Facet-inducing inequalities and a cut-and-branch algorithm for the bandwidth coloring polytope based on orientation model

Bruno Dias, **Rosiane de Freitas**, Javier Marenco, Nelson Maculan

UFAM/UFRJ, Brazil and UNGS/UBA, Argentina

IX Latin and American Algorithms, Graphs and Optimization Symposium

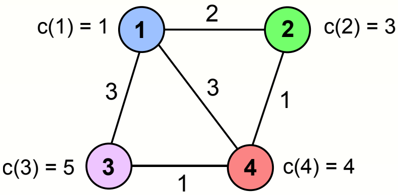
September 15th, 2017
Marseille - France

LAGOS 2017

Introduction

Definition 1: Bandwidth Coloring Problem (BCP).

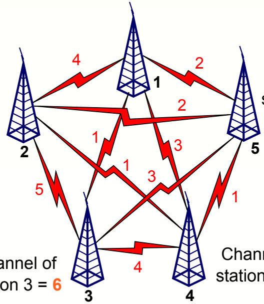
Let $G = (V, E)$ be an undirected graph and $d : E \rightarrow \mathbb{Z}_+$. A feasible coloring of G and d for the BCP is an assignment of colors $c : V \rightarrow \mathbb{Z}_+$ such that for each $(i, j) \in E$, the condition $|c(i) - c(j)| \geq d(i, j)$ is true. The span, defined as $\max_{i \in V} c(i)$, must be the minimum possible.

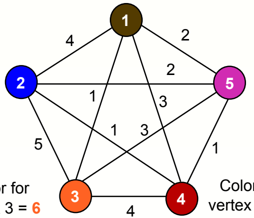


$$\max_{i \in V} c(i) = \max\{1, 3, 4, 5\} \rightarrow \textbf{span = 5.}$$

Introduction

- Important application: channel assignment in mobile wireless networks.
  - Network consists of a number of transmitters, each responsible for calls in its area.
  - Channels must respect interference constraints.
  - Spectrum usage must be minimized.





Introduction

- BCP is a **particular case** of **T-coloring**, which asks for a coloring $c : V \to \mathbb{Z}_+$ such that $|c(i) - c(j)| \notin T_{i,j}$ for every $(i, j) \in E$.
  - $T_{i,j}$: forbidden sets [Hale, 1980].

Introduction

- BCP is a **particular case** of **T-coloring**, which asks for a coloring $c : V \to \mathbb{Z}_+$ such that $|c(i) - c(j)| \notin T_{i,j}$ for every $(i, j) \in E$.
  - $T_{i,j}$: forbidden sets [Hale, 1980].
  - BCP is equivalent to T-coloring with $T_{i,j} = \{0, 1, \ldots, d_{i,j} - 1\}$ for all $(i, j) \in E$.

Introduction

- BCP is a **particular case** of **T-coloring**, which asks for a coloring $c : V \to \mathbb{Z}_+$ such that $|c(i) - c(j)| \notin T_{i,j}$ for every $(i, j) \in E$.
  - $T_{i,j}$: forbidden sets [Hale, 1980].
  - BCP is equivalent to T-coloring with $T_{i,j} = \{0, 1, \ldots, d_{i,j} - 1\}$ for all $(i, j) \in E$.
- If $d_{i,j} = 1$ for every $(i, j) \in E$, then the BCP is equivalent to the classic k-coloring problem.

Introduction

- BCP is a **particular case** of **T-coloring**, which asks for a coloring $c : V \rightarrow \mathbb{Z}_+$ such that $|c(i) - c(j)| \notin T_{i,j}$ for every $(i, j) \in E$.
  - $T_{i,j}$: forbidden sets [Hale, 1980].
  - BCP is equivalent to T-coloring with $T_{i,j} = \{0, 1, \ldots, d_{i,j} - 1\}$ for all $(i, j) \in E$.
- If $d_{i,j} = 1$ for every $(i, j) \in E$, then the BCP is equivalent to the classic k-coloring problem.
  - In this case, the span $\max_{i \in V} c(i)$ is equivalent to the number of used colors.

Introduction

- Previous works on the BCP include:

Introduction

- Previous works on the BCP include:
  - tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];

Introduction

- Previous works on the BCP include:
  - tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
  - local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];

Introduction

- Previous works on the BCP include:
  - tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
  - local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
  - heuristic framework [Phan and Skiena, 2002];

Introduction

- Previous works on the BCP include:
  - tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
  - local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
  - heuristic framework [Phan and Skiena, 2002];
  - integer programming [Mak, 2007];

Introduction

- Previous works on the BCP include:
  - tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
  - local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
  - heuristic framework [Phan and Skiena, 2002];
  - integer programming [Mak, 2007];
  - evolutionary algorithms [Malaguti et al., 2008];

Introduction

- Previous works on the BCP include:
  - tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
  - local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
  - heuristic framework [Phan and Skiena, 2002];
  - integer programming [Mak, 2007];
  - evolutionary algorithms [Malaguti et al., 2008];
  - constraint programming [Prestwich, 2008],;

Introduction

- Previous works on the BCP include:
  - tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
  - local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
  - heuristic framework [Phan and Skiena, 2002];
  - integer programming [Mak, 2007];
  - evolutionary algorithms [Malaguti et al., 2008];
  - constraint programming [Prestwich, 2008],;
  - GRASP [Marti et al., 2010].

Our research

- Some of our results:

Our research

- Some of our results:
    - simulated annealing [Dias, Freitas and Maculan, 2013],

Our research

- Some of our results:
  - simulated annealing [Dias, Freitas and Maculan, 2013],
  - constraint programming [Dias, Freitas, Maculan and Michelon, 2016];

Our research

- Some of our results:
  - simulated annealing [Dias, Freitas and Maculan, 2013],
  - constraint programming [Dias, Freitas, Maculan and Michelon, 2016];
  - graph theoretical properties [Freitas, Dias, Maculan and Szwarcfiter, 2016],

