A 3-approximation algorithm for the maximum leaf k-forest problem

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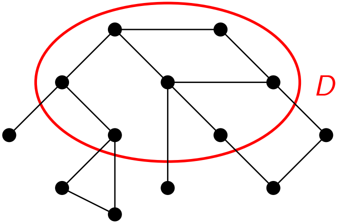
Joint work with M.F. Reis and M.C. San Felice and F. Usberti

Supported by FAPESP

Minimum connected dominating set

Let $G = (V, E)$ be a connected graph.



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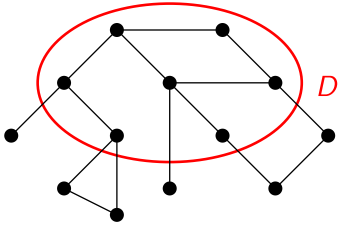

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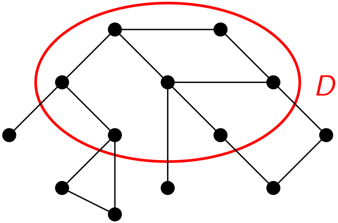

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Minimum connected dominating set problem: find a smallest connected dominating set in G.

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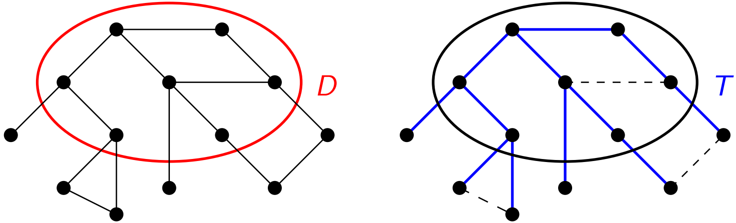

Maximum leaf spanning tree: find a spanning tree T with maximum number of **leaves**.

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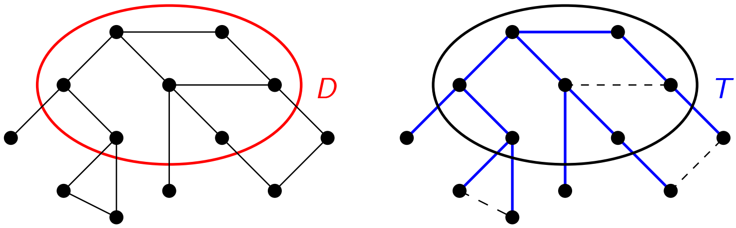

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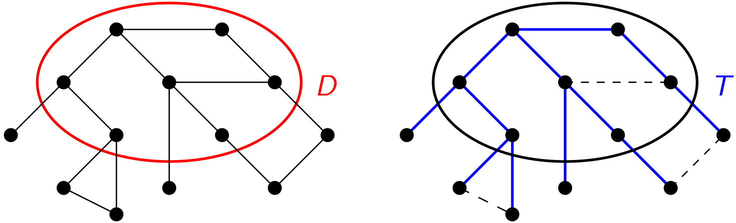

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Equivalent to minimum connected dominating set problem.

D corresponds to the internal vertices of T.

Minimum dominating set with at most k components

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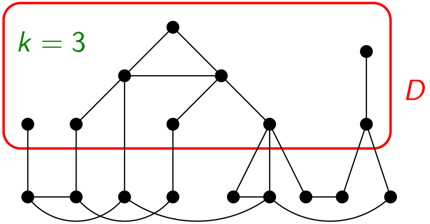

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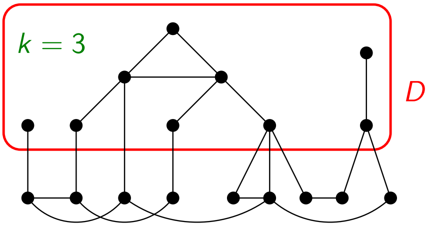

Problem: find a smallest dominating set D such that $G[D]$ has at most k components.

