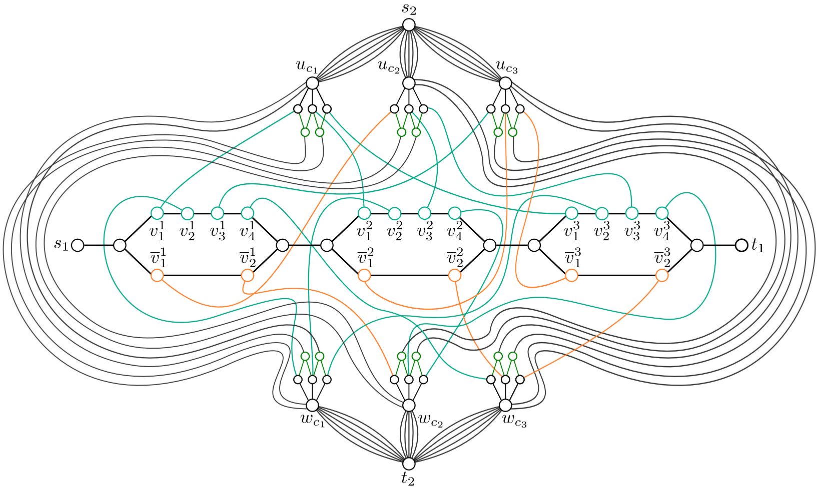
Theorem

The SIMPLE U2CIF *problem is NP-complete even if the* **demand** *of one commodity is* **unitary***.*

An example

$$I = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x}_3)$$



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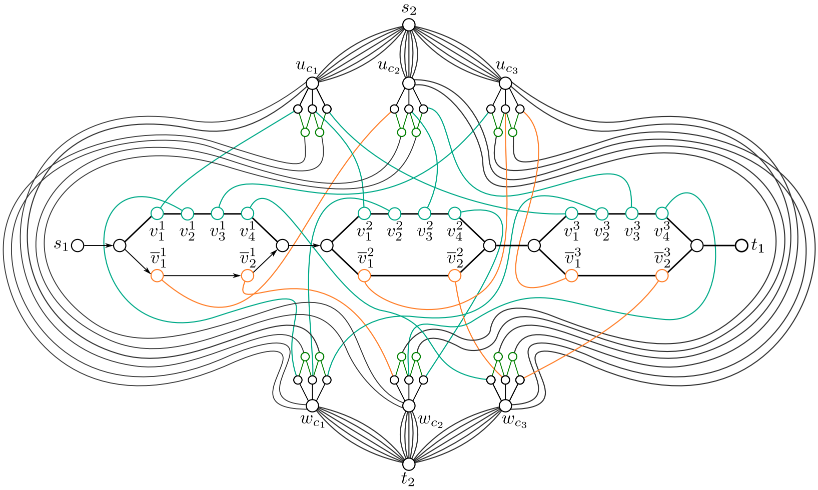

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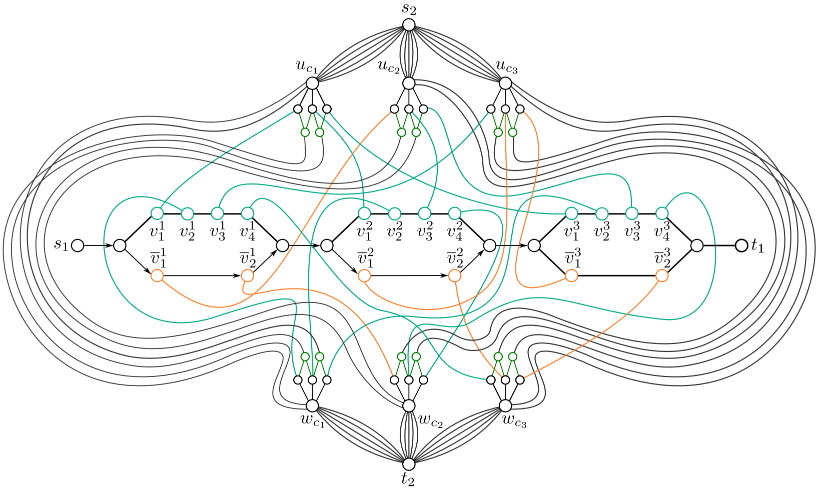

