THE MINIMUM CHROMATIC VIOLATION PROBLEM: A POLYHEDRAL APPROACH

M. Braga[a] D. Delle Donne [a] J. Marenco [a]
M. Escalante[b,c] M.E. Ugarte[c] M.C. Varaldo [c]

[a]ICI, Universidad Nacional de General Sarmiento
[b]CONICET
[c]FCEIA, Universidad Nacional de Rosario

ARGENTINA

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- Chromatic violation problem in a graph.

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- Polyhedral study of $P_{CV}(G)$.
- General Lifting Procedure for generating valid inequalities
- Families of new facets without using Lifting Procedure

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- k-coloring of $G = (V, E)$: partition of V into k stable sets.
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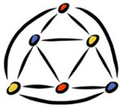

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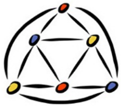

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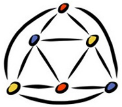

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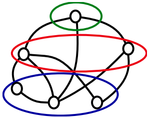

Our problem:

Given $G = (V, E)$, $\mathscr{C}$ colors, $F \subset E$ *weak edges*

Minimum chromatic violation problem (MCVP)

Find $\mathscr{C}$-coloring of $G' = (V, E \setminus F)$ minimizing the weak edges with both endpoints at the same color.

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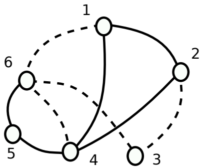

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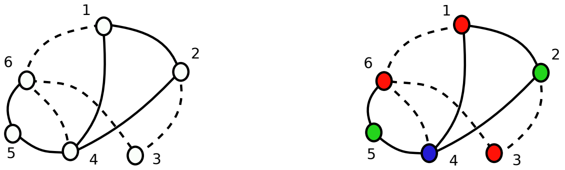

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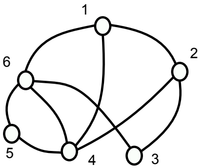
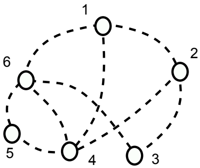


$F = \emptyset$ (VCP)



$F = E$ (k-P)

For $i \in V$ and $c \in \mathscr{C}$ let

$$x_{ic} = \begin{cases} 1 & \text{if } i \text{ colored by } c \\ 0 & \text{otherwise} \end{cases}$$

For $ij \in F$ let

$$z_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ have the same color} \\ 0 & \text{otherwise} \end{cases}$$

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The MCVP is

$$\min \sum_{ij \in F} z_{ij}$$

subject to

$$\begin{array}{ll}
\sum_{c \in \mathscr{C}} x_{ic} = 1 & i \in V \\
x_{ic} + x_{jc} \leq 1 & ij \in E \setminus F, c \in \mathscr{C} \\
x_{ic} + x_{jc} \leq 1 + z_{ij} & ij \in F, c \in \mathscr{C} \\
x_{ic}, x_{jc}, z_{ij} \in \{0, 1\} & i \in V, j \in V, ij \in F, c \in \mathscr{C}.
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Chromatic violation polytope

$$P_{CV}(G,F,\mathscr{C}) = \text{conv}\left\{ (x,z) \in \{0,1\}^s : \begin{array}{lll} \sum\limits_{c\in\mathscr{C}} x_{ic} & = 1 & i \in V \\ x_{ic} + x_{jc} & \leq 1 & ij \in E \setminus F, c \in \mathscr{C} \\ x_{ic} + x_{jc} & \leq 1 + z_{ij} & ij \in F, c \in \mathscr{C} \end{array} \right\}$$

where $s = |V||\mathscr{C}| + |F|$.

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Observe that

- $P_{col}(G,\mathscr{C}) = P_{CV}(G,\emptyset,\mathscr{C})$ where

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Observe that

- $P_{col}(G,\mathscr{C}) = P_{CV}(G,\emptyset,\mathscr{C})$
- $P_k(G) \subset P_{CV}(G,E,\mathscr{C})$ where

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LEMMA

If $|\mathscr{C}| > \chi(G-F)$ *then*

- $\sum\limits_{c\in\mathscr{C}} x_{ic} = 1$, $i \in V$ *minimal equation system for* $P_{CV}(G)$

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If $|\mathscr{C}| > \chi(G-F)$ then

- $\sum\limits_{c\in\mathscr{C}} x_{ic} = 1$, $i \in V$ *minimal equation system for* $P_{CV}(G)$
- $\dim(P_{CV}(G)) = |V|(|\mathscr{C}| - 1) + |F|$.

