The Solitaire Clobber game and correducibility

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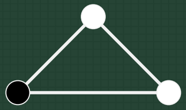

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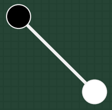

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Reducibility of a graph G:

$$r(G) := \min \{k \geq 1 \mid G \text{ is } k\text{-reducible.}\} .$$

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- Albert, M. H., Grossman, J. P., Nowakowski, R. J., Wolfe, D. (2005);
- Dorbec, P., Duchêne, E., Gravier, S. (2008): Hamming graphs;
- Blondel, V. D., Hendrickx, J. M., Jungers, R. M. (2008): optimization problem;
- Duchêne, E., Gravier, S., Moncel, J. (2009); Dantas, S., Gravier, S., Pará, T. (2009, 2011); Beaudou, L., Duchêne, E., Gravier, S. (2015).

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k-correducible graph G: For every non-monochromatic initial configuration of stones and for every subset $S \subset V(G)$ of cardinality at most k, there exists a Solitaire Clobber game on G that empties S.

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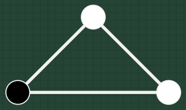

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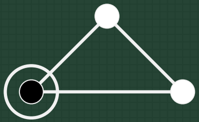

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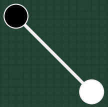

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Corredducibility of a graph G:

$$cr(G) := \max\{k \in \mathbb{N} \mid G \text{ is } k\text{-corredducible}\}.$$

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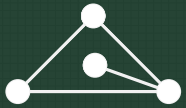

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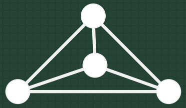

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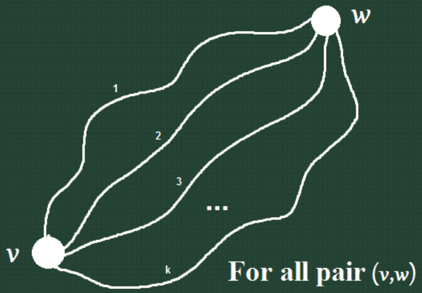

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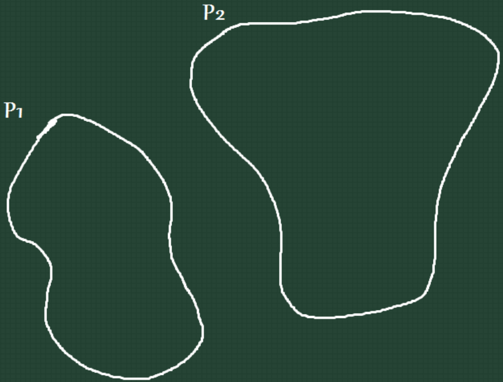

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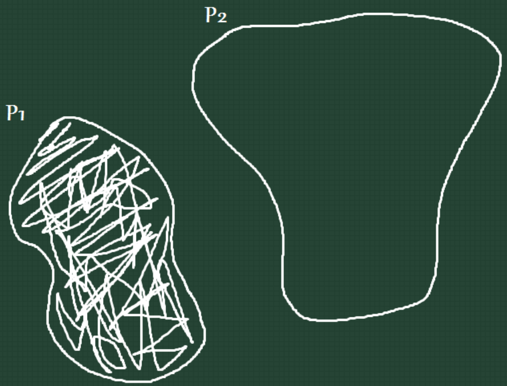

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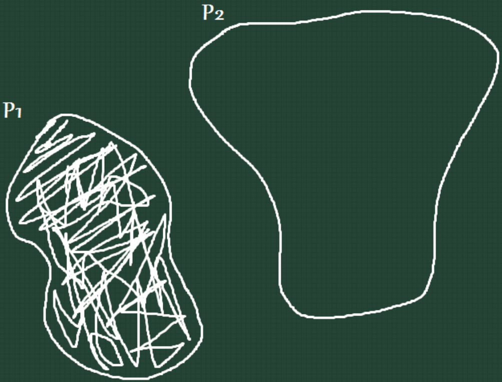

Proposition

A graph G is 1-correducible if and only if it is 1-connected.

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- Our main theorem states that every other k-connected graph is k-correducible.

Theorem

Let $k \geq 1$, and let G be a graph with $|G| \geq k+2$. If G is k-connected, then it is k-correducible.

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Lemma
Let $k \geq 1$, and let G be a k-connected graph such that $|G| \geq k+2$. Given any initial configuration $\Phi_1 : V(G) \to \{0,1\}$ with $|\Phi_1^{-1}(0)| = 1$ and a subset S of $V(G)$ with $|S| = k$, there exists a Solitaire Clobber game on G that empties S.

