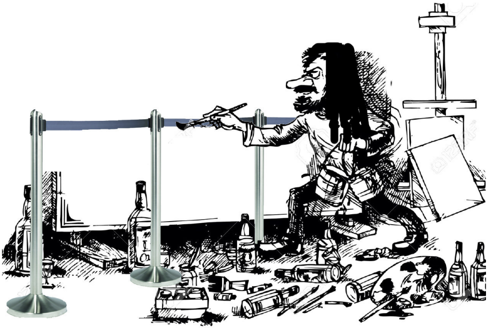
Möbius Stanchion Systems

Lucas Isenmann, Timothée Pecatte



Stanchion system

What are **stanchions**?

Problem: paint both sides of every strips of a stanchion system.

Stanchion painting problem

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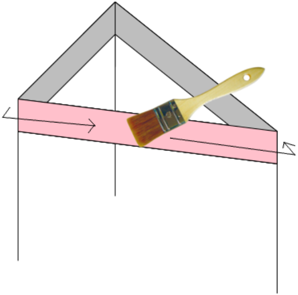

Stanchion painting problem

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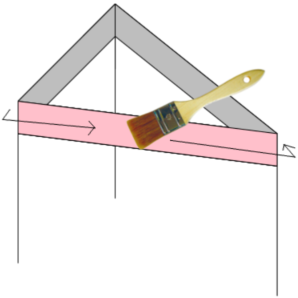
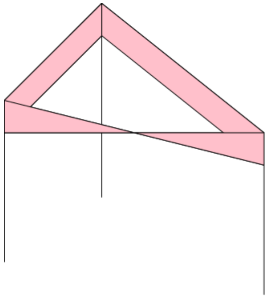


Condition: we do not want to lift up the brush.

Stanchion painting problem

Solution: twist a strip!



Stanchion painting problem

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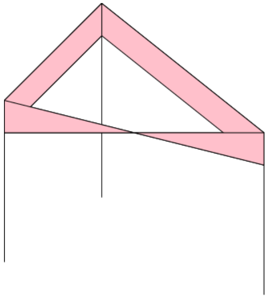
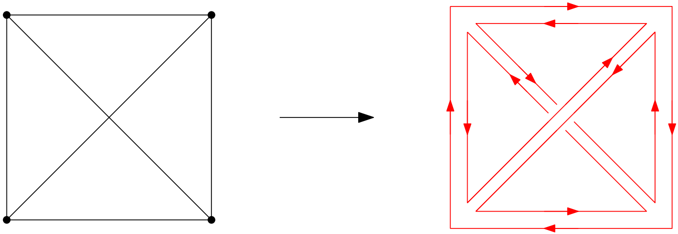
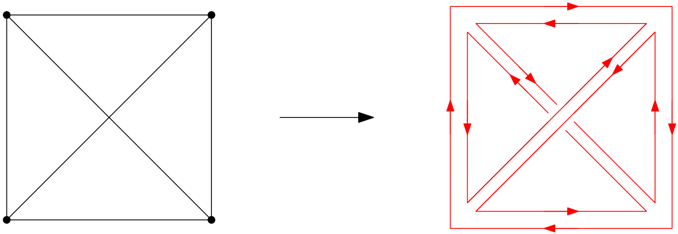


The painter can paint **without lifting up** the brush!

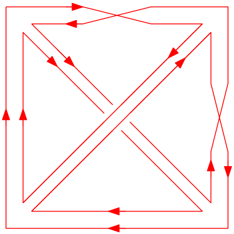
A ribbon graph



A ribbon graph

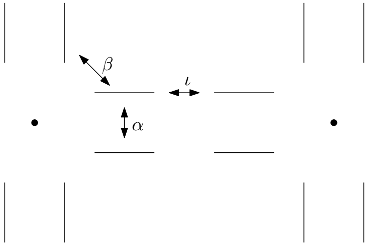


With twisted strips:



Modelisation

Combinatorial map: ι, α and β are involution on quarter-edges.