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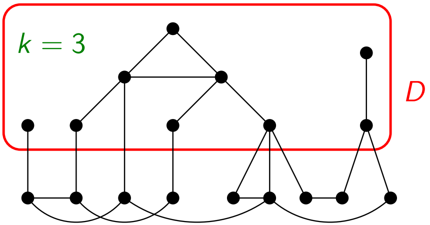

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$k = 1$ corresponds to minimum connected dominating set problem.

Maximum leaf (spanning) k-forest

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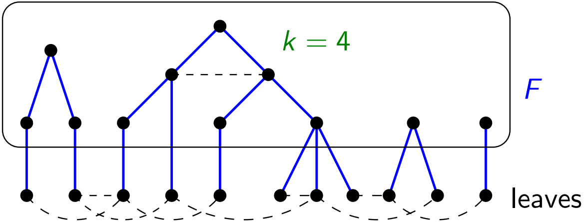

k-forest: forest with at most k components (trees).

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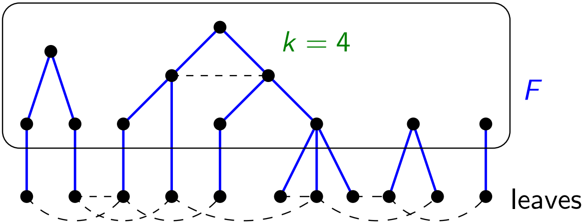

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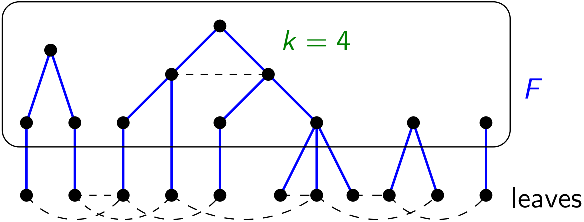

k-forest: forest with at most k components (trees).

Maximum leaf k-forest: find a spanning k-forest with maximum number of **leaves**.

Remark: a tree in F with exactly two vertices has only one leaf.

Known results

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- 3-approximation algorithm (Lu and Ravi, 1996).
- 2-approximation algorithm (Solis-Oba, 1998; Solis-Oba, Bonsma and Lowski, 2017).

Our contribution

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- The algorithm is inspired on Lu and Ravi's 3-approximation algorithm for maximum leaf spanning tree.
- A key ingredient in Lu and Ravi's algorithm (and ours) is the concept of leafy forest.

Leafy forest

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- T contains at least a vertex of degree 3, and
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A leafy forest is **maximal** if it cannot be extended to a larger (more edges or vertices) leafy forest.

Building a maximal leafy tree (Lu and Ravi)

Start a new tree T by adding a vertex with degree ≥ 3 and all its neighbors.

Repeat one of the following expansions until it is no longer possible:

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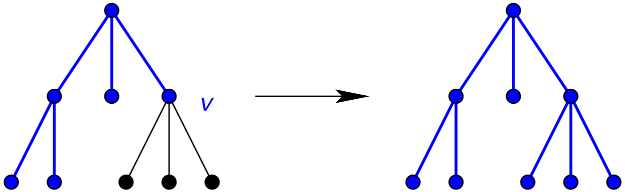

Add all those neighbors to T.

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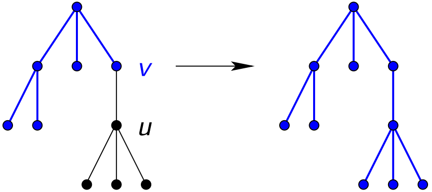
Expansion Type 2: a leaf v of T is adjacent to a vertex u in $G - V(T)$ with degree ≥ 3



Add u and all its neighbors in $G - V(T)$ to T.

