On the local density problem for graphs of given odd-girth

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Universität Hamburg

Joint work with Guilherme Oliveira Mota, Christian Reiher and Mathias Schacht

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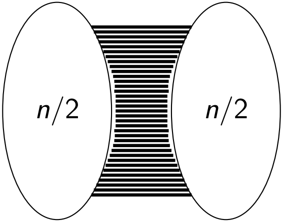

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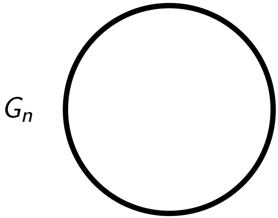

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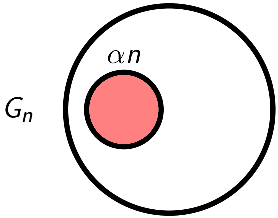

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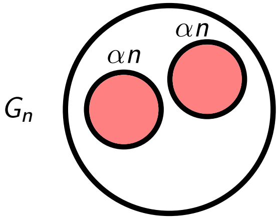

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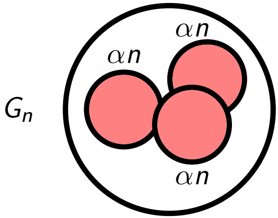

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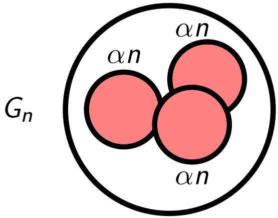

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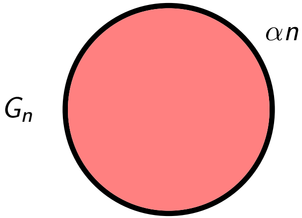
- **Example:** For $\alpha = 1, \quad \beta = 1/4$.



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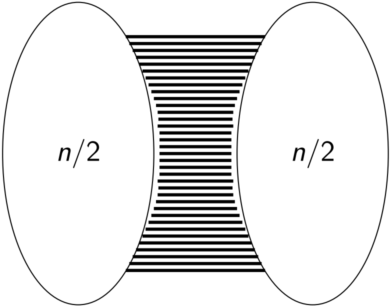

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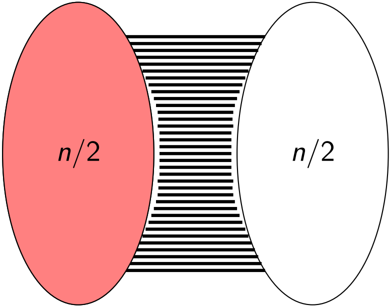

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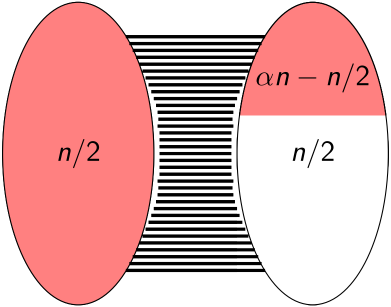

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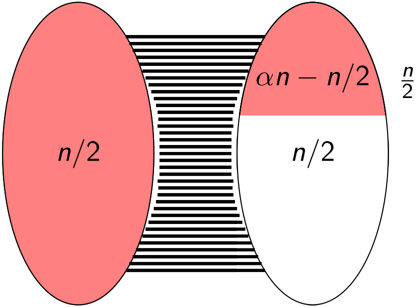

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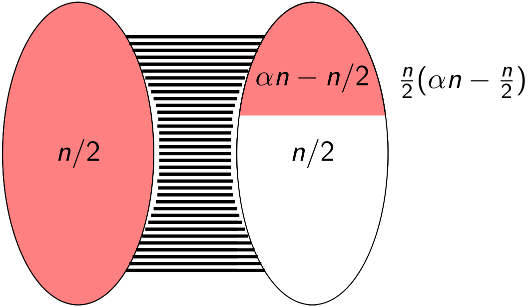