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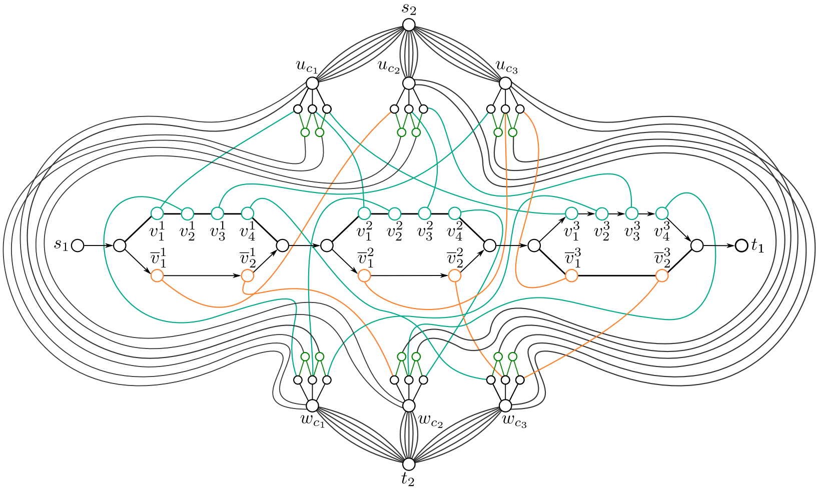

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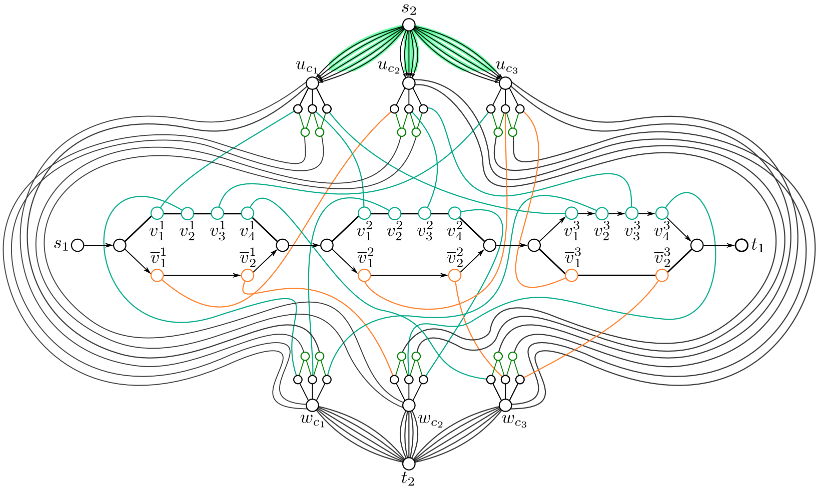

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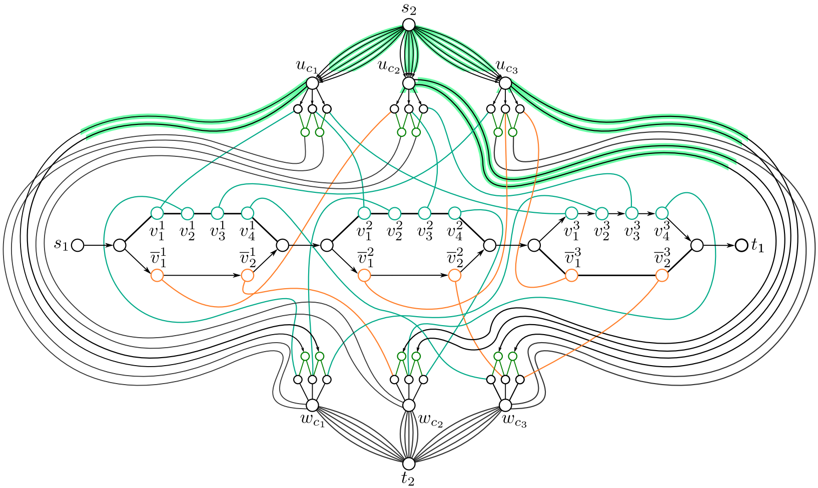