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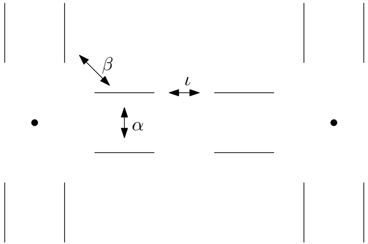

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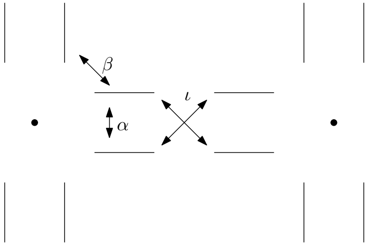
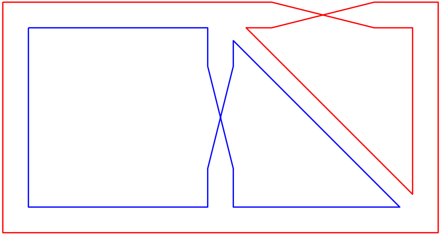


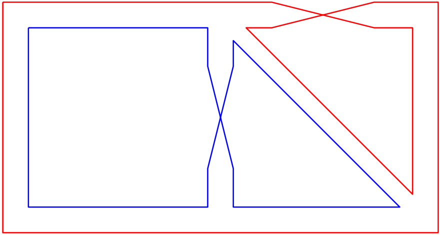
Question

For this graph, the walk of the brush makes 2 cycles:



Question

For this graph, the walk of the brush makes 2 cycles:

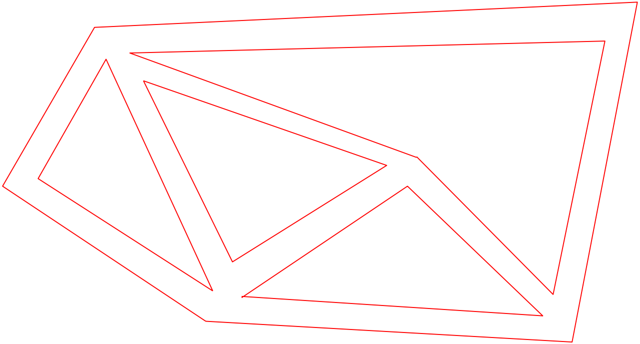


How to twist strips so that there is only **one** cycle?

A solution = Möbius stanchion system

Your turn!

Can you twist some edges to solve the problem for this graph?



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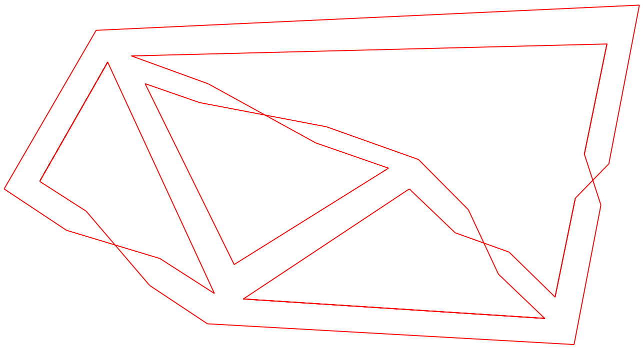

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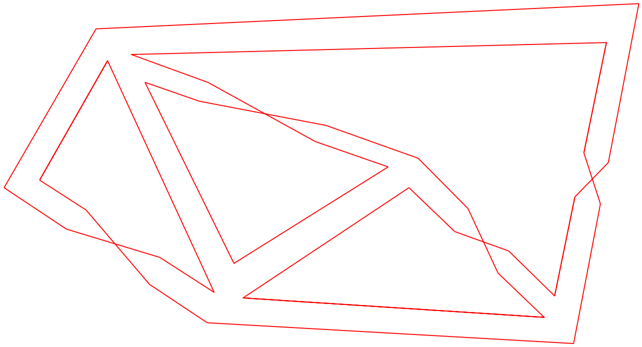
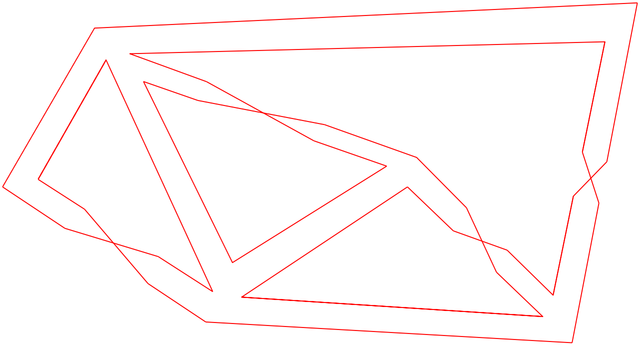


Claim: 4 twists is the minimum

Dual graph

Here, the minimum number is the number of **interior faces**.



