# Weighted Upper Domination

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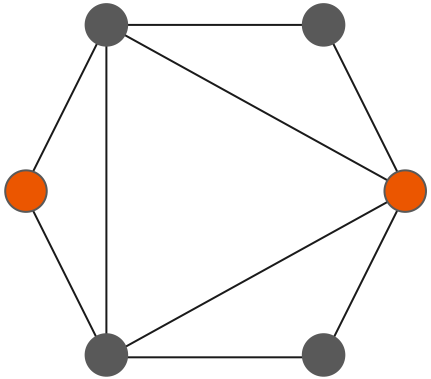
# Abstract

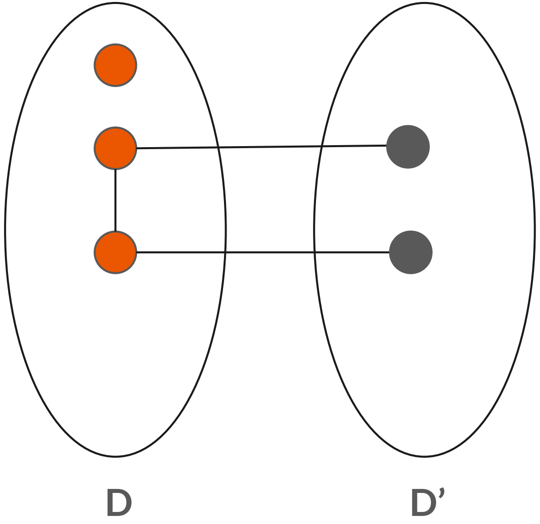

- Upper Domination Number: the cardinality of maximum minimal dominating set
- NP-complete even in planar and co-bipartite graphs.
- Polynomial-time solvable in strongly perfect graphs (bipartite, split).
- We show that the weighted version is NP-complete even in very restricted cases of bipartite and split graphs.

A vertex set D is **dominating** if all vertices outside of D have a neighbour in D.

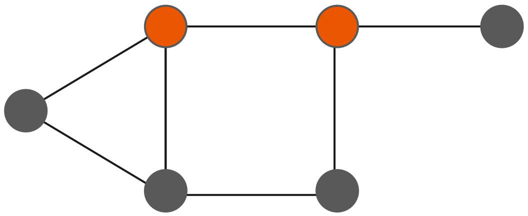
D is **minimal dominating set** if D is dominating and minimal for inclusion.



**Domination**
**+**
**Private Neighbor (possibly itself)**

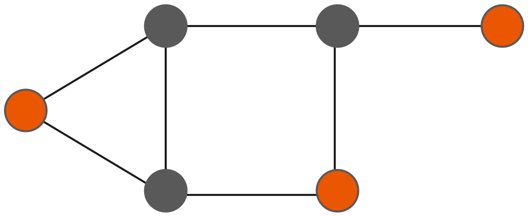


A graph may have dominating sets of different cardinalities.