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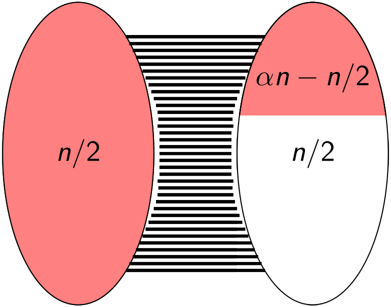

$\frac{n}{2}(\alpha n - \frac{n}{2}) = \frac{(2\alpha - 1)}{4} n^2$

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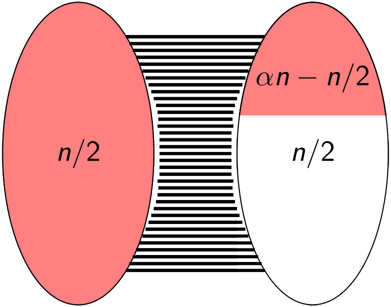

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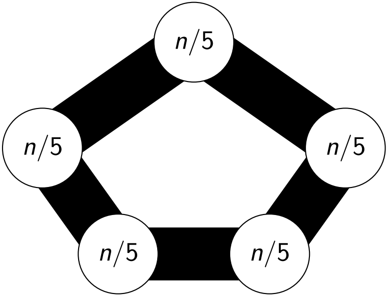

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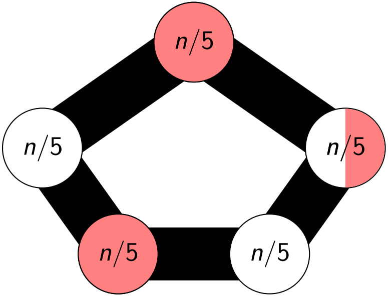

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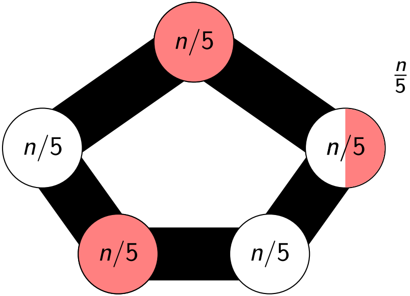

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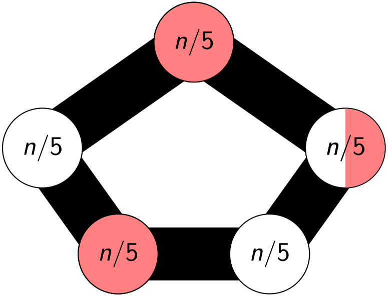

$\frac{n}{5}(\alpha n - \frac{2n}{5})$

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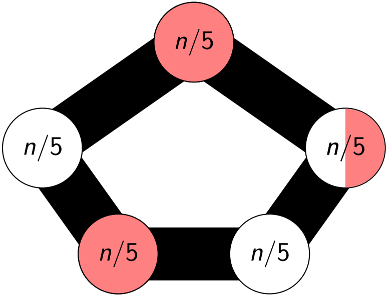

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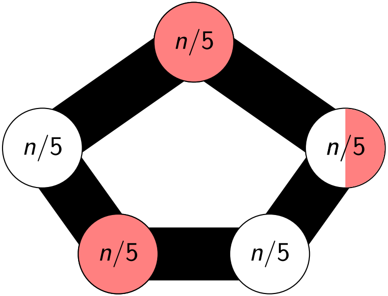

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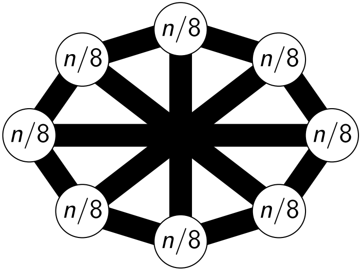

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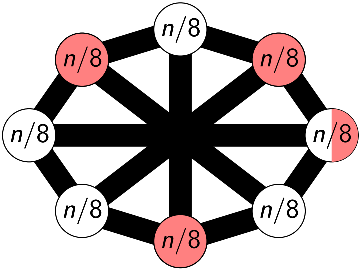