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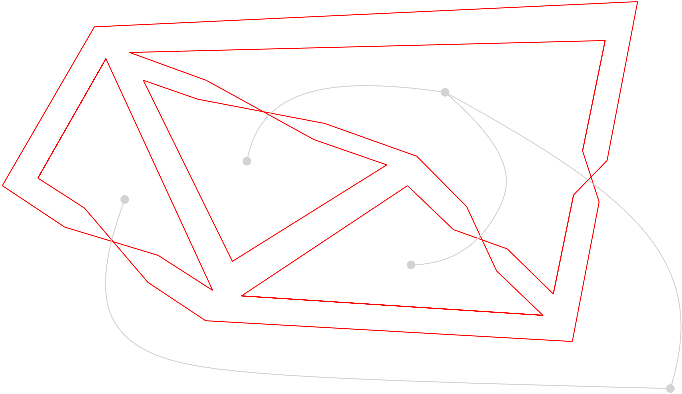
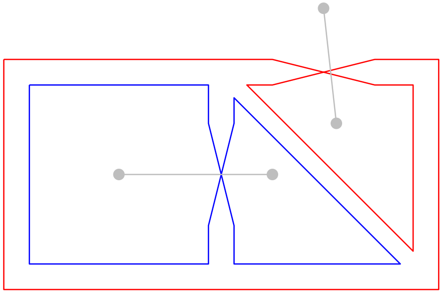


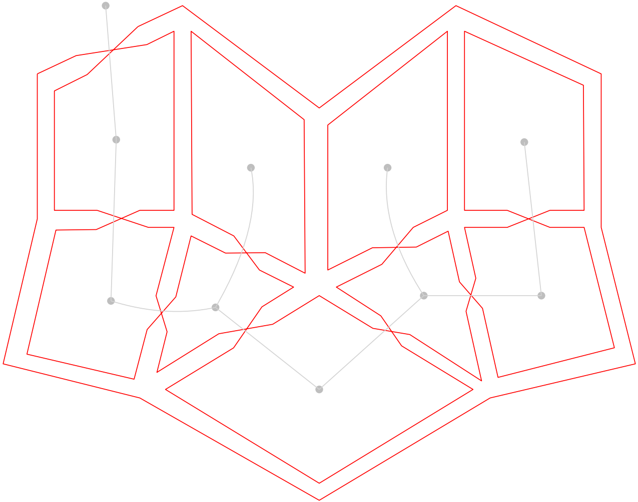
Twisted strips **connect** faces → subgraph of the dual graph

In a solution, all faces should be **connected** = connected spanning subgraph of the dual graph



Minimal solutions

A spanning tree of the dual graph gives a solution



Minimal solutions

Theorem

Spanning trees of the dual are the **minimal solutions** *with* $f - 1$ *twisted edges (*f *is the number of faces).*
Therefore the painter needs **at least** *to twist as much as there are interior faces.*

Are there other solutions?

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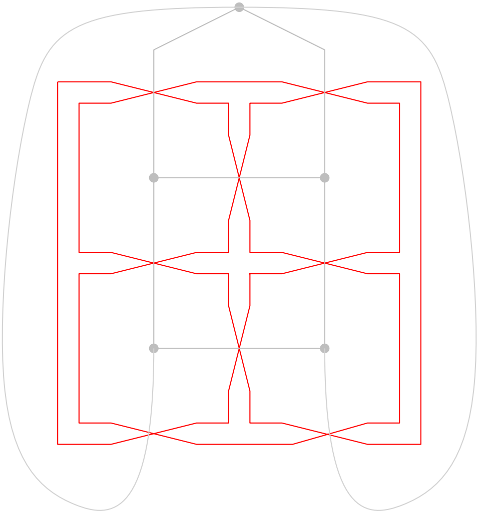

Yes!

How to get other solutions

Start with a solution given by a spanning tree

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and try to modify it using some **preserving rules**.

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Two elementary operations :

- single twist

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Two elementary operations :

- single twist
- double twist

Elementary operations

Single twist = twist an edge

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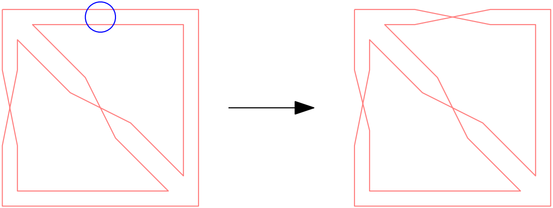
Some edges are twistable → another solution