Proposition

If $|\mathscr{C}| > \chi(G - F)$ *then*

- $x_{ic} \geq 0, \quad i \in V, c \in \mathscr{C}$
- $z_{ij} \leq 1, \quad ij \in F$
- $z_{ij} \geq 0, \quad ij \in F$ *such that* $|\mathscr{C}| > \chi(G - (F \setminus \{ij\}))$,
- $x_{ic} + x_{jc} \leq 1 + z_{ij}, \quad ij \in F$ *maximal clique in* $G - (F \setminus \{ij\})$
- $x_{ic} + x_{jc} \leq 1, \quad ij \in E \setminus F$ *maximal clique in* $G - F$

are facet defining inequalities for $P_{CV}(G)$.

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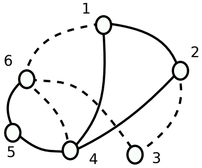
$G \quad F = \{23, 36, 46, 16\}$



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- $x_{ic} + x_{jc} \leq 1 + z_{ij}, \quad ij \in F$ *maximal clique in* $G-(F \setminus \{ij\})$
- $x_{ic} + x_{jc} \leq 1, \quad ij \in E \setminus F$ *maximal clique in* $G-F$

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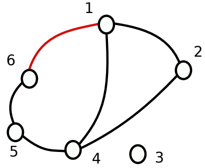
$G-(F \setminus \{16\})$



$x_{1c} + x_{6c} \leq 1 + z_{16}$ facet $\forall c$

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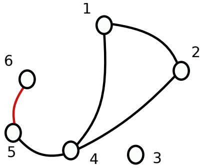
$G-F$



$x_{5c} + x_{6c} \leq 1$ facet $\forall c$

LEMMA

If $\lambda x + \mu z \leq \lambda_0$ *non-boolean facet of* $P_{CV}(G)$ *then* $\mu \leq 0$.

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An instance $(G_1, F_1, \mathscr{C}_1)$ of MCVP is stronger than $(G_2, F_2, \mathscr{C}_2)$ if
$G_1 = G_2$, $\mathscr{C}_1 = \mathscr{C}_2$ and $F_1 \subset F_2$.

LEMMA

If $\lambda x + \mu z \leq \lambda_0$ *non-boolean facet of* $P_{CV}(G)$ *then* $\mu \leq 0$.

THEOREM

Let $H \subset F$.

$\lambda x + \mu_H z_H \leq \lambda_0$ *facet of* $P_{CV}(G, F) \Longleftrightarrow \lambda x + \mu_H z_H \leq \lambda_0$ *facet of* $P_{CV}(G', H)$
where $G' = G - (F \setminus H)$

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Note that: $\lambda x \leq \lambda_0$ facet of $P_{col}(G') \Leftrightarrow$ facet of $P_{CV}(G, F)$ where $G' = (V, E \setminus F)$.

LEMMA

If $\lambda x + \mu z \leq \lambda_0$ *non-boolean facet of* $P_{CV}(G)$ *then* $\mu \leq 0$*.*

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Relationship with the k-partition problem.

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Relationship with the k-partition problem.

Recall that $P_k(G) \subset P_{CV}(G,E)$.

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Relationship with the k-partition problem.

Recall that $P_k(G) \subset P_{CV}(G, E)$.

LEMMA

Let $\lambda x + \mu z \leq \lambda_0$ valid for $P_k(G)$.

- *Facet for $P_k(G)$ and valid for $P_{CV}(G, E) \Rightarrow$ facet for $P_{CV}(G, E)$.*

LEMMA

If $\lambda x + \mu z \leq \lambda_0$ *non-boolean facet of* $P_{CV}(G)$ *then* $\mu \leq 0$.

THEOREM

Let $H \subset F$.

$\lambda x + \mu_H z_H \leq \lambda_0$ *facet of* $P_{CV}(G,F) \Longleftrightarrow \lambda x + \mu_H z_H \leq \lambda_0$ *facet of* $P_{CV}(G',H)$ *where* $G' = G - (F \setminus H)$

Relationship with the k-partition problem.

Recall that $P_k(G) \subset P_{CV}(G,E)$.

LEMMA

Let $\lambda x + \mu z \leq \lambda_0$ *valid for* $P_k(G)$.

- *Facet for* $P_k(G)$ *and valid for* $P_{CV}(G,E) \Rightarrow$ *facet for* $P_{CV}(G,E)$.
- $\mu \leq 0 \Rightarrow$ *valid for* $P_{CV}(G,E)$.