$\gamma(G) = 2$

Minimum Domination



$\Gamma(G) = 3$

Upper Domination

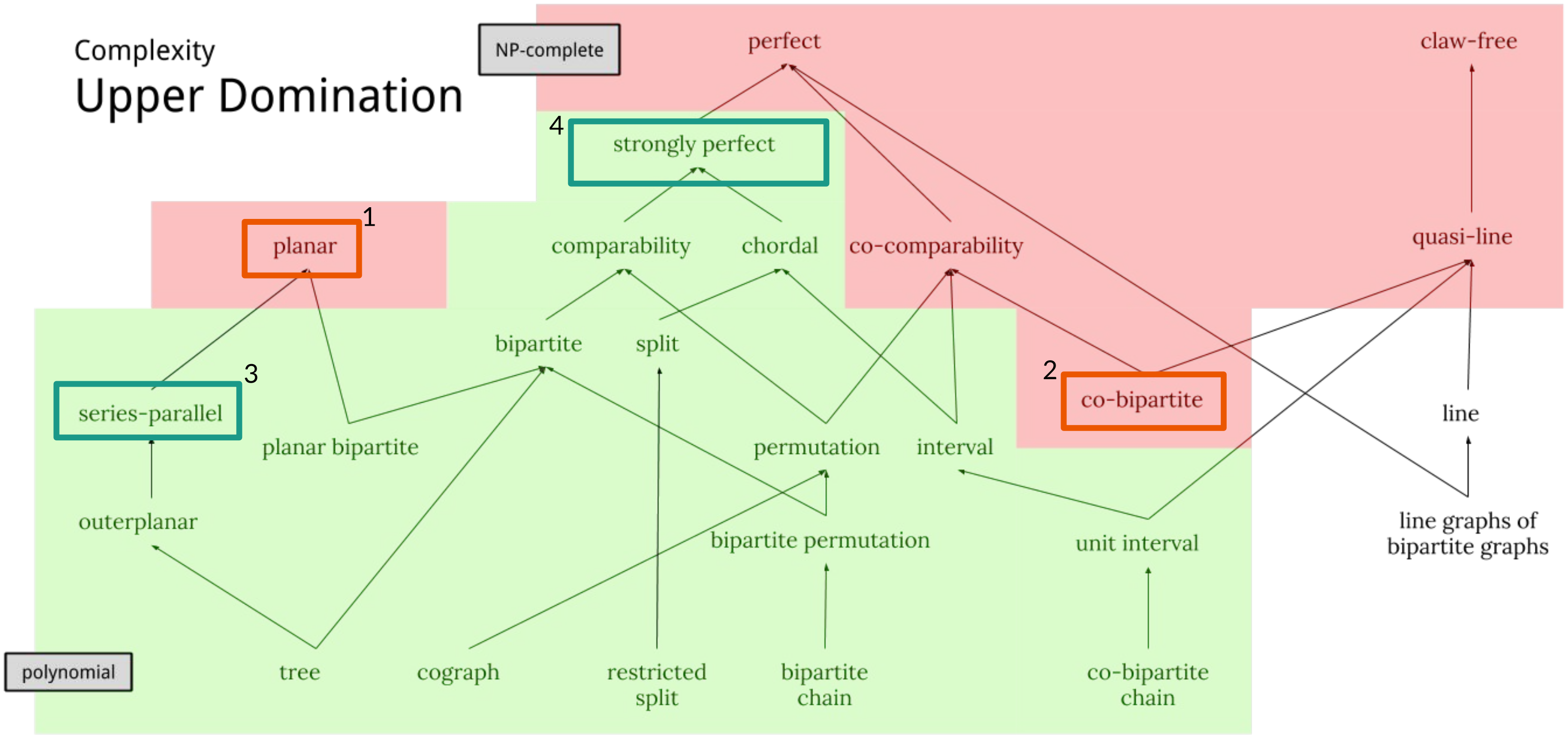
# Maximum Minimal Dominating Set (a.k.a. Upper Domination)

Some Other MaxMin / MinMax Problems Considered in the Literature:

- Minimum Maximal Matching
- Minimum Maximal Independent Set
- Maximum Minimal Vertex Cover

Complexity

# Upper Domination

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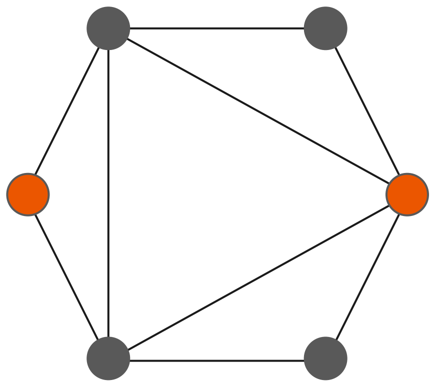
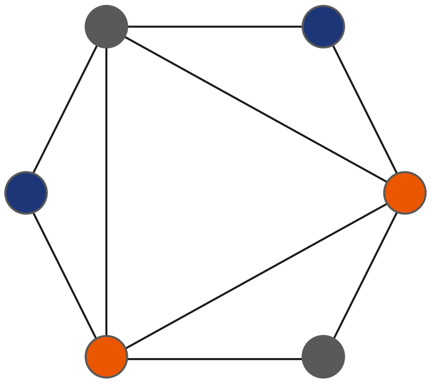
A verset set I is **independent** if none of the vertices in I is adjacent to none of the vertices of I.