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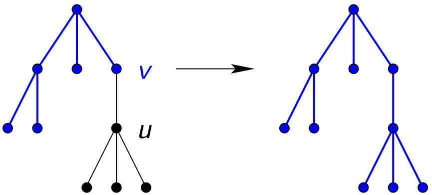
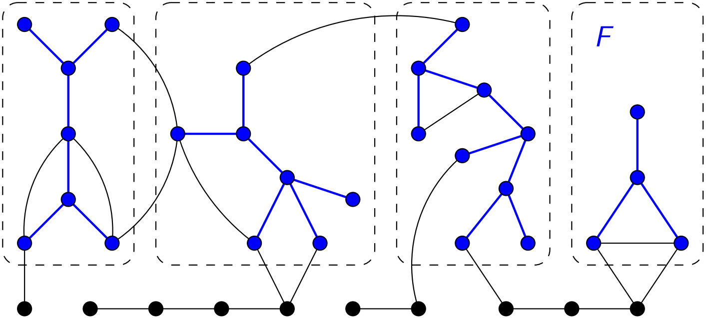


Add u and all its neighbors in $G - V(T)$ to T.

Algorithm: repeatedly build a new leafy tree until no vertex of degree ≥ 3 remains.

Maximal leafy forest



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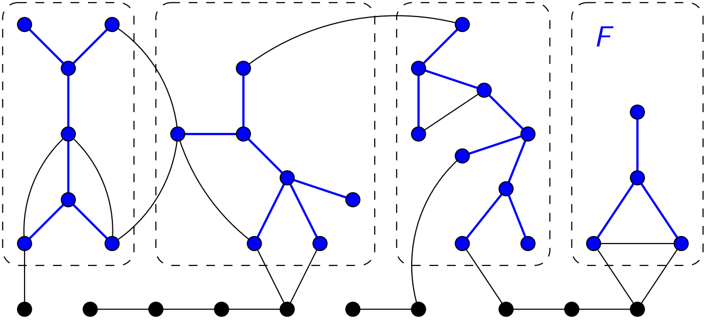

- Each component of $G - V(F)$ is a **path**.

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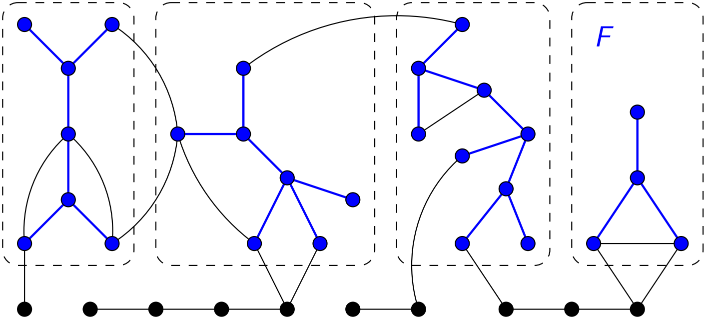

- Each component of $G - V(F)$ is a **path**.
- Only **leaves** of F can have neighbors in $G - V(F)$.

Lu and Ravi's 3-approximation algorithm

Type	Name	#trees	#leaves
maximal leafy forest	F	q	$\ell(F)$
optimum spanning tree	T^*	1	$\ell(T^*)$
Lu and Ravi's tree	T'	1	$\ell(T')$

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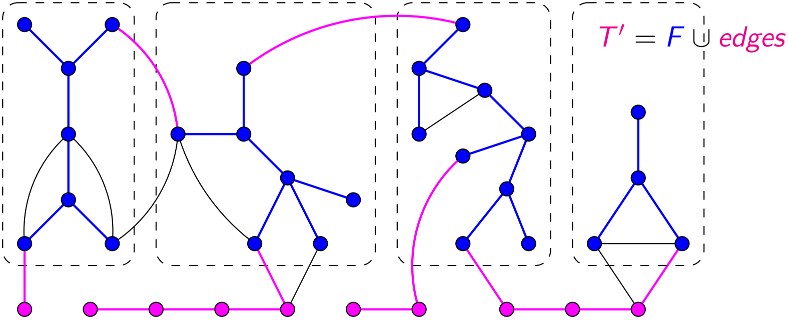

Build T' by connecting trees in F through edges/paths and then connnect the forest to the remaining outer paths.