LEMMA

$\mathscr{F} = \{(x,z) \in P_{CV}(G, F \setminus \{ij\}) : \lambda x + \mu z = \lambda_0\}$ *non-empty face. Then*

$$\lambda x + \mu z \leq \lambda_0 + \lambda^* z_{ij}, \tag{1}$$

with $\lambda^* = \max\{|\lambda_{vc_1} - \lambda_{vc_2}| : v \in \{i,j\}$ *and* $c_1, c_2 \in \mathscr{C}\}$ *valid for* $P_{CV}(G,F)$.

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with $\lambda^* = \max\{|\lambda_{vc_1} - \lambda_{vc_2}| : v \in \{i,j\} \text{ and } c_1, c_2 \in \mathscr{C}\}$ *valid for* $P_{CV}(G,F)$.
If $\mathscr{F}$ *facet and* $\exists (x,z) \in \mathscr{F}$, $v \in V$ *and* $c_1, c_2 \in \mathscr{C}$ *such that*

- $x_{vc_1} = 1$ *and* $\lambda_{vc_2} - \lambda_{vc_1} = \lambda^*$,
- $x_{uc_2} = 0, \quad \forall u \in \Gamma_s(v)$,
- $x_{uc_2} = 0$ *or* $\mu_{vu} = 0$ *or* $z_{vu} = 1, \ \forall u \in \Gamma_w(v)$

then (1) *defines facet of* $P_{CV}(G)$.

COROLLARY

Let $K \subseteq V$ clique in G. For $c \in \mathscr{C}$, the semi-clique inequality

$$\sum_{v \in K} x_{vc} \leq 1 + \sum_{e \in F(K)} z_e$$

is valid for $P_{CV}(G)$.

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is valid for $P_{CV}(G)$.
If K maximal clique in $G - (F \setminus F(K))$ and $|\mathscr{C}| > \chi(G - (F \setminus F(K)))$ then it defines a facet of $P_{CV}(G)$.

COROLLARY

Let $K \subseteq V$ clique in G. For $c \in \mathscr{C}$, the semi-clique inequality

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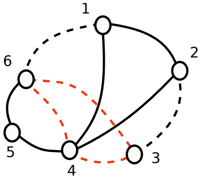
is valid for $P_{CV}(G)$.
If K maximal clique in $G-(F \setminus F(K))$ and $|\mathscr{C}| > \chi(G-(F \setminus F(K)))$ then it defines a facet of $P_{CV}(G)$.

$K = \{3, 4, 6\}$



COROLLARY

Let $K \subseteq V$ clique in G. For $c \in \mathscr{C}$, the semi-clique inequality

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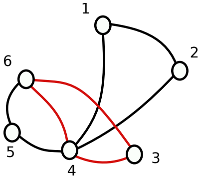

$$x_{3c} + x_{4c} + x_{6c} \leq 1 + z_{34} + z_{36} + z_{46},$$

$$|\mathscr{C}| > 3$$

facet of $P_{CV}(G)$.

Recursively applying the Lifting Lemma

COROLLARY

$G' \subset_{SG} G$ and $F(G')$ weak edges in G'. For $T \subset \mathscr{C}$, the multirank inequality

$$\sum_{t\in T}\sum_{i\in V'} x_{it} \leq \alpha(G')|T| + \sum_{e\in F(G')} z_e$$

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For general G' is not easy to analyze facetness.
Two particular structures: cliques and odd holes.

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PROPOSITION

Let $K \subset V$ clique, $T \subset \mathscr{C}$. For $|\mathscr{C}| > \chi(G - (F \setminus F(K))) + 1$ and $1 \leq |T| \leq |K| \leq |\mathscr{C}| + |T|$, the multicolor clique inequality (MKI)

$$\sum_{t \in T} \sum_{i \in V'} x_{it} \leq |T| + \sum_{e \in F(K)} z_e$$

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Corollary

$G' \subset_{SG} G$ and $F(G')$ weak edges in G'. For $T \subset \mathscr{C}$, the multirank inequality

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Proposition

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Facet of $P_{CV}(G)$ $\Leftrightarrow$

- $1 \le |T| < |K| < |\mathscr{C}|+|T|$
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- $|\mathscr{C}| > 2$,
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PROPOSITION

Let $K \subseteq V$ clique with $F(K) = E(K)$, $T \subseteq \mathscr{C}$ and $q_t \in \mathbb{N}$ for each $t \in T$, the multicolor combinatorial clique inequality

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If $|\mathscr{C}| > \chi(G-(F \setminus F(K)))+1$, it is facet of $P_{CV}(G) \Leftrightarrow$

- $\nexists w \in V \setminus K$ *with* $K \subseteq \Gamma_s(w)$
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Note that

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- Not a generalization of MKI: the complete graph has *no* strong edges

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- Polyhedral study of the minimum chromatic violation problem

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Thanks for your attention!

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