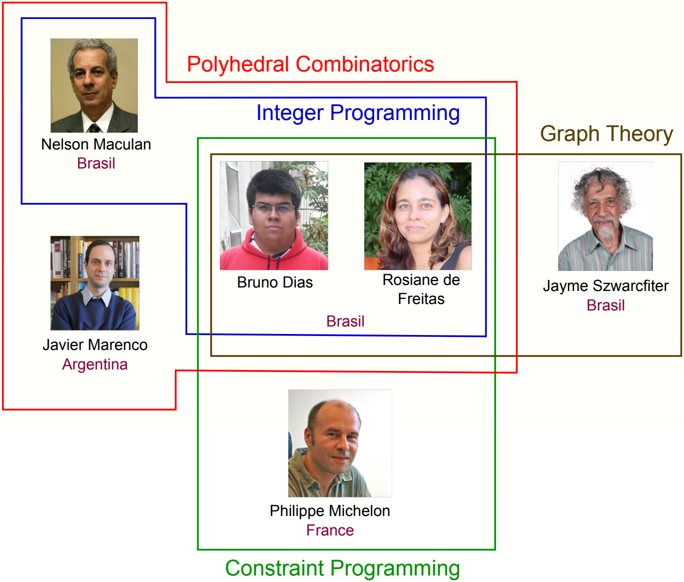
Our research

- Some of our results:
  - simulated annealing [Dias, Freitas and Maculan, 2013],
  - constraint programming [Dias, Freitas, Maculan and Michelon, 2016];
  - graph theoretical properties [Freitas, Dias, Maculan and Szwarcfiter, 2016],
  - integer programming [Dias, Freitas, Maculan and Michelon, 2016];

Our research

- Some of our results:
  - simulated annealing [Dias, Freitas and Maculan, 2013],
  - constraint programming [Dias, Freitas, Maculan and Michelon, 2016];
  - graph theoretical properties [Freitas, Dias, Maculan and Szwarcfiter, 2016],
  - integer programming [Dias, Freitas, Maculan and Michelon, 2016];
  - current work: polyhedral combinatorics [Dias, Freitas, Marenco and Maculan, 2017].

Our research



Happy 75 years old, Jayme!!

Happy 75 years old, Jayme!!

At Jayme's 65 years old

Aconcagua (Andes) 2007/2008

Happy 75 years old, Jayme!!

At Jayme's 65 years old
Aconcagua (Andes) 2007/2008

At Jayme's 70 years old
Elbrus (Russian Caucashs) 2012

Happy 75 years old, Jayme!!

At Jayme's 65 years old
Aconcagua (Andes) 2007/2008

At Jayme's 70 years old
Elbrus (Russian Caucashs) 2012

I owe a high mountain for your 75 years, Jayme!!

Integer programming models

Standard IP model

Existing integer programming model [Koster, 1999]:

Standard IP model

Existing integer programming model [Koster, 1999]:

- Variables:
  - $x_{ik} = \begin{cases} 1 & \text{if color } k \text{ is assigned to vertex } i, \\ 0 & \text{otherwise.} \end{cases}$

Standard IP model

Existing integer programming model [Koster, 1999]:

- Variables:
  - $x_{ik} = \begin{cases} 1 & \text{if color } k \text{ is assigned to vertex } i, \\ 0 & \text{otherwise.} \end{cases}$
  - $y_k = \begin{cases} 1 & \text{if color } k \text{ is given to any vertex,} \\ 0 & \text{otherwise} \end{cases}$

Standard IP model

Existing integer programming model [Koster, 1999]:

- Variables:
  - $x_{ik} = \begin{cases} 1 & \text{if color } k \text{ is assigned to vertex } i, \\ 0 & \text{otherwise.} \end{cases}$
  - $y_k = \begin{cases} 1 & \text{if color } k \text{ is given to any vertex,} \\ 0 & \text{otherwise} \end{cases}$
  - z_{max} = maximum used color ()channel).

- C = set of possible colors.

Standard IP model

Minimize $z_{\max}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k,m \in C : |k-m| < d_{i,j})$
3. $x_{ik} \leq y_k \quad (\forall i \in V;\ \forall k \in C)$
4. $z_{\max} \geq k y_k \quad (\forall k \in C)$
5. $x_{ik} \in \{0,1\} \quad (\forall i \in V;\ \forall k \in C)$
6. $y_k \in \{0,1\} \quad (\forall k \in C)$
7. $z_{\max} \in \mathbb{Z}_+$

Standard IP model

Minimize $z_{\max}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$ Color demands
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k,m \in C : |k-m| < d_{i,j})$
3. $x_{ik} \leq y_k \quad (\forall i \in V;\ \forall k \in C)$
4. $z_{\max} \geq k y_k \quad (\forall k \in C)$
5. $x_{ik} \in \{0,1\} \quad (\forall i \in V;\ \forall k \in C)$
6. $y_k \in \{0,1\} \quad (\forall k \in C)$
7. $z_{\max} \in \mathbb{Z}_+$

Standard IP model

Minimize $z_{\max}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$ Color demands
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k,m \in C : |k-m| < d_{i,j})$ Distance constraints
3. $x_{ik} \leq y_k \quad (\forall i \in V;\ \forall k \in C)$
4. $z_{\max} \geq k y_k \quad (\forall k \in C)$
5. $x_{ik} \in \{0,1\} \quad (\forall i \in V;\ \forall k \in C)$
6. $y_k \in \{0,1\} \quad (\forall k \in C)$
7. $z_{\max} \in \mathbb{Z}_+$

Standard IP model

Minimize $z_{\max}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$ Color demands
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k,m \in C : |k-m| < d_{i,j})$ Distance constraints
3. $x_{ik} \leq y_k \quad (\forall i \in V;\ \forall k \in C)$ Color usage
4. $z_{\max} \geq k y_k \quad (\forall k \in C)$
5. $x_{ik} \in \{0,1\} \quad (\forall i \in V;\ \forall k \in C)$
6. $y_k \in \{0,1\} \quad (\forall k \in C)$
7. $z_{\max} \in \mathbb{Z}_+$

Standard IP model

Minimize $z_{\max}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$ Color demands
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k,m \in C : |k-m| < d_{i,j})$ Distance constraints
3. $x_{ik} \leq y_k \quad (\forall i \in V;\ \forall k \in C)$ Color usage
4. $z_{\max} \geq k y_k \quad (\forall k \in C)$ Maximum used color
5. $x_{ik} \in \{0,1\} \quad (\forall i \in V;\ \forall k \in C)$
6. $y_k \in \{0,1\} \quad (\forall k \in C)$
7. $z_{\max} \in \mathbb{Z}_+$

Standard IP model

Minimize z_{max}

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$ — Color demands
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k,m \in C : |k - m| < d_{i,j})$ — Distance constraints
3. $x_{ik} \leq y_k \quad (\forall i \in V;\ \forall k \in C)$ — Color usage
4. $z_{max} \geq k y_k \quad (\forall k \in C)$ — Maximum used color
5. $x_{ik} \in \{0, 1\} \quad (\forall i \in V;\ \forall k \in C)$
6. $y_k \in \{0, 1\} \quad (\forall k \in C)$
7. $z_{max} \in \mathbb{Z}_+$