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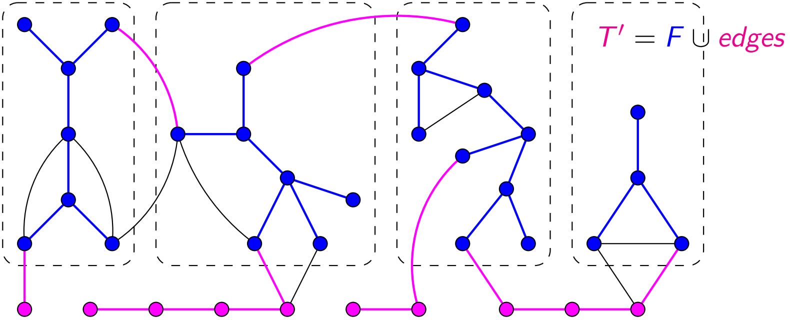

We might lose leaves by connecting trees through edges/paths.
We do not lose leaves by connecting F to an outer path.

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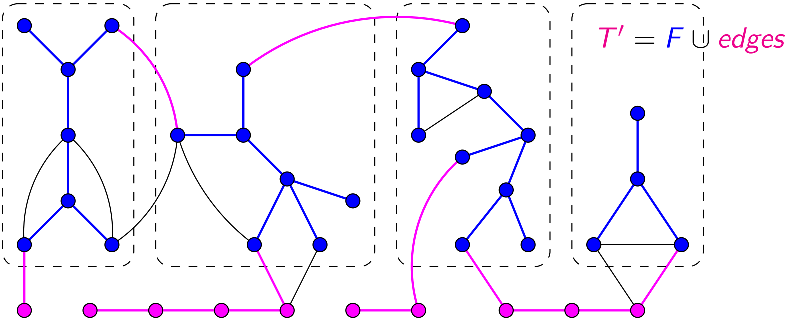

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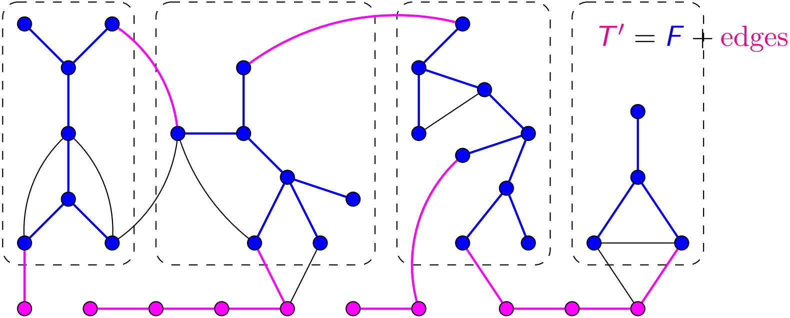

$3\ell(T') \geq 3[\ell(F) - 2(q-1)]$

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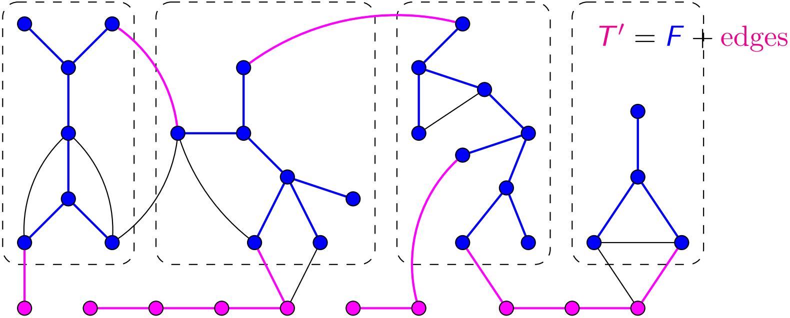

$3\ell(T') \geq 3[\ell(F) - 2(q-1)] = 3\ell(F) - 6q + 6 > \ell(T^*)$
(3-approximation)