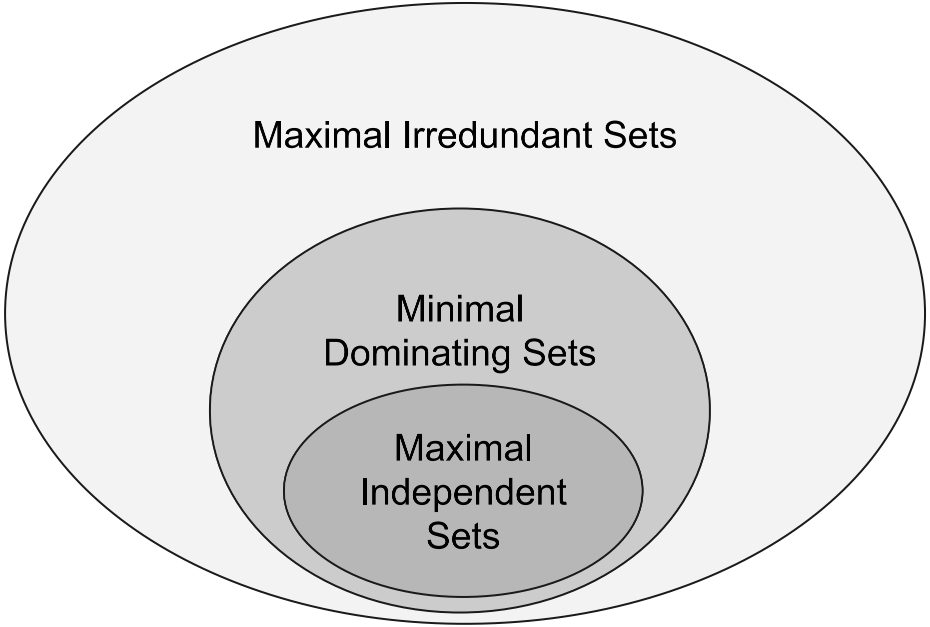
A vertex set S is **irredundant** if every element of S has a private neighbor.

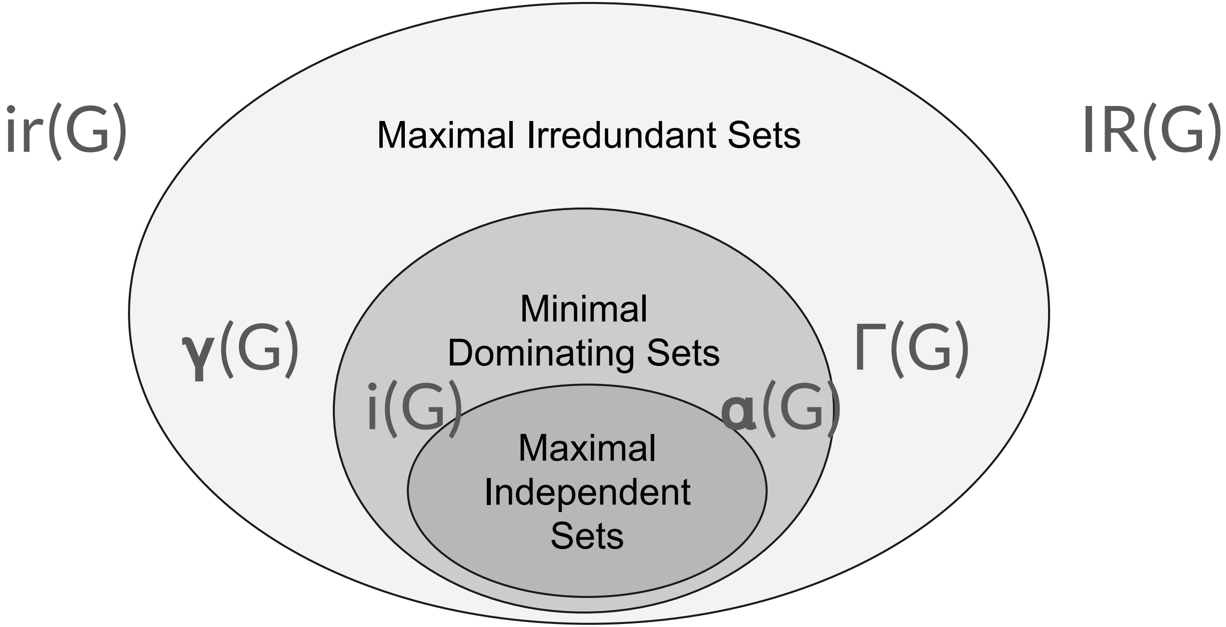




Every maximal independent set is a minimal dominating set.