Integrality constraints

Improving the standard IP model

Improved IP model with less constraints and variables [Dias et al., 2016]:

Minimize z_{max}

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k, m \in C : |k - m| < d_{i,j})$
3. $z_{\text{max}} \geq k x_{ik} \quad (\forall i \in V;\ \forall k \in C)$
4. $x_{ik} \in \{0, 1\} \quad (\forall i \in V;\ \forall k \in C)$

Improving the standard IP model

Improved IP model with less constraints and variables [Dias et al., 2016]:

Minimize $z_{\max}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$ Color demands
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k,m \in C : |k-m| < d_{i,j})$
3. $z_{\max} \geq k x_{ik} \quad (\forall i \in V;\ \forall k \in C)$
4. $x_{ik} \in \{0,1\} \quad (\forall i \in V;\ \forall k \in C)$

Improving the standard IP model

Improved IP model with less constraints and variables [Dias et al., 2016]:

Minimize $z_{\max}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$ Color demands
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k,m \in C : |k-m| < d_{i,j})$ Distance constraints
3. $z_{\max} \geq k x_{ik} \quad (\forall i \in V;\ \forall k \in C)$
4. $x_{ik} \in \{0,1\} \quad (\forall i \in V;\ \forall k \in C)$

Improving the standard IP model

Improved IP model with less constraints and variables [Dias et al., 2016]:

Minimize $z_{\max}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$ — Color demands
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k,m \in C : |k-m| < d_{i,j})$ — Distance constraints
3. $z_{\max} \geq k x_{ik} \quad (\forall i \in V;\ \forall k \in C)$ — Max. used color and color usage
4. $x_{ik} \in \{0,1\} \quad (\forall i \in V;\ \forall k \in C)$

Improving the standard IP model

Improved IP model with less constraints and variables [Dias et al., 2016]:

Minimize $z_{\max}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$ Color demands
2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E;\ \forall k,m \in C : |k - m| < d_{i,j})$ Distance constraints
3. $z_{\max} \geq k x_{ik} \quad (\forall i \in V;\ \forall k \in C)$ Max. used color and color usage
4. $x_{ik} \in \{0, 1\} \quad (\forall i \in V;\ \forall k \in C)$ Integrality constraints

Orientation model for the BCP

Orientation IP formulation

Based on the model by [Borndörfer et al., 1998] for the classical vertex coloring problem.

Orientation IP formulation

Based on the model by [Borndörfer et al., 1998] for the classical vertex coloring problem.

- Variables:
  - x_i = color assigned to vertex i ($x_i \in \mathbb{N}$).

Orientation IP formulation

Based on the model by [Borndörfer et al., 1998] for the classical vertex coloring problem.

- Variables:
  - x_i = color assigned to vertex i ($x_i \in \mathbb{N}$).
  - $y_{ij} = \begin{cases} 1 & \text{if } x_i < x_j, \\ 0 & \text{otherwise.} \end{cases}$

Orientation IP formulation

Based on the model by [Borndörfer et al., 1998] for the classical vertex coloring problem.

- Variables:
  - x_i = color assigned to vertex i ($x_i \in \mathbb{N}$).
  - $y_{ij} = \begin{cases} 1 & \text{if } x_i < x_j, \\ 0 & \text{otherwise.} \end{cases}$
  - z_{max} = maximum used color ()channel).
- C = set of possible colors.

Orientation IP formulation

Minimize $z_{\max}$

Subject to:

1. $x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \quad (\forall (i,j) \in E \mid i < j)$
2. $x_j + d_{i,j} \leq x_i + U y_{ij} \quad (\forall (i,j) \in E \mid i < j)$
3. $z_{\max} \geq x_i \quad (\forall i \in V)$
4. $x_i \in \mathbb{Z}_+ \quad (\forall i \in V)$
5. $y_{ij} \in \{0, 1\} \quad (\forall i, j \in V \mid i < j)$

Orientation IP formulation

Minimize z_{max}

Subject to:

1. $x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \quad (\forall (i,j) \in E \mid i < j)$

Distance constraint:
If $x_j - x_i \geq d_{i,j}$, then $x_i < x_j$:
set $y_{ij} = 1$

2. $x_j + d_{i,j} \leq x_i + U y_{ij} \quad (\forall (i,j) \in E \mid i < j)$

3. $z_{max} \geq x_i \quad (\forall i \in V)$

4. $x_i \in \mathbb{Z}_+ \quad (\forall i \in V)$

5. $y_{ij} \in \{0, 1\} \quad (\forall i, j \in V \mid i < j)$

Orientation IP formulation

Minimize z_{max}

Subject to:

1. $x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \quad (\forall (i,j) \in E \mid i < j)$

Distance constraint:
If $x_j - x_i \geq d_{i,j}$, then $x_i < x_j$:
set $y_{ij} = 1$

2. $x_j + d_{i,j} \leq x_i + U y_{ij} \quad (\forall (i,j) \in E \mid i < j)$

Distance constraint:
If $x_i - x_j \geq d_{i,j}$, then $x_j < x_i$:
set $y_{ij} = 0$

3. $z_{max} \geq x_i \quad (\forall i \in V)$
4. $x_i \in \mathbb{Z}_+ \quad (\forall i \in V)$
5. $y_{ij} \in \{0, 1\} \quad (\forall i, j \in V \mid i < j)$

Orientation IP formulation

Minimize z_{max}

Subject to:

1. $x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \quad (\forall (i,j) \in E \mid i < j)$

Distance constraint:
If $x_j - x_i \geq d_{i,j}$, then $x_i < x_j$:
set $y_{ij} = 1$

2. $x_j + d_{i,j} \leq x_i + U y_{ij} \quad (\forall (i,j) \in E \mid i < j)$

Distance constraint:
If $x_i - x_j \geq d_{i,j}$, then $x_j < x_i$:
set $y_{ij} = 0$

3. $z_{max} \geq x_i \quad (\forall i \in V)$

Maximum used color

4. $x_i \in \mathbb{Z}_+ \quad (\forall i \in V)$

5. $y_{ij} \in \{0,1\} \quad (\forall i,j \in V \mid i < j)$

Orientation IP formulation

Minimize $z_{\max}$

Subject to:

1 $x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \quad (\forall (i,j) \in E \mid i < j)$

Distance constraint: If $x_j - x_i \geq d_{i,j}$, then $x_i < x_j$: set $y_{ij} = 1$