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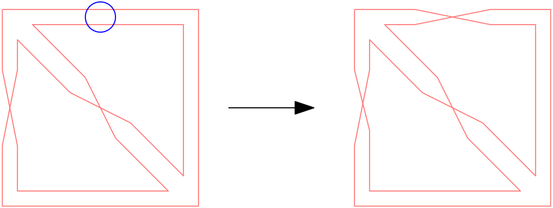
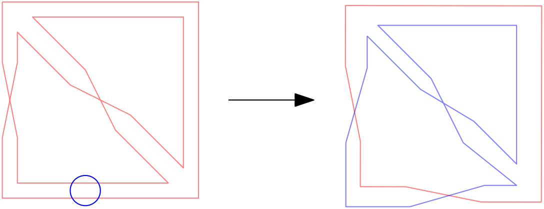


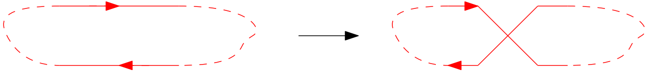
Some others are not → not a solution



Elementary operations

The good edges are **crossed two-ways**

Single twist of a good edge $\rightarrow$ **still a solution**



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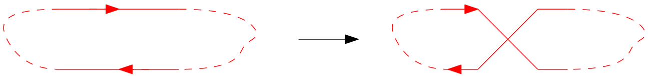
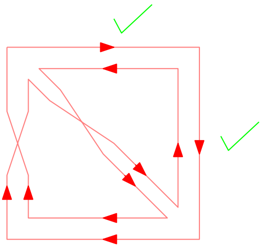


Example:



Elementary operations

Double twist = twist two edges simultaneously

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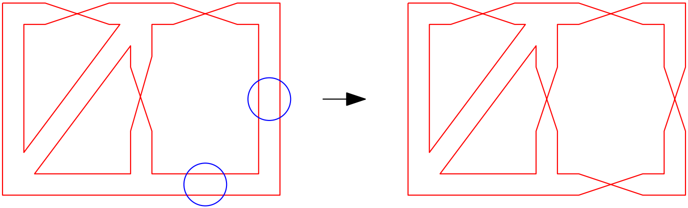
Some pair of edges seems to be twistable:



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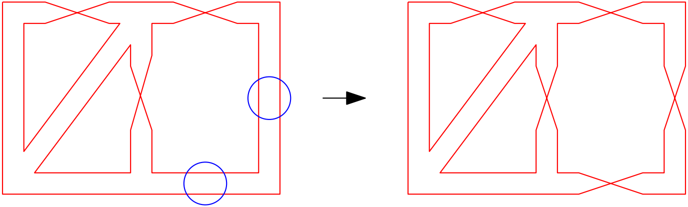
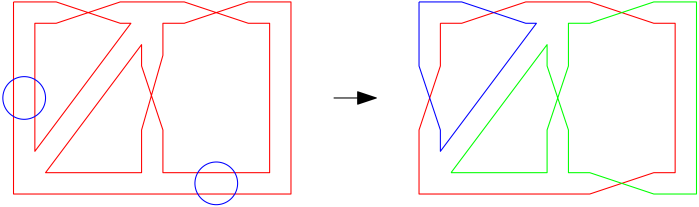
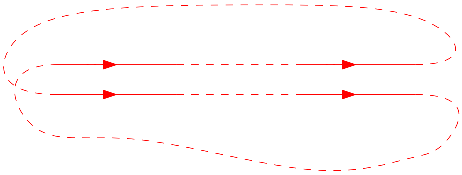


But some others not:



Elementary operations

A **good pair** of strips need the following connexions:



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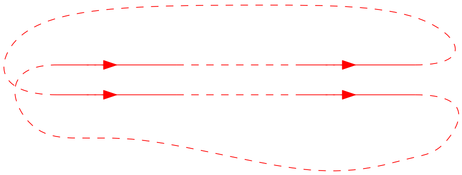
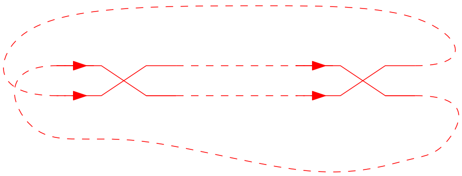


When twisted:



This **double twist** still gives a solution.