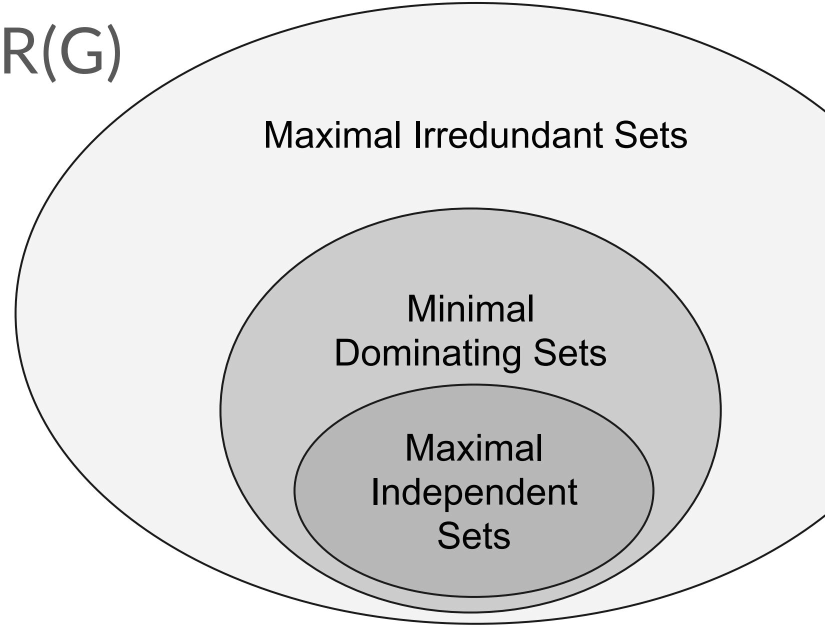
Every minimal dominating set is a maximal irredundant set.

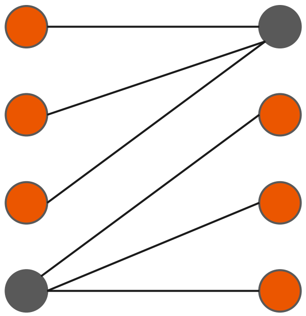



ir(G)
Maximal Irredundant Sets
IR(G)
γ(G)
Minimal
Dominating Sets
Γ(G)
i(G)
α(G)
Maximal
Independent
Sets

# Well-known domination chain

$ir(G) \leq \boldsymbol{\gamma}(G) \leq i(G) \leq \boldsymbol{\alpha}(G) \leq \Gamma(G) \leq IR(G)$





A graph where $\boldsymbol{\alpha}(G) = \Gamma(G) = IR(G)$

# When $\alpha(G) = \Gamma(G) = IR(G)$ ?

**Most general answer is:**
**If G is a strongly perfect graph*.**

And fortunately,
**maximum independent set** in strongly perfect graphs
is solvable in **polynomial-time**.

*Every induced subgraph H has an independent vertex set meeting all maximal cliques of H.

# Weighted Upper Domination Set (WUDS)

# We may have $\alpha_w(G) < \Gamma_w(G)$ even for a simple path graph

# Main results

**Theorem**: Computing WUDS is strongly NP-hard for 3-subregular split graphs, even for bi-valued weights.

**Corollary**: For any $\varepsilon>0$, WUDS is not $O(n^{1-\varepsilon})$-approximable in split graphs on n vertices, even for bi-valued weights, unless **NP ≠ ZPP**

**Theorem**: Computing WUDS is strongly NP-hard for planar bipartite graphs of maximum degree 4, even for tri-valued weights.

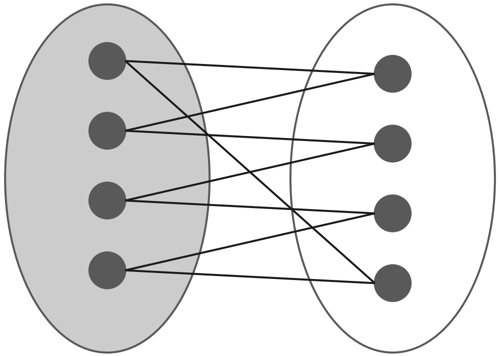
**Corollary**: WUDS is **APX**-complete in bipartite graphs.