2 $x_j + d_{i,j} \leq x_i + U y_{ij} \quad (\forall (i,j) \in E \mid i < j)$

Distance constraint: If $x_i - x_j \geq d_{i,j}$, then $x_j < x_i$: set $y_{ij} = 0$

3 $z_{\max} \geq x_i \quad (\forall i \in V)$

Maximum used color

4 $x_i \in \mathbb{Z}_+ \quad (\forall i \in V)$

5 $y_{ij} \in \{0,1\} \quad (\forall i,j \in V \mid i < j)$

Integrality constraints

Orientation IP formulation

Minimize $z_{\max}$

Subject to:

$$x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \quad (\forall (i,j) \in E \mid i < j) \tag{1}$$

Distance constraint: If $x_j - x_i \geq d_{i,j}$, then $x_i < x_j$: set $y_{ij} = 1$

$$x_j + d_{i,j} \leq x_i + U y_{ij} \quad (\forall (i,j) \in E \mid i < j) \tag{2}$$

Distance constraint: If $x_i - x_j \geq d_{i,j}$, then $x_j < x_i$: set $y_{ij} = 0$

$$z_{\max} \geq x_i \quad (\forall i \in V) \tag{3}$$

Maximum used color

$$x_i \in \mathbb{Z}_+ \quad (\forall i \in V) \tag{4}$$

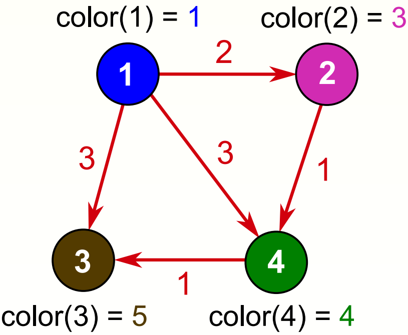
$$y_{ij} \in \{0, 1\} \quad (\forall i, j \in V \mid i < j) \tag{5}$$

Integrality constraints

This is a new formulation for BCP!

Orientation IP formulation

The formulation induces an orientation on the input graph, according to colors.



Values of variables in the optimal solution:

$z_{max} = 5$

	$y_{12} = 1$
$x_1 = 1$	$y_{13} = 1$
$x_2 = 3$	$y_{14} = 1$
$x_3 = 5$	$y_{24} = 1$
$x_4 = 4$	$y_{34} = 0$

Facet-defining inequalities for orientation model

Polyhedral study

- The parameter $U \in \mathbb{Z}_+$ in the distance constraints must be an upper bound on available colors.

Polyhedral study

- The parameter $U \in \mathbb{Z}_+$ in the distance constraints must be an upper bound on available colors.
- If no such a bound is imposed, then the convex hull of the resulting feasible solutions is not a polytope, and an integer programming formulation is not possible in this case.

Polyhedral study

- The parameter $U \in \mathbb{Z}_+$ in the distance constraints must be an upper bound on available colors.
- If no such a bound is imposed, then the convex hull of the resulting feasible solutions is not a polytope, and an integer programming formulation is not possible in this case.

Definition 2: BCP Polytope .

We define $PO(G, d, U)$ to be the convex hull of feasible solutions to the previous formulation.

Polyhedral study

- The parameter $U \in \mathbb{Z}_+$ in the distance constraints must be an upper bound on available colors.
- If no such a bound is imposed, then the convex hull of the resulting feasible solutions is not a polytope, and an integer programming formulation is not possible in this case.

Definition 2: BCP Polytope .

We define $PO(G, d, U)$ to be the convex hull of feasible solutions to the previous formulation.

Theorem 1. If $U \geq \chi(G, d) + 2d_{max}$, then the polytope $PO(G, d, U)$ is full-dimensional.

Orientation model - Valid inequalities and facets

- There are some valid inequalities (facets) proved to the Orientation model - k-coloring:

Orientation model - Valid inequalities and facets

- There are some valid inequalities (facets) proved to the Orientation model - k-coloring:
  - Clique inequalities

Orientation model - Valid inequalities and facets

- There are some valid inequalities (facets) proved to the Orientation model - k-coloring:
  - Clique inequalities
  - Double clique inequalities

Orientation model - Valid inequalities and facets

- For the BCP: additional constraints and variables are needed to use the inequalities:

Orientation model - Valid inequalities and facets

- For the BCP: additional constraints and variables are needed to use the inequalities:
  - Variables $y_{ji} = \begin{cases} 1 & \text{if } x_i < x_j, \\ 0 & \text{otherwise.} \end{cases}$

Orientation model - Valid inequalities and facets

- For the BCP: additional constraints and variables are needed to use the inequalities:
  - Variables $y_{ji} = \begin{cases} 1 & \text{if } x_i < x_j, \\ 0 & \text{otherwise.} \end{cases}$
  - Constraints $y_{ij} + y_{ji} = 1$ $(\forall i, j \in V)$.

Clique cuts

Definition 3. Let $i \in V$ and consider a clique $K \subseteq N(i)$. We define the *clique inequality* associated with i and K to be

$$\sum_{j \in K} y_{ji} \ \leq \ x_i .$$

- The clique inequalities strengthen the bounds $x_i \geq 0$.

Generalized clique cuts

Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be

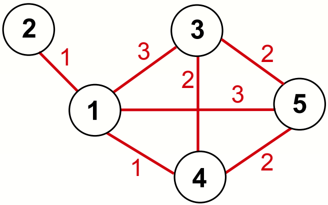
$$\sum_{j \in K} \delta^i_K(j)\, y_{ji} \;\leq\; x_i.$$

- The generalized clique cuts introduce the distance constraints to the clique cuts.

Generalized clique cuts - example

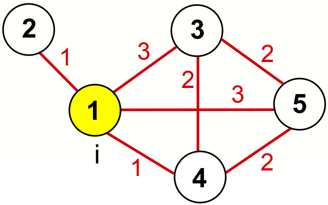
Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be

$$\sum_{j \in K} \delta^i_K(j)\, y_{ji} \;\leq\; x_i.$$



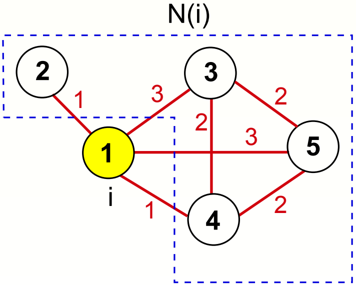
Generalized clique cuts - example

Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be $\sum_{j \in K} \delta^i_K(j)\, y_{ji} \;\leq\; x_i$.