Our algorithm for maximum leaf k-forest

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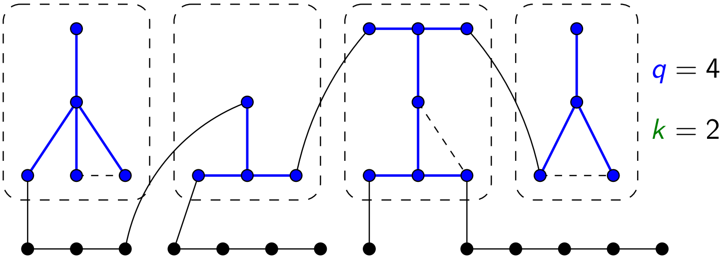

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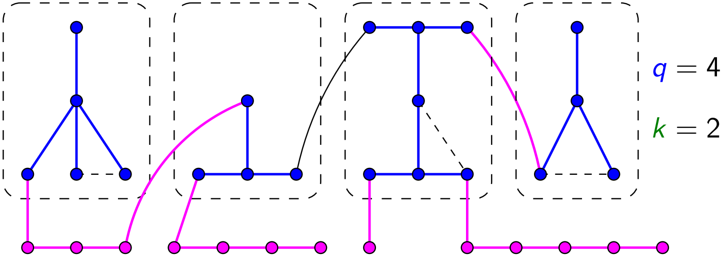

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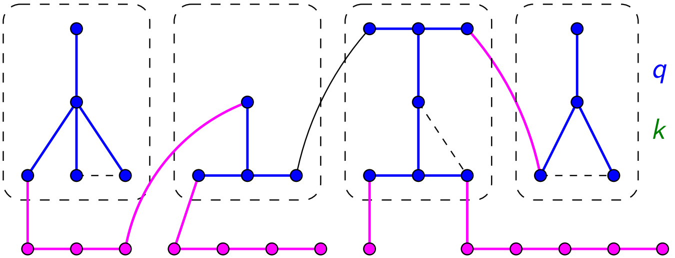

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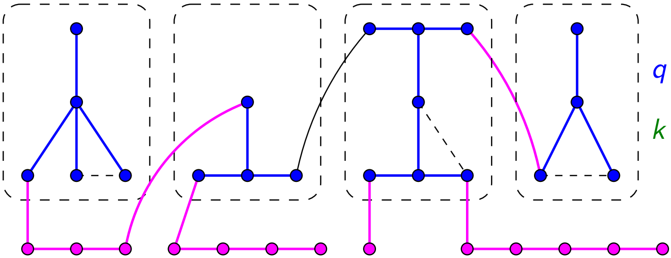

$q = 4$

$k = 2$

$3\ell(F') \geq 3\ell(F) - 6(q - k)$

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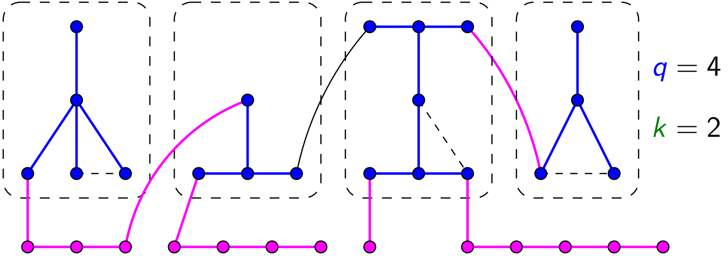
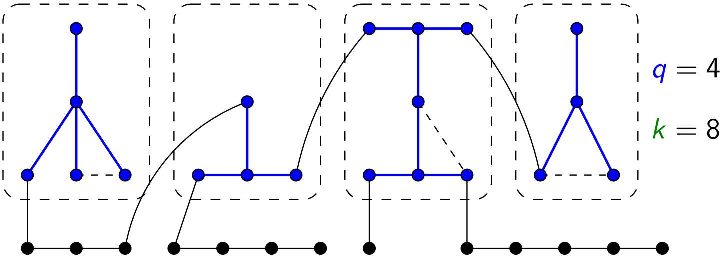


$$3\ell(F') \geq 3\ell(F) - 6(q-k) \geq \ell(F^*) + 6q - 2q^* + 1 - 6q + 6k \geq \ell(F^*)$$