Elementary operations

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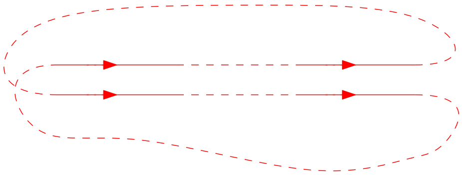

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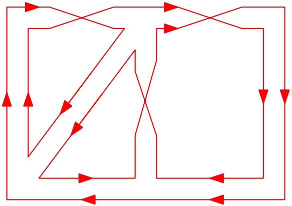
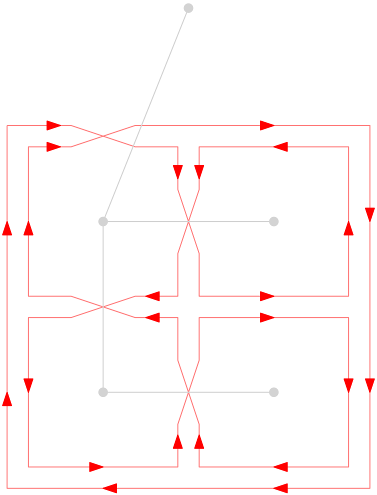
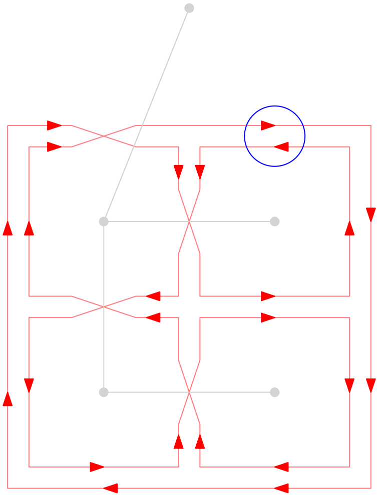


Example

Let's start with this spanning tree solution:

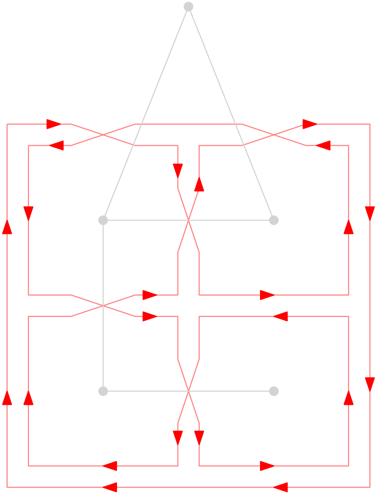


We can **single twist** this strip:



Example

We get a new solution:



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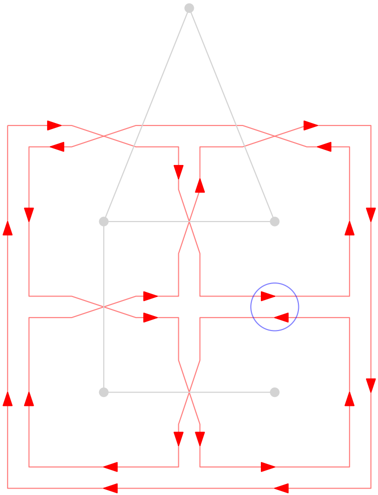
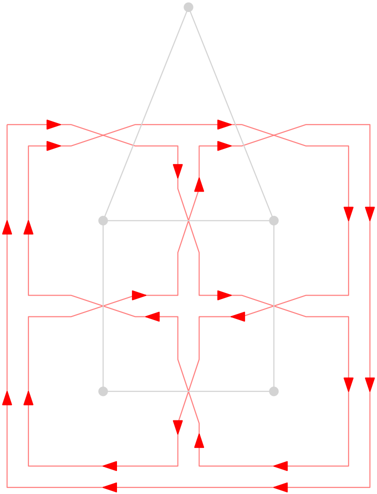


Example

Another new solution:



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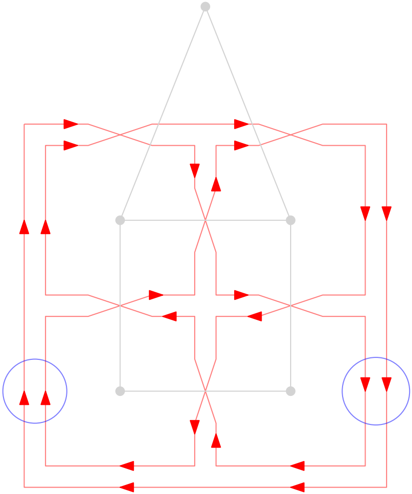