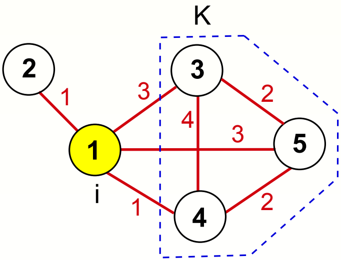
Generalized clique cuts - example

Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be $\sum_{j \in K} \delta^i_K(j)\, y_{ji} \ \leq \ x_i$.



Generalized clique cuts - example

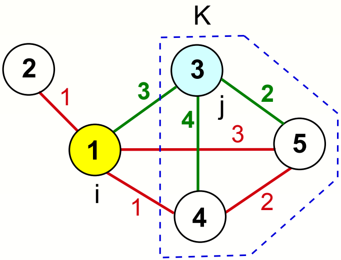
Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be $\sum_{j \in K} \delta^i_K(j)\, y_{ji} \ \leq \ x_i$.



$$\boxed{\delta^1_K(3) y_{31} + \delta^1_K(4) y_{41} + \delta^1_K(5) y_{51} \leq x_1}$$

Generalized clique cuts - example

Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be $\sum_{j \in K} \delta^i_K(j)\, y_{ji} \;\leq\; x_i$.

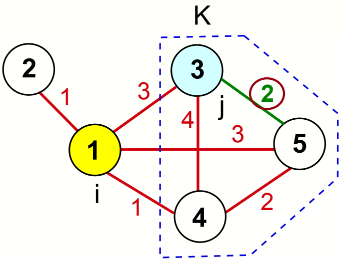


$$\boxed{\delta^1_K(3) y_{31} + \delta^1_K(4) y_{41} + \delta^1_K(5) y_{51} \leq x_1}$$

Generalized clique cuts - example

Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be

$$\sum_{j \in K} \delta^i_K(j)\, y_{ji} \;\leq\; x_i.$$

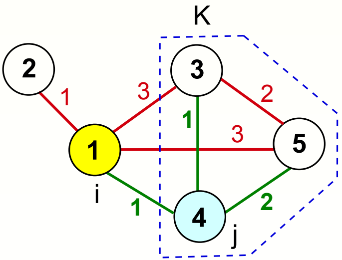


$$\boxed{\mathbf{2y_{31} + \delta^1_K(4)y_{41} + \delta^1_K(5)y_{51} \leq x_1}}$$

Generalized clique cuts - example

Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be

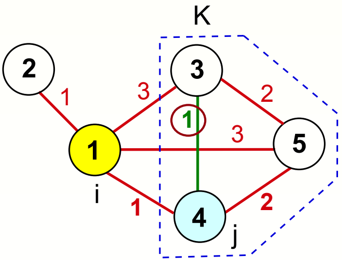
$$\sum_{j \in K} \delta^i_K(j)\, y_{ji} \;\leq\; x_i.$$



$$\boxed{\mathbf{2y_{31} + \delta^1_K(4)y_{41} + \delta^1_K(5)y_{51} \leq x_1}}$$

Generalized clique cuts - example

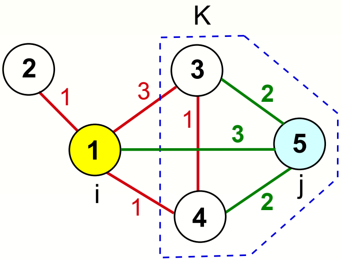
Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be $\sum_{j \in K} \delta^i_K(j)\, y_{ji} \;\leq\; x_i$.



$$\boxed{\mathbf{2y_{31} + 1y_{41} + \delta^1_K(5)y_{51} \leq x_1}}$$

Generalized clique cuts - example

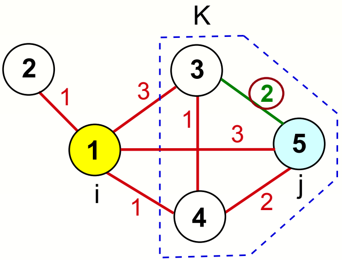
Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be $\sum_{j \in K} \delta^i_K(j)\, y_{ji} \;\leq\; x_i$.



$$\boxed{\mathbf{2y_{31} + 1y_{41} + \delta^1_K(5)y_{51} \leq x_1}}$$

Generalized clique cuts - example

Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex i and the clique K to be $\sum_{j \in K} \delta^i_K(j)\, y_{ji} \;\leq\; x_i$.



$$\boxed{2y_{31} + 1y_{41} + 1y_{51} \leq x_1}$$

Generalized clique cuts

Theorem 2. The generalized clique inequality is valid for $PO(G, d, U)$. If

(a) $U \geq \chi(G, d) + 3d_{\max}$,

(b) $d_{ij} = \delta^i_K(j)$ for every $j \in K$, and

(c) for every $t \in N(i) \setminus K$ there exists $j \in K$ with $jt \notin E$ and $d_{it} \leq d_{ij}$,

then the generalized clique inequality induces a facet of $PO(G, d, U)$.

Double clique cuts

Definition 5. Let $(i, j) \in E$ and consider a clique $K \subseteq N(i) \cap N(j)$. We define

$$x_i + 1 + \sum_{v \in K} (y_{ik} - y_{jk}) \ \leq \ x_j + U(1 - y_{ij}).$$

to be the *double clique inequality* associated with the edge (i, j) and the clique K.

Double clique cuts

Definition 5. Let $(i, j) \in E$ and consider a clique $K \subseteq N(i) \cap N(j)$. We define

$$x_i + 1 + \sum_{v \in K} (y_{ik} - y_{jk}) \ \leq \ x_j + U(1 - y_{ij}).$$

to be the *double clique inequality* associated with the edge (i, j) and the clique K.

- The double-clique inequalities strengthen the model constraints $x_i + d_{i,j} \leq x_j + U(1 - y_{ij})$.

Generalized double clique cuts

Definition 6. Let $(i,j) \in E$ and consider a clique $K \subseteq N(i) \cap N(j)$. For $v \in K$, define $\delta_K^{ij}(v) := \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Also, fix a vertex $p \in K$. We define

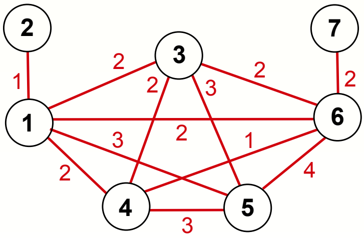
$$x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \;\leq\; x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$$

to be the *generalized double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.

- The generalized clique cuts introduce the distance constraints to the double clique cuts.