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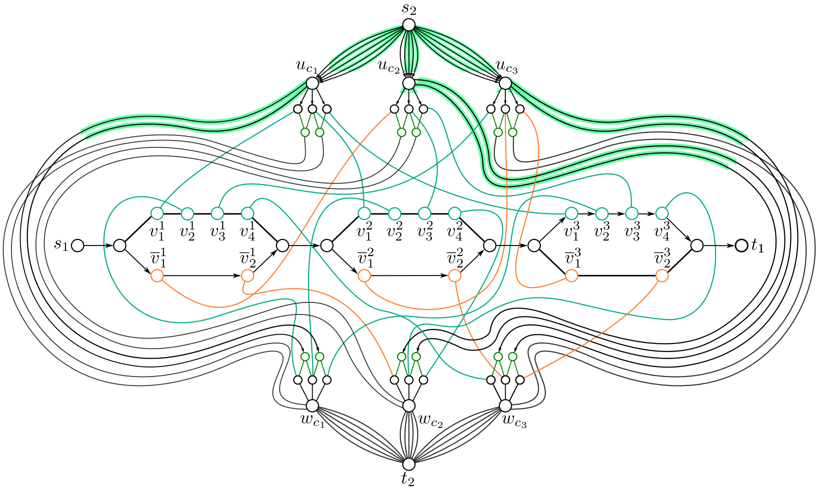

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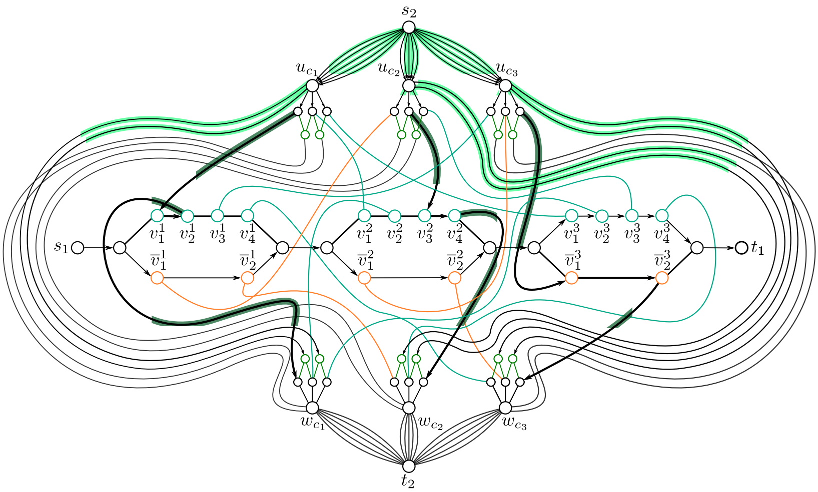

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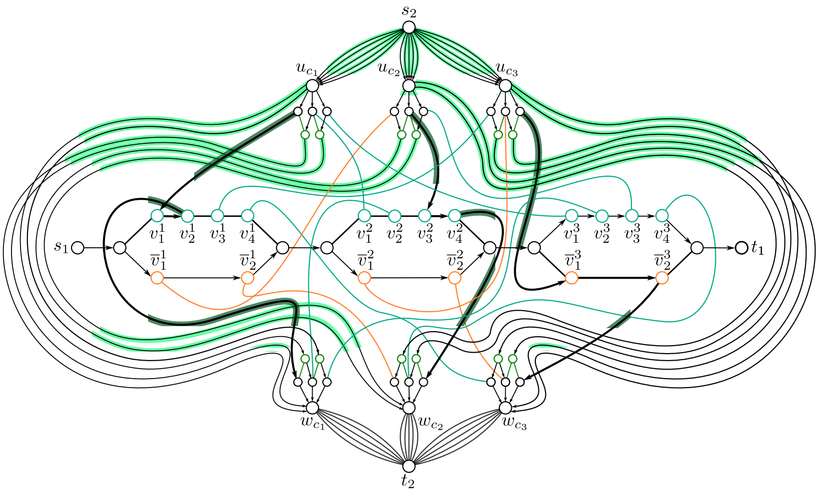