# Remark 1

**Theorem**. Computing WUDS is strongly NP-hard for 3-subregular split graphs, even for bi-valued weights.

The strongest hardness result that we can produce.

A 2-subregular split graph.



# Remark 2

WUDS with performance ratio O(n) is always possible.

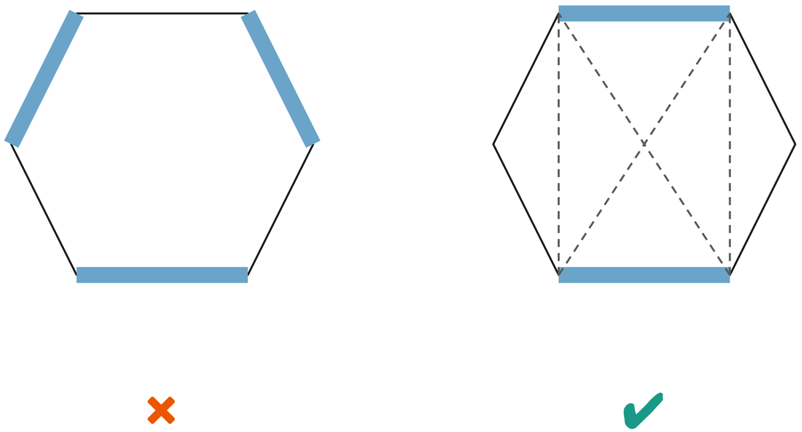
By taking any maximal independent set containing a vertex of maximum weight $w_{max}$.

By **Corollary**: For any $\varepsilon>0$, WUDS is not $O(n^{1-\varepsilon})$-approximable in split graphs on n vertices, even for bi-valued weights, unless **NP ≠ ZPP**.

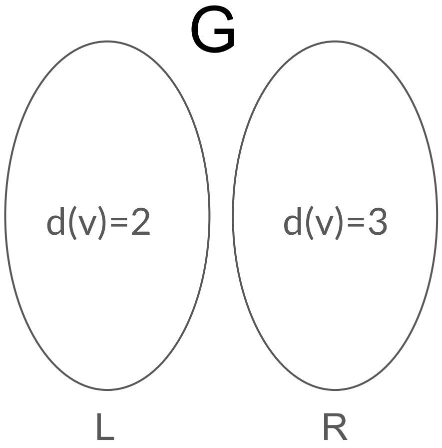
This is the best we can have for split graphs.

**Theorem.** Computing WUDS is strongly NP-hard for 3-subregular split graphs, even for bi-valued weights.

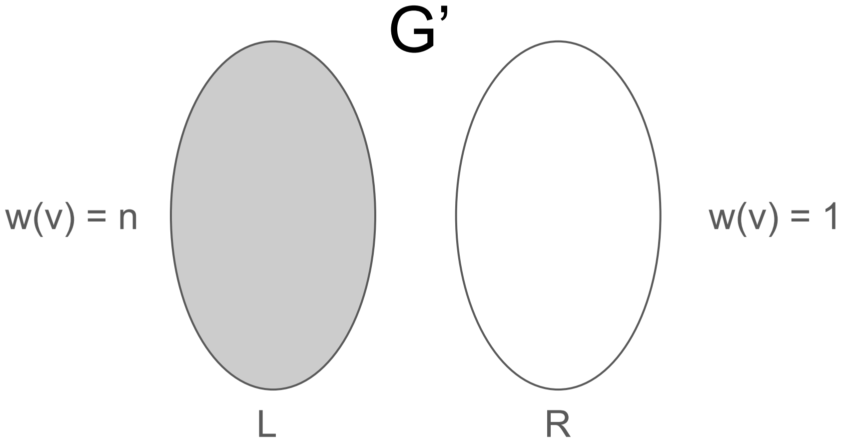
Reduction is done from THE MAXIMUM INDUCED MATCHING PROBLEM.