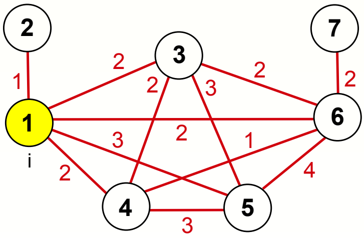
Generalized double clique cuts - example

Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.



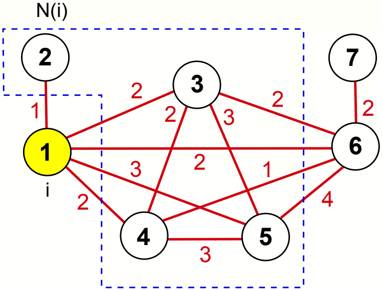
Generalized double clique cuts - example

Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.



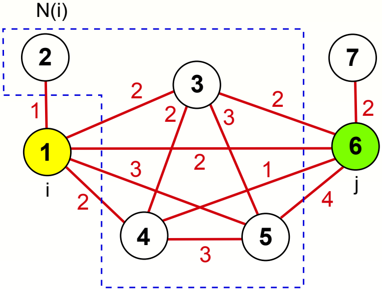
Generalized double clique cuts - example

Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.



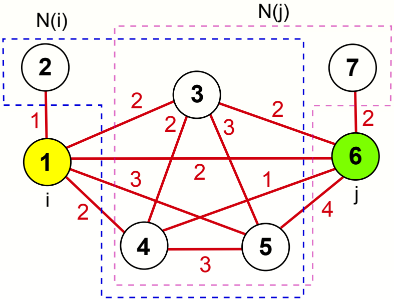
Generalized double clique cuts - example

Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.



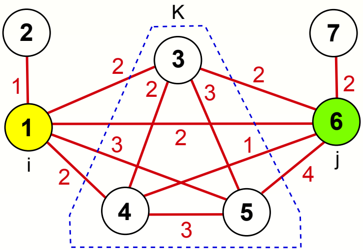
Generalized double clique cuts - example

Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta^{ij}_K(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta^{ij}_K(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta^{ij}_K(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.



Generalized double clique cuts - example

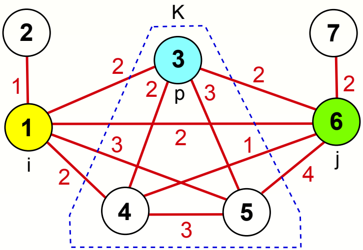
Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.



$$\boxed{\mathbf{x_1 + 2 + \gamma_3(y_{13} - y_{63}) + \gamma_4(y_{14} - y_{64}) + \gamma_5(y_{15} - y_{65}) \leq x_6 + (U + 2 - \gamma_K) y_{61}}}$$

Generalized double clique cuts - example

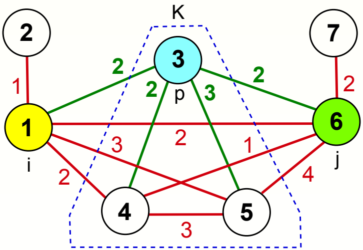
Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.



$$\boxed{\mathbf{x_1 + 2 + \gamma_3(y_{13} - y_{63}) + \gamma_4(y_{14} - y_{64}) + \gamma_5(y_{15} - y_{65}) \leq x_6 + (U + 2 - \gamma_K) y_{61}}}$$

Generalized double clique cuts - example

Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.

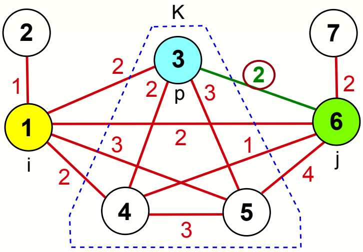


γ(p) = γ(3) = max(0, (2 · $\delta^{16}_K(3)$) - $d_{1,6}$)

$$\boxed{\mathbf{x_1 + 2 + \gamma_3(y_{13} - y_{63}) + \gamma_4(y_{14} - y_{64}) + \gamma_5(y_{15} - y_{65}) \leq x_6 + (U + 2 - \gamma_K) y_{61}}}$$

Generalized double clique cuts - example

Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.

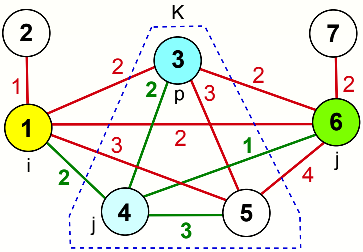


γ(p) = γ(3) = max(0, (2 · 2) - 2) = 2

$$\boxed{\mathbf{x_1 + 2 + 2(y_{13} - y_{63}) + 0(y_{14} - y_{64}) + \gamma_5(y_{15} - y_{65}) \leq x_6 + (U + 2 - \gamma_K) y_{61}}}$$

Generalized double clique cuts - example

Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.

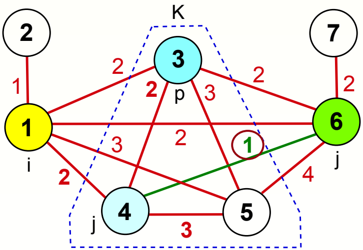


$\gamma(4) = \max(0, \delta^{16}_K(4) - d_{1,6})$

$$\boxed{x_1 + 2 + 2(y_{13} - y_{63}) + 0(y_{14} - y_{64}) + \gamma_5(y_{15} - y_{65}) \leq x_6 + (U + 2 - \gamma_K) y_{61}}$$

Generalized double clique cuts - example

Definition 6. Let $(i, j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i, j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.

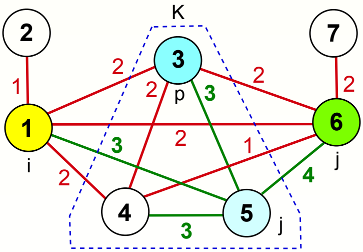


γ(4) = max(0, 1 - 2) = 0

$$\mathbf{x_1 + 2 + 2(y_{13} - y_{63}) + 0(y_{14} - y_{64}) + 1(y_{15} - y_{65}) \leq x_6 + (U + 2 - \gamma_K) y_{61}}$$

Generalized double clique cuts - example

Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.