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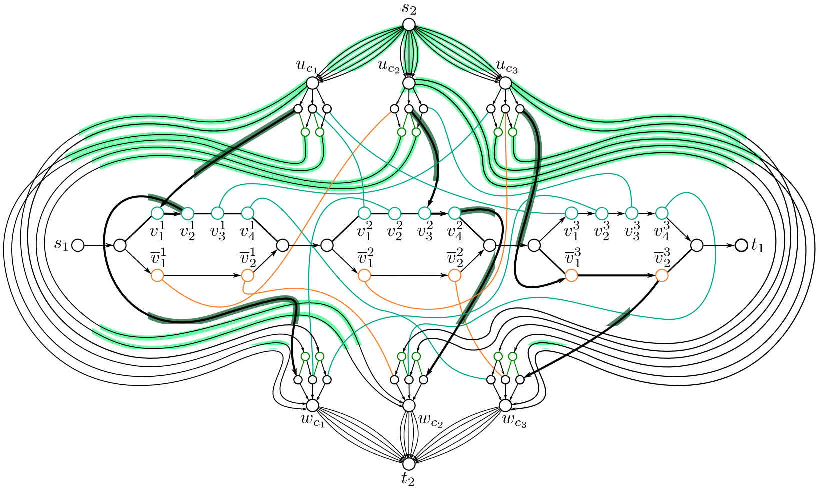

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K+1 VERTEX-DISJOINT PATHS

Disjoint paths problems

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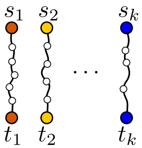
  - It can be reduced to the k-EDGE-DISJOINT PATHS problem.



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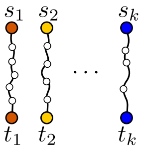

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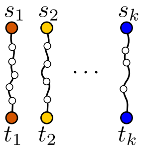

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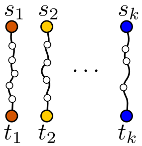

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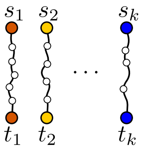

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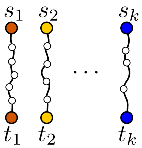

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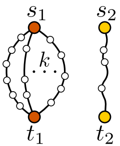
- Consequently, SIMPLE U2CIF is polynomial-time solvable if **both demands** are bounded by constants.

The $k+1$ disjoint paths problems

- Besides SIMPLE U2CIF to be related to *k*-EDGE-DISJOINT PATHS problem, note that SIMPLE U2CIF **coincides** with the $k+1$ EDGE-DISJOINT PATHS problem.



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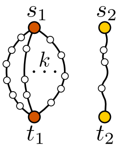
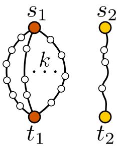


- In this work, we additionally analyse the complexity of $k + 1$ VERTEX-DISJOINT PATHS problem.

The $k+1$ disjoint paths problems

- Besides Simple U2CIF to be related to k-edge-disjoint paths problem, note that Simple U2CIF **coincides** with the $k+1$ edge-disjoint paths problem.



- In this work, we additionally analyse the complexity of $k+1$ vertex-disjoint paths problem.

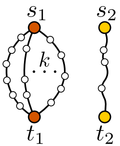
$k+1$ vertex-disjoint paths ($k+1$ VDP)

Input: An undirected graph G, two unordered pairs $\{s_1, t_1\}$ and $\{s_2, t_2\}$ of vertices of G and $k \in \mathbb{Z}^+$.

Question: Does G admit $k+1$ **vertex-disjoint paths**, such that one path is between s_1 and t_1 and k paths are between s_2 and t_2?