Case 2: $q < k$



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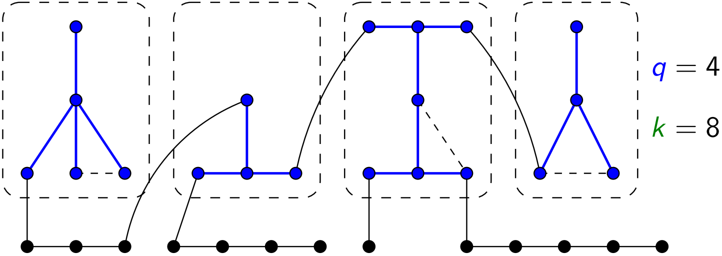

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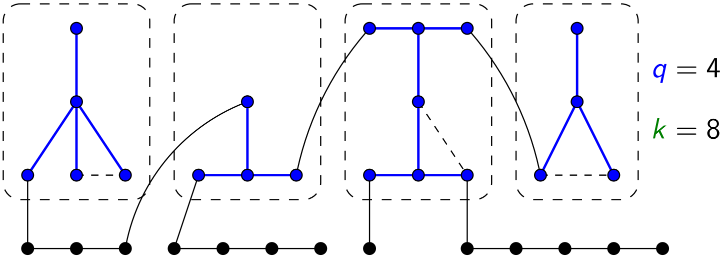

- **Natural try:** connect F to the outer paths and return the k-forest.
- This may not work (few leaves compared to the optimum).
- Instead we try to increase the number of trees to k as follows.

Our algorithm for maximum leaf k-forest

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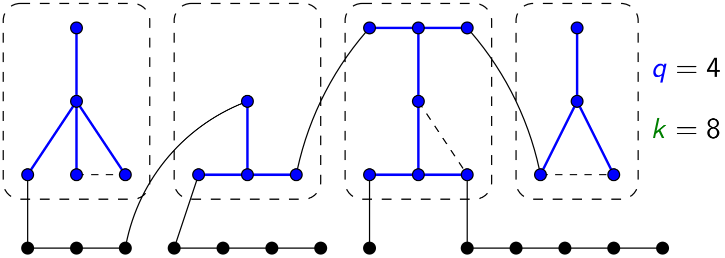

While there are less than k trees in F try to

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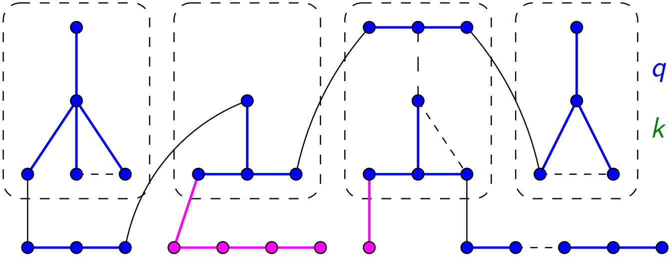
Case 2a: $q < k$ and **we added** $(k - q)$ **new trees**



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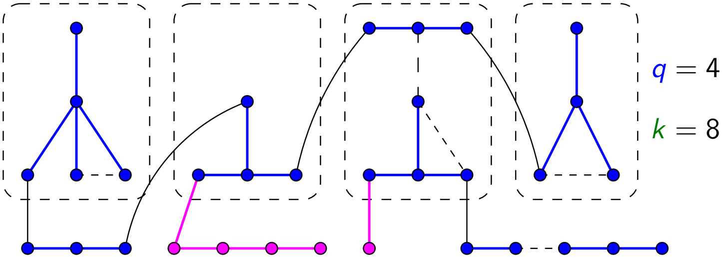

$\ell(F') \geq \ell(F) + (k - q)$ (gained $(k - q)$ leaves)