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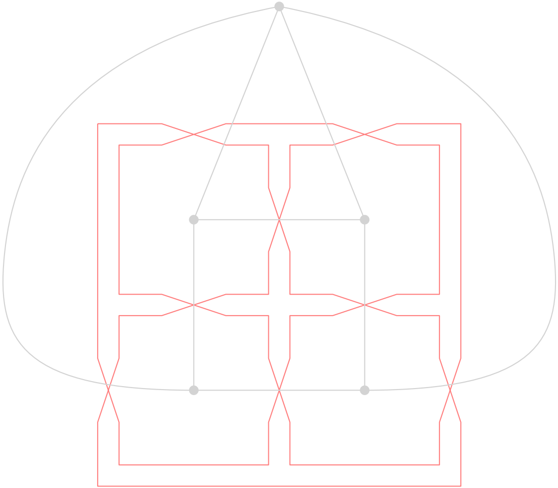

Elementary operations

Theorem

From a solution we can **get any other solution** *by applying successive single and double twists.*

Sketch of proof by induction

Ok for trees ($f - 1$ twisted edges).

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Take a solution $\geq f$ twisted edges.

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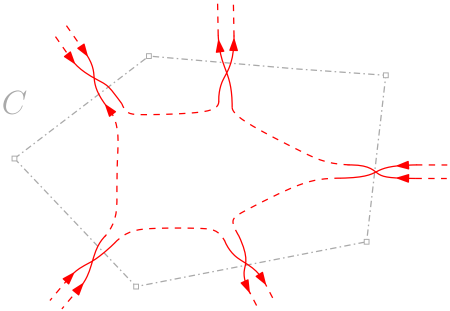

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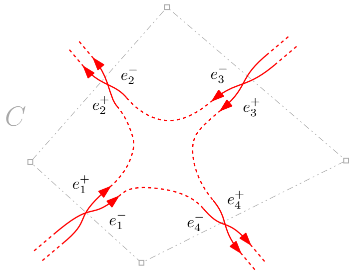

cyclic order : $[e_1^+, e_2^+, e_2^-, e_1^-]$, $[e_2^+, e_3^+, e_3^-, e_2^-]$, ...
$[e_1^+, \ldots, e_4^+, e_1^+, e_1^-, e_4^-, \ldots, e_1^-]$ gives a contradiction

Theorem

Let G be a planar graph. The Möbius stanchions systems of G are independent of the chosen embedding for G in the plane.

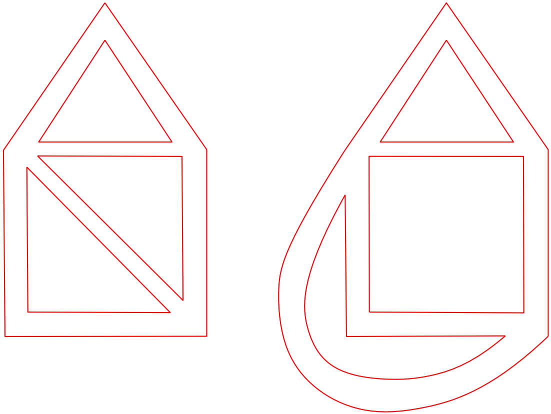

Figure: two non-isomorphic embeddings

Sketch of proof

Planar 3-connected $\Rightarrow$ only one embedding.

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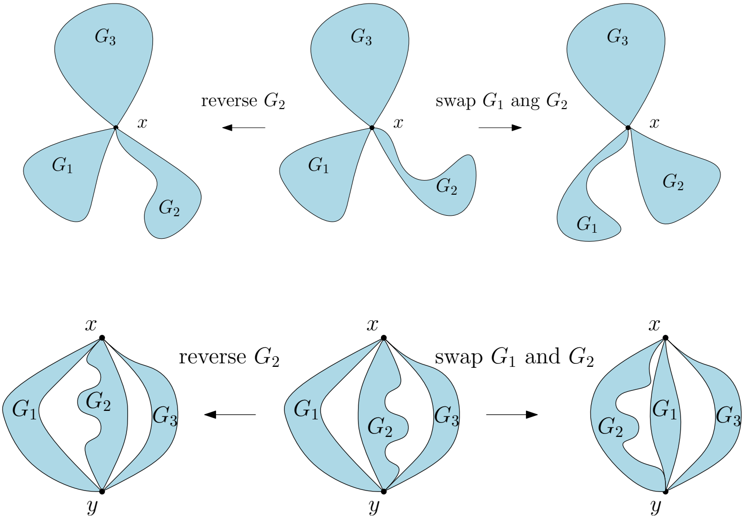
Not 3-connected : **swap and flip** operations connect all the embeddings



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- are the results the same on the torus?
- how to generalize to higher dimensions?
- how can it help to understand unicellular embeddings?

Thank you for your attention!