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Now, we may always assume that $|\Phi^{-1}(0)| > 1$ and $|\Phi^{-1}(1)| > 1$.

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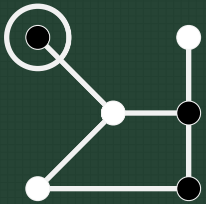

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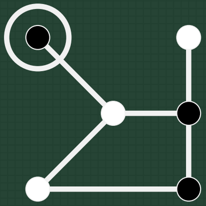

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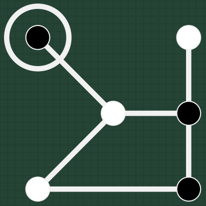

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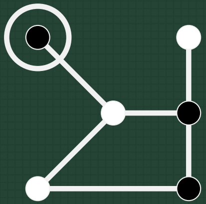

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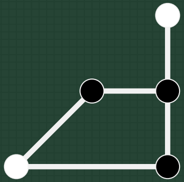

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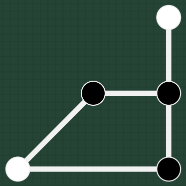

- $|G_2| \geq k + 1 = (k - 1) + 2$;
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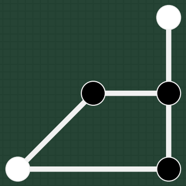

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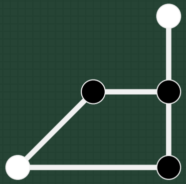

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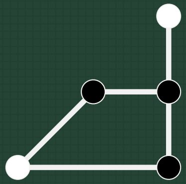

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- Φ_2 is non-monochromatic.

Now assume that for each vertex $v \in S$ and any vertex u adjacent to v, $\Phi_1(u) = \Phi_1(v)$.

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Theorem (Dirac '60)

If G is a k-connected graph (with $k \geq 2$), and S is a set of k vertices in G, then G has a cycle **C** including S in its vertex set.

Theorem (Fan Lemma, Dirac '60)

Let G be a k-connected graph, let x be a vertex of G, and let $Y \subseteq V(G) \setminus \{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y (that is, a family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct).

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We may assume that the restriction of Φ_1 to $V(\mathbf{C})$ is non-monochromatic.

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We may assume that the restriction of Φ_1 to $V(\mathbf{C})$ is non-monochromatic.

If $\Phi_1(x) = \Phi_1(y)$ for all $x, y \in S$, then there is an obvious Solitaire Clobber game on the cycle **C** that empties S.

If both S and C are non-monochromatic, we use the following lemma which extends Dirac's Theorem.

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Lemma

Let $k \geq 2$, G be a k-connected graph, $S \subseteq V(G)$ with $|S| = k$, and $T_i = \{v_{i,1}, \ldots, v_{i,s_i}\}$, $1 \leq i \leq m$, m pairwise disjoint subsets of S. Suppose that G contains a cycle **C** that satisfies the following condition: $(*)$ For each $1 \leq i \leq m$, $(v_{i,1}, \ldots, v_{i,s_i})$ is a path in $\mathbf{C}$. Then G contains a cycle that includes S in its vertex set and satisfies $(*)$.

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Let G be a graph with $|G| \geq 4$. Then G is 2-connected if and only if it is 2-correducible.

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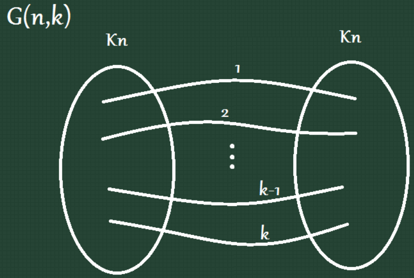

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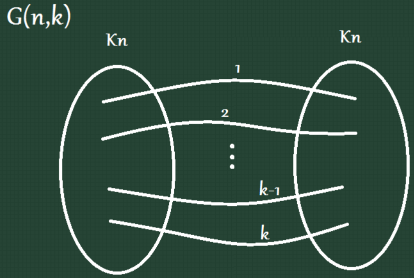

Proposition

For all $n \geq 2$ and $k \leq n-1$, $k(G(n,k)) = k$. In contrast, for a fixed k, $\lim_{n\to\infty} cr(G(n,k)) = \infty$.

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- Find graphs with $k(G)=cr(G)$;
- Study other types of connectivity;
- Determine the correducibility of some interesting graphs (eg.: grids, tori, hypercubes, ...)

Thank you for your attention!

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