# Construction




Bipartite Planar Subcubic

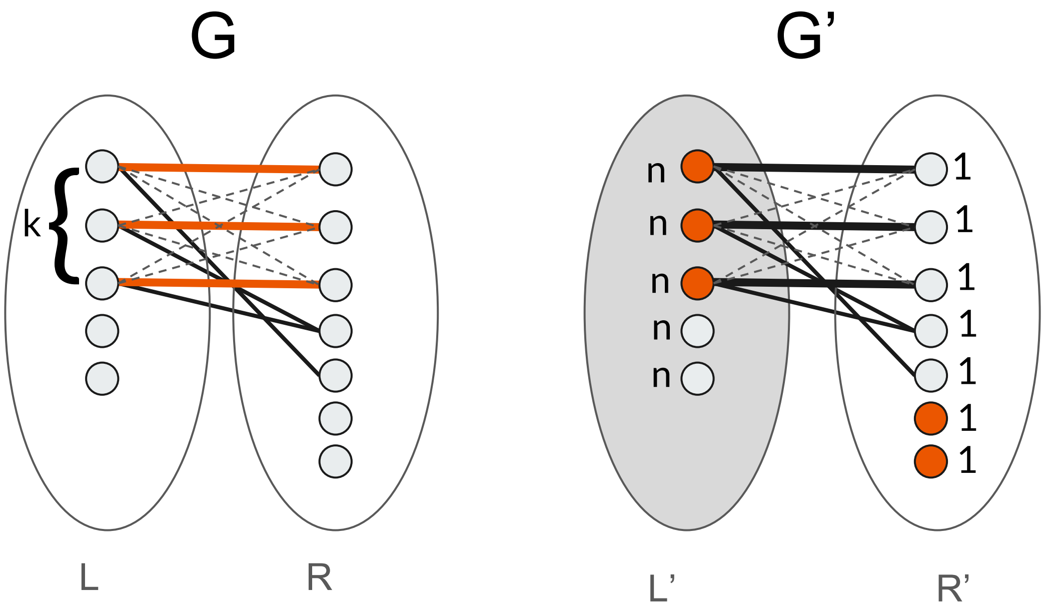



Split 3-subregular

# Claim

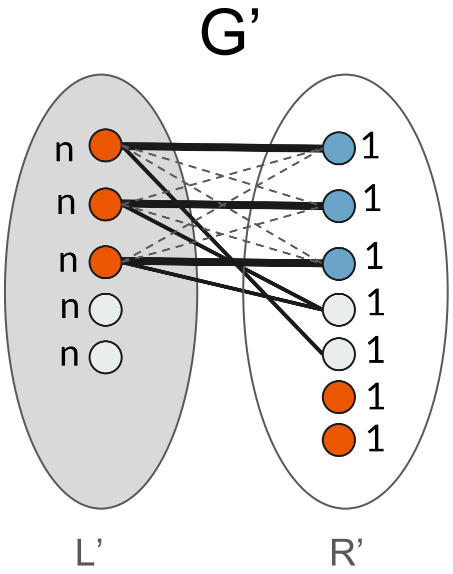
MIM of G of size at least $k \geq 1 \Leftrightarrow$ UDS of G’ of weight at least $nk \geq n$

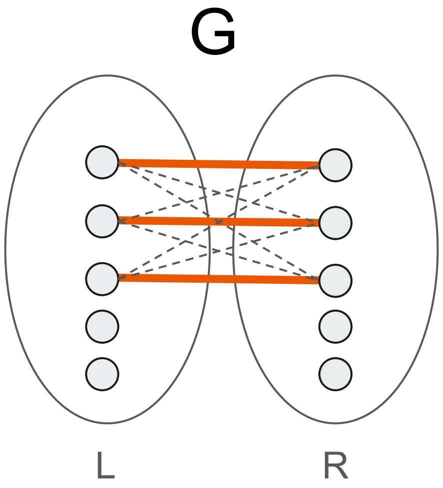
## MIM of G of size at least k ≥ 1 ⇒ UDS of G' of weight at least nk ≥ n




1 - D dominates G'.
2 - D is minimal.
3 - w(D) ≥ nk.