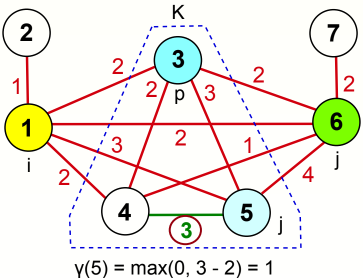


$\gamma(5) = \max(0, \delta^{16}_K(5) - d_{1,6})$

$$\mathbf{x_1 + 2 + 2(y_{13} - y_{63}) + 0(y_{14} - y_{64}) + 1(y_{15} - y_{65}) \leq x_6 + (U + 2 - \gamma_K) y_{61}}$$

Generalized double clique cuts - example

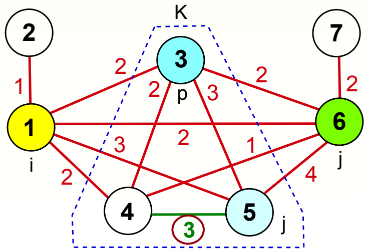
Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.



$$\mathbf{x_1 + 2 + 2(y_{13} - y_{63}) + 0(y_{14} - y_{64}) + 1(y_{15} - y_{65}) \leq x_6 + (U + 2 - \gamma_K) y_{61}}$$

Generalized double clique cuts - example

Definition 6. Let $(i,j) \in E$, $K \subseteq N(i) \cap N(j)$ be a clique. For $v \in K$, define $\delta_K^{ij}(v) = \min_{\ell \in K \cup \{i,j\} \setminus \{v\}} d_{v,\ell}$. Fix a vertex $p \in K$. We define $x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}$ to be the *double clique inequality* associated with the edge (i,j), the clique K, and the vertex p, where $\gamma_p = \max\{0, 2\delta_K^{ij}(p) - d_{i,j}\}$, $\gamma_v = \max\{0, \delta_K^{ij}(v) - d_{i,j}\}$ for $v \in K \setminus \{p\}$, and $\gamma(K) = \sum_{v \in K} \gamma_v$.



γ(5) = max(0, 3 - 2) = 1

$$\boxed{\mathbf{x_1 + 2 + 2(y_{13} - y_{63}) + 0(y_{14} - y_{64}) + 1(y_{15} - y_{65}) \leq x_6 + (U + 2 - 3) y_{61}}}$$

Generalized double clique cuts

Theorem 3. The generalized double clique inequality is valid for $PO(G, d, U)$. If

(a) $U \geq \chi(G, d) + 4d_{\max}$, and

(b) $d_{i,v} = d_{j,v} = \delta_K^{ij}(v)$ for every $v \in K$,

(c) $d_{p,v} = d_{p,j}$ for every $v \in K \backslash \{p\}$,

(d) $d_{i,j} \leq \delta_K^{ij}(v)$ for every $v \in K \backslash \{p\}$, and

(e) $(t, p) \notin E$ and $d_{i,t} + d_{t,j} \leq d_{i,j}$ for every $t \in [N(i) \cap N(j)] \backslash K$

then the generalized double clique inequality induces a facet of $PO(G, d, U)$.

Computational experiments

Computational experiments

- We implemented a preliminary cut-and-branch (C&B) algorithm in order to test these inequalities.

Computational experiments

- We implemented a preliminary cut-and-branch (C&B) algorithm in order to test these inequalities.
- Experiments executed on a computer with:
  - Intel Core i7-3770 (3.4GHz), 8 cores.
  - 8GB of RAM.
  - Ubuntu Linux 16.04.2 LTS.

Computational experiments

- We implemented a preliminary cut-and-branch (C&B) algorithm in order to test these inequalities.
- Experiments executed on a computer with:
  - Intel Core i7-3770 (3.4GHz), 8 cores.
  - 8GB of RAM.
  - Ubuntu Linux 16.04.2 LTS.
- Solver software: IBM/ILOG CPLEX 12.6.

Computational experiments

- We implemented a preliminary cut-and-branch (C&B) algorithm in order to test these inequalities.
- Experiments executed on a computer with:
  - Intel Core i7-3770 (3.4GHz), 8 cores.
  - 8GB of RAM.
  - Ubuntu Linux 16.04.2 LTS.
- Solver software: IBM/ILOG CPLEX 12.6.
- Time limit: 3600 seconds (1 hour).

Computational experiments

- We implemented a preliminary cut-and-branch (C&B) algorithm in order to test these inequalities.
- Experiments executed on a computer with:
  - Intel Core i7-3770 (3.4GHz), 8 cores.
  - 8GB of RAM.
  - Ubuntu Linux 16.04.2 LTS.
- Solver software: IBM/ILOG CPLEX 12.6.
- Time limit: 3600 seconds (1 hour).
- Instances used: GEOM set (without multicoloring demands) [Trick et al., 2002].

Cut-and-branch for BCP - pseudocode

Require: graph $G = (V, E)$, distances $d : E \to \mathbb{Z}_{\geq 0}$.

```
function CutAndBranch-BCP-Orientation(G = (V, E), d)
    lpOrient ← AssembleLPModel-Relaxation(G)
    (x, y, z_max) ← LPSolver(lpOrient)
    if (x, y, z_max) is not integer then
        H ← GenerateMaximalCliques(G)
        for each clique K ∈ V do
            Add-GeneralCliqueCut(lpOrient, (x, y, z_max), K)
            Add-GeneralDblCliqueCut(lpOrient, (x, y, z_max), K)
            (x, y, z_max) ← LPSolver(lpOrient)
            if (x, y, z_max) is integer then
                break
        if (x, y, z_max) is not integer then
            mipOrient ← ChangeVarsToInt(lpOrient)
            (x, y, z_max) ← B&C-MIPSolver(mipOrient)
    return (x, y, z_max)
```

Computational experiments - Orientation model

Instance	Standard model		Orientation model	
	Best	Time	Best	Time
GEOM20	21	0.33	21	
GEOM30	28	0.88	28	
GEOM40	28	1.97	28	
GEOM50	28	21.44	28	
GEOM60	33	45.73	33	
GEOM70	38	533.53	38	
GEOM80	41	3019.18	41	

Computational experiments - Orientation model

Instance	Standard model		Orientation model	
	Best	Time	Best	Time
GEOM20	21	0.33	21	0.03
GEOM30	28	0.88	28	0.22
GEOM40	28	1.97	28	0.19
GEOM50	28	21.44	28	4.26
GEOM60	33	45.73	33	149.60
GEOM70	38	533.53	38	121.61
GEOM80	41	3019.18	41	2167.43

Computational experiments - Orientation model

Instance	Standard model		Orientation model	
	Best	Time	Best	Time
GEOM20	21	0.33	21	0.03
GEOM30	28	0.88	28	0.22
GEOM40	28	1.97	28	0.19
GEOM50	28	21.44	28	4.26
GEOM60	33	45.73	33	149.60
GEOM70	38	533.53	38	121.61
GEOM80	41	3019.18	41	2167.43

- The orientation model has advantages on most problems in comparison with the standard IP model.