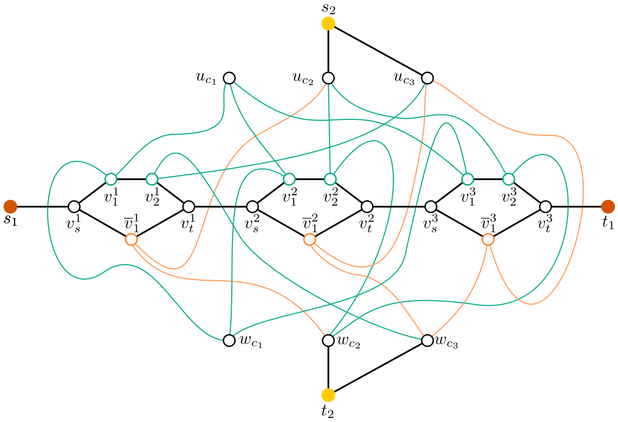
The $k+1$ disjoint paths problems

- Besides SIMPLE U2CIF to be related to k-EDGE-DISJOINT PATHS problem, note that SIMPLE U2CIF **coincides** with the $k+1$ EDGE-DISJOINT PATHS problem.

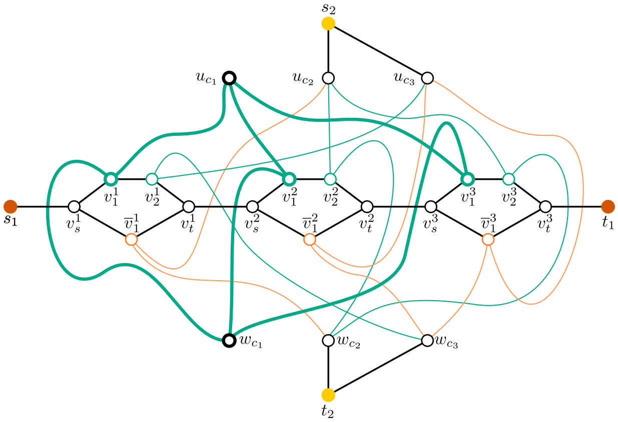


- In this work, we additionally analyse the complexity of $k+1$ VERTEX-DISJOINT PATHS problem.
- Similarly to SIMPLE U2CIF, we prove that $k+1$ VERTEX-DISJOINT PATHS is a NP-complete problem
  - By a polynomial-time reduction from **3-SAT**.

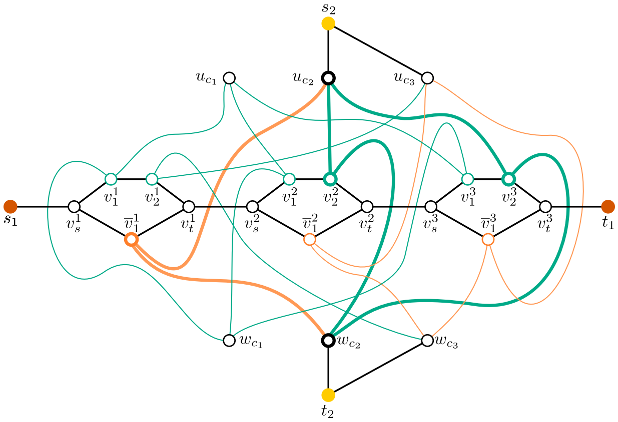
$$I = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x}_3)$$



$$I = (\boldsymbol{x_1} \vee \boldsymbol{x_2} \vee \boldsymbol{x_3}) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x}_3)$$