## UDS of G’ of weight at least nk ≥ n ⇒ MIM of G of size at least k ≥ 1

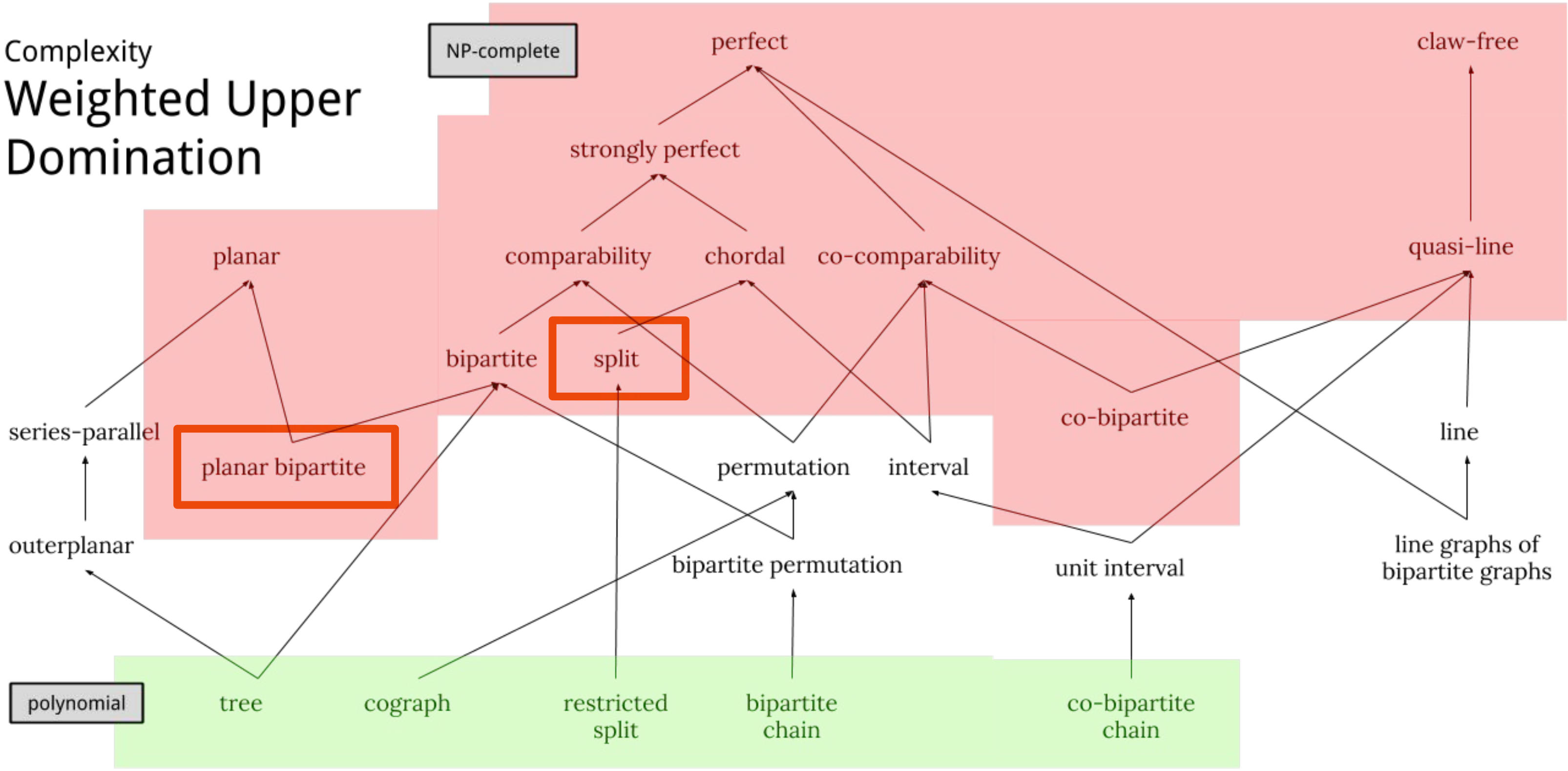



1 - |L’∩D| is at least k.
2 - M is an induced matching.