Computational experiments - Orientation model: cuts

Instance	CPLEX		C&B: CPLEX + cuts	
	Best	Time (s)	Best	Time (s)
GEOM20	21	0.03	21	0.02
GEOM30	28	0.22	28	0.25
GEOM40	28	0.19	28	0.87
GEOM50	28	4.26	28	43.63
GEOM60	33	149.60	33	18.07
GEOM70	38	121.61	38	11.66
GEOM80	41	2167.43	41	22.76

- The clique and double-clique inequalities are quite useful within the cut-and-branch procedure.

Computational experiments - Orientation model: cuts

Instance	CPLEX				
	Cliques	Imp.Bnd.	MI Round.	0-Half	Gomory
GEOM20	0	7	14	1	1
GEOM30	0	12	52	8	2
GEOM40	0	13	49	14	4
GEOM50	0	53	123	40	4
GEOM60	0	45	233	54	1
GEOM70	0	135	213	51	1
GEOM80	0	182	318	69	1

Instance	C&B: CPLEX + cuts				
	Cliques	Imp.Bnd.	MI Round.	0-Half	Gomory
GEOM20	0	6	1	4	2
GEOM30	1	24	13	3	7
GEOM40	3	33	37	5	6
GEOM50	17	47	72	16	22
GEOM60	21	85	102	23	26
GEOM70	22	94	161	44	23
GEOM80	16	98	170	43	30

- More CPLEX clique and Gomory cuts are added when the valid inequalities are included.

Orientation model and cut-and-branch: discussions

Orientation model and cut-and-branch: discussions

- The clique and double-clique inequalities proved to be useful within a cut-and-branch procedure.
- More CPLEX clique and Gomory cuts are added when the valid inequalities are included.
- The orientation model seems to be a better platform than the standard model for tackling BCP with integer programming.

Orientation model and cut-and-branch: discussions

- The clique and double-clique inequalities proved to be useful within a cut-and-branch procedure.
- More CPLEX clique and Gomory cuts are added when the valid inequalities are included.
- The orientation model seems to be a better platform than the standard model for tackling BCP with integer programming.
- Further cuts are possible, which indicates that the orientation model is a very competitive approach to BCP.

Concluding remarks

Concluding remarks

- We proposed a new orientation-based IP formulation for the Bandwidth Coloring Problem, which seems to be a better platform than the standard model for tackling BCP with integer programming.

Concluding remarks

- We proposed a new orientation-based IP formulation for the Bandwidth Coloring Problem, which seems to be a better platform than the standard model for tackling BCP with integer programming.
- We made a preliminary study of the polytope associated to the new model, including the definition of facet-defining cutting planes for it.

Concluding remarks

- We proposed a new orientation-based IP formulation for the Bandwidth Coloring Problem, which seems to be a better platform than the standard model for tackling BCP with integer programming.
- We made a preliminary study of the polytope associated to the new model, including the definition of facet-defining cutting planes for it.
- **Current work:**

Concluding remarks

- We proposed a new orientation-based IP formulation for the Bandwidth Coloring Problem, which seems to be a better platform than the standard model for tackling BCP with integer programming.
- We made a preliminary study of the polytope associated to the new model, including the definition of facet-defining cutting planes for it.
- **Current work:**
  - Search for more facet-inducing inequalities, and implement a branch-and-cut procedure for BCP.

Concluding remarks

- We proposed a new orientation-based IP formulation for the Bandwidth Coloring Problem, which seems to be a better platform than the standard model for tackling BCP with integer programming.
- We made a preliminary study of the polytope associated to the new model, including the definition of facet-defining cutting planes for it.
- **Current work:**
  - Search for more facet-inducing inequalities, and implement a branch-and-cut procedure for BCP.
  - Try to apply the distance model for the BCP.

References

References

- Borndörfer, R., Eisenblätter, A., Grötschel, M., and Martin, A. (1998).
  The orientation model for frequency assignment problems.
  Technical report, ZIB Berlin.

- Dias, B., de Freitas, R., Maculan, N., and Michelon, P. (2016).
  Solving the bandwidth coloring problem applying constraint and integer programming techniques.
  Optimization Online (e-print).

- Dorne, R. and Hao, J. (1998).
  Tabu search for graph coloring, t-coloring and set t-colorings.
  In Voss, S., Martello, S., Osman, I., and Roucairol, C., editors, *Metaheuristics: advances and trends in local search paradigms for optimization*, pages 77–92. Kluver Academic Publishers.

- Galinier, P. and Hertz, A. (2006).
  A survey of local search methods for graph coloring.
  Computers and Operations Research, 33(9):2547–2562.

References

Hale, W. (1980).
Frequency assignment: Theory and applications.
Proceedings of the IEEE, 25:1497–1514.

Koster, A. (1999).
Frequency assignment: models and algorithms.
PhD thesis, Universiteit Maastricht, the Netherlands.

Lai, X. and Lu, Z. (2013).
Multistart Iterated Tabu Search for Bandwidth Coloring Problem.
Computers & Operations Research, 40:1401–1409.

Lau, T. L. and Tsang, E. P. K. (1998).
Solving the Radio Link Frequency Assignment Problem using Guided Local Search.
In *NATO Symposium on Radio Length Frequency Assignment*.

References

Mak, V. (2007).
Polyhedral studies for minimum-span graph labelling with integer distance constraints.
International Transactions in Operational Research, 14(2):105–121.

Malaguti, E., Monaci, M., and Toth, P. (2008).
A Metaheuristic Approach for the Vertex Coloring Problem.
INFORMS Journal on Computing, 20(2):302–316.

Marti, R., Gortazar, F., and Duarte, A. (2010).
Heuristics for the bandwidth colouring problem.
International Journal of Metaheuristics, 1:11.

Phan, V. and Skiena, S. (2002).
Coloring graphs with a general heuristic search engine.
In *Computational Symposium on Graph Coloring and Its Generalizations*, pages 92–99.

References

Prestwich, S. (2008).
Generalised graph colouring by a hybrid of local search and constraint programming.
Discrete Applied Mathematics, 156(2):148–158.

Trick, M., Mehrotra, A., and Johnson, D. (2002).
COLOR02/03/04: Graph Coloring and its Generalizations.
http://mat.gsia.cmu.edu/COLOR02/.

Merci!

Thank you!

Rosiane de Freitas
rosiane@icomp.ufam.edu.br

Supported by:

CAPES

CNPq
Conselho Nacional de Desenvolvimento Científico e Tecnológico

FAPEAM
Fundação de Amparo à Pesquisa do Estado do Amazonas

FAPERJ
Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro

FAPESP