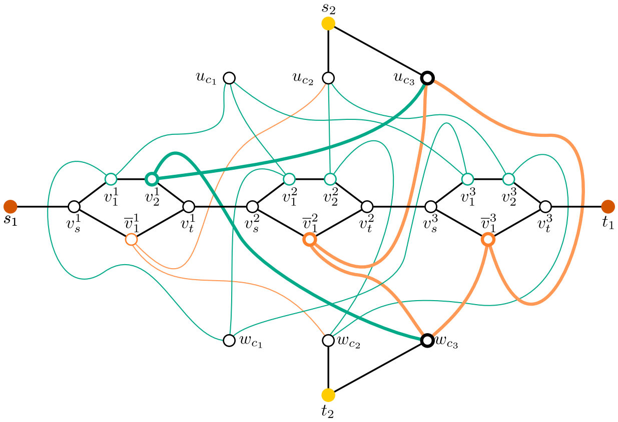


$$I = (x_1 \vee x_2 \vee x_3) \wedge (\overline{\boldsymbol{x}}_{\mathbf{1}} \vee \boldsymbol{x}_{\mathbf{2}} \vee \boldsymbol{x}_{\mathbf{3}}) \wedge (x_1 \vee \overline{x_2} \vee \overline{x}_3)$$



$k + 1$ VDP: construction of the instance $g(I)$

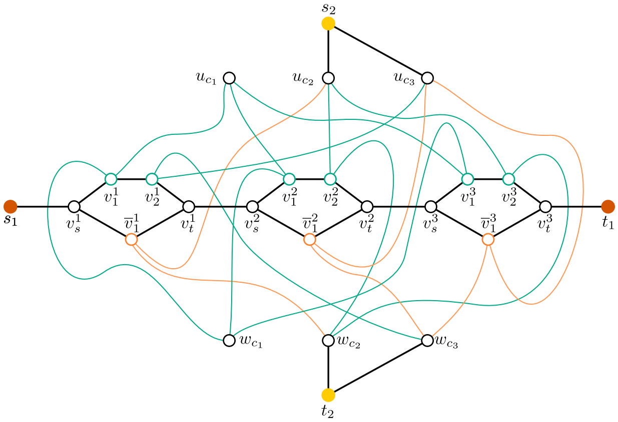
$$I = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (\boldsymbol{x_1} \vee \overline{\boldsymbol{x_2}} \vee \overline{\boldsymbol{x}}_{\boldsymbol{3}})$$



$k+1$ VDP: construction of the instance $g(I)$

$$I = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x}_3)$$

- $k = |\mathcal{C}|$ paths between s_2 and t_2 and 1 path between s_1 and t_1.



Conclusion

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Conclusion

- In this work, we have proved that SIMPLE U2CIF is NP-complete even if the demand of one commodity is unitary, closing a forty-year complexity gap.
- Additionally, we have proved that $k+1$ VERTEX-DISJOINT PATHS is also a NP-complete problem.
- As future work, we intend to analyse the complexity of these problems when they are restricted to some specific graphs, *e.g.* planar graphs.

Thank you for your attention!

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References I

- Cormen, H. T., Leiserson, E. C., Rivest, L. R and Stein, C., *Introduction to Algorithms*, MIT Press, third edition, 2009.
- Even, S., Itai, A. and Shamir, A., *On the complexity of timetable and multicommodity flow problems*, SIAM J. Comput. **5:4** (1976), 691–703.
- Edmonds, J. and Karp, R. M., *Theoretical improvements in algorithmic efficiency for network flow problems*, J. ACM **19:2** (1972), 248–264.
- Fortune, S., Hopcroft, J. and Wyllie, J., *The directed subgraph homeomorphism problem*, Theoret. Comput. Sci. **10** (1980), 111–121.
- Karp, R. M., *On the computational complexity of combinatorial problems*, Networks **5** (1975), 45–68.
- Kawarabayashi, K., Kobayashi, Y. and Reed, B., *The disjoint paths problem in quadratic time*, J. Combin. Theory Ser. B **102** (2012), 424–435
- Perl, Y. and Shiloach, Y., *Finding two disjoint paths between two pairs of vertices in a graph*, J. ACM **25:1** (1978), 1–9.
- Roberson, N. and Seymour, P. D., *Graph minors. XIII. The disjoint paths problem*, J. Combin. Theory Ser. B **63:1** (1995), 651–10.
- Tali, E., *The disjoint shortest paths problem*, Discrete Appl. Math. **85** (1998), 113–138.