Complexity

# Weighted Upper Domination

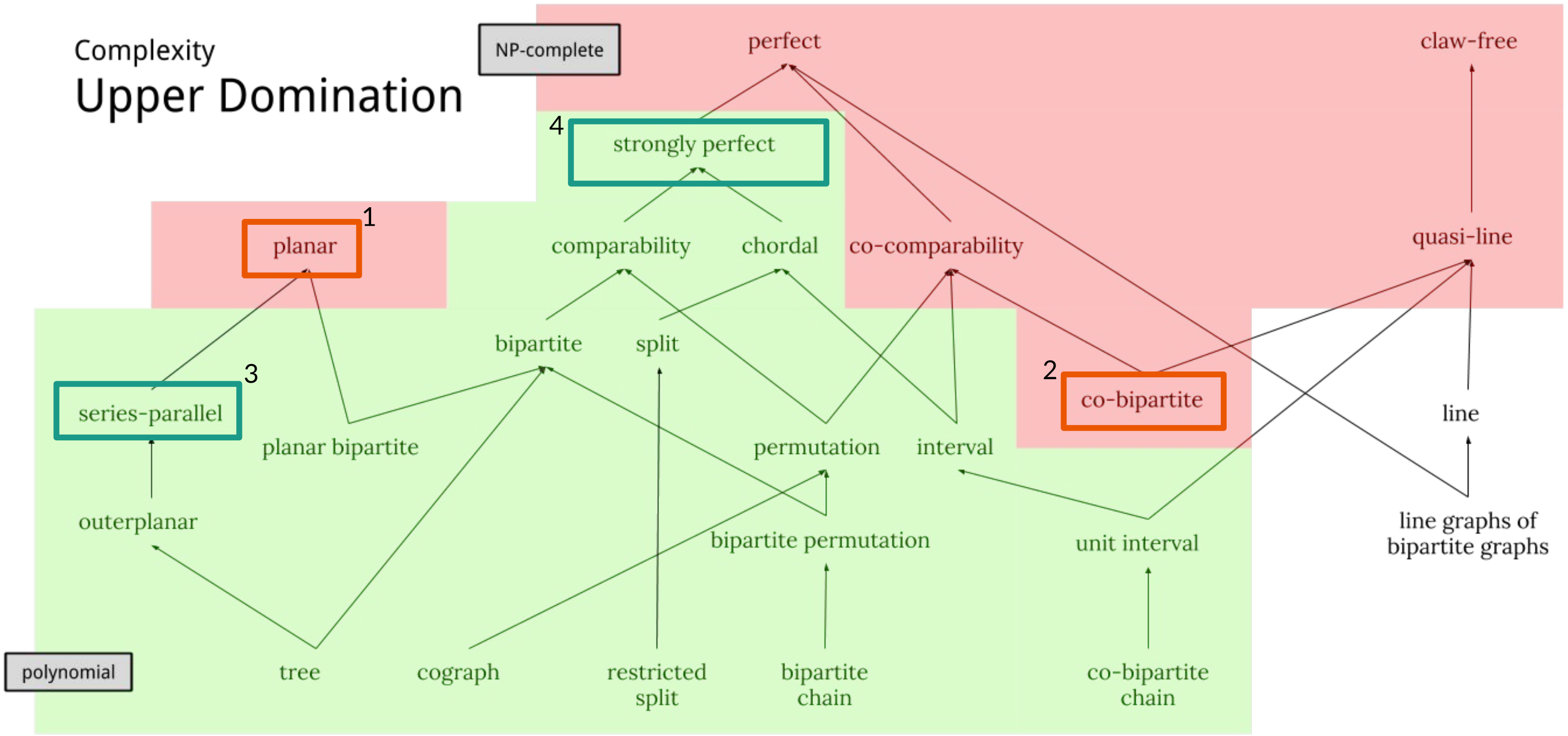


# Open Problems

- Complexity of UDS in special graph classes
  - Unweighted: line graphs, d-regular graphs for $d>4$,
  - Weighted: series-parallel, permutation, interval and many more.
- Equivalent problems:
  - Min Max Matching ~ Min Edge Dominating Set
  - Min Max Independent Set ~ Min Independent Dominating Set
- Recognition of well-dominated graphs
- For every graph G in series - parallel graphs, do we have $\alpha(G) = \Gamma(G)$?

Complexity

# Upper Domination

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# Thank you.

# Dynamic Programming for WUDS in trees

**Theorem**: UDS is not $O(n^{1-\epsilon})$-approximable even in co-bipartite graphs.

A split graph is called a p-subregular if for $l \in L$, $d_G(l) - |L| + 1 \leq p$ and for $r \in R$, $d_G(r) \leq p$.

A 2-subregular split graph.