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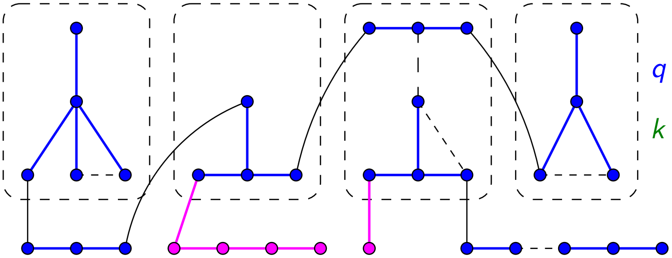

$$3\ell(F') \geq 3\ell(F) + 3(k - q)$$

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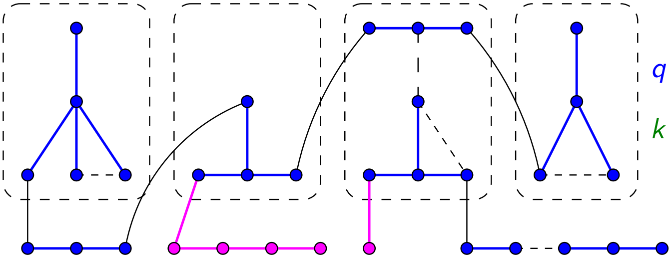
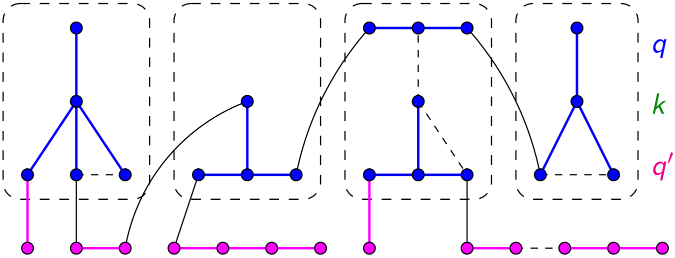


$$3\ell(F') \geq 3\ell(F) + 3(k - q) \geq \ell(F^*) + 6q - 2q^* + 1 + 3k - 3q \geq \ell(F^*)$$

Case 2b: $q < k$ and **we did not add** $(k - q)$ **new trees**



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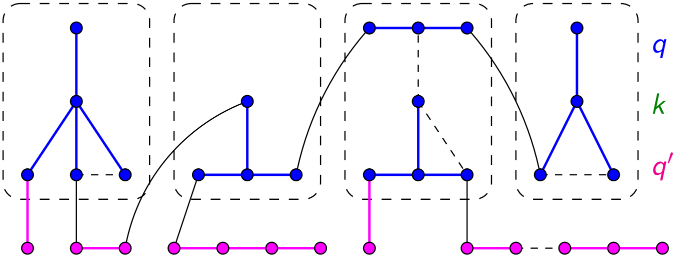


$q = 4$

$k = 20$

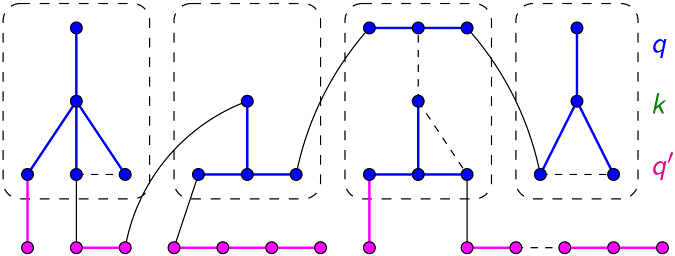
$q' = 9$

Every degree 2 vertex of F' is adjacent to a leaf.

So $3\ell(T') \geq |V(T')|$ for each tree T' in F'.

Our algorithm for maximum leaf k-forest

Case 2b: $q < k$ and **we did not add $(k-q)$ new trees**



Every degree 2 vertex of F' is adjacent to a leaf.

So $3\ell(T') \geq |V(T')|$ for each tree T' in F'.

Thus, $3\ell(F') \geq |V(G)| \geq \ell(F^*)$.

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Thank you!