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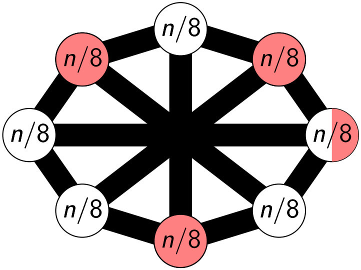

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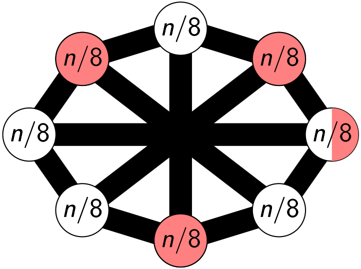

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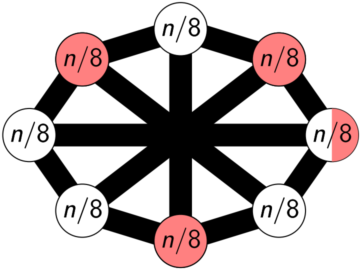

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Conjectures

Conjecture (Erdős 1976)

Let $53/120 \leq \alpha \leq 1$ *and let* G_n *be an n-vertex graph. If n is sufficiently large and* G_n *is* (α, β)*-dense with*

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- Interesting case: $\alpha = 1/2, \quad \beta \geq 1/50.$

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Stated in the contrapositive...

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Every triangle-free graph G_n contains a subset of $\lfloor n/2 \rfloor$ vertices that induces at most $n^2/50$ edges.

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Theorem (Norin–Yepremian 2015)

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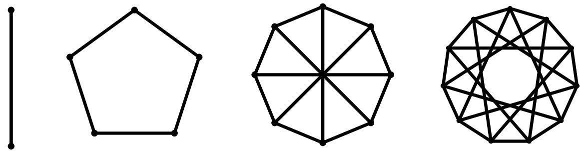
If G_n is triangle-free with $\chi(G_n) \leq 3$ and $\delta(G_n) > n/3$, then G_n is not $(1/2, 1/50)$-dense.

Important structures: Andrásfai graphs



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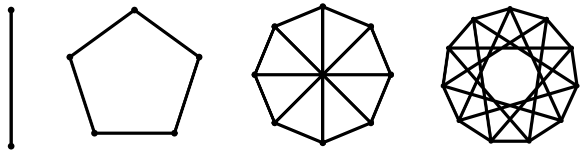

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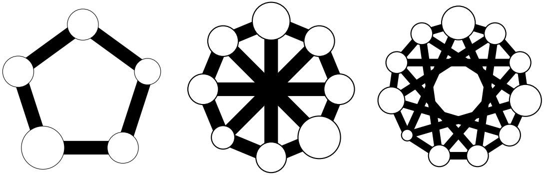
Properties of F_d

- $3d - 1$ vertices
- Triangle-free
- d-regular
- $\alpha(F_d) = d$

Why are Andrásfai graphs important to the problem?

Andrásfai graphs - Blow-ups



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Triangle-free graphs with large minimum degree

Theorem (Andrásfai–Erdős–Sós 1974)

If $\delta(G_n) > 2n/5$ *and* $K_3 \not\subseteq G_n$*, then* G_n *is a subgraph of a blow-up of* F_1

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Theorem (Chen–Jin–Koh 1997)

If $\delta(G_n) > \big((d+1)/(3d+2)\big)n$*,* $K_3 \not\subseteq G_n$ *and* $\chi(G_n) \leq 3$*, then* G_n *a subgraph of a blow-up of* F_d*.*

Obtaining the main results

Theorem (B.–Mota–Reiher–Schacht 2017+)

If G_n is a subgraph of a blow-up of F_d for some integer $d \geq 1$, then G_n is not $(1/2, 1/50)$-dense.

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Corollary (B.–Mota–Reiher–Schacht 2017+)

If G_n is triangle-free with $\delta(G_n) > n/3$ and $\chi(G_n) \leq 3$, then G_n is not $(1/2, 1/50)$-dense.

Proof for blow-ups of F_3

Theorem (B.–Mota–Reiher–Schacht 2017+)

If G_n is a subgraph of a blow-up of F_3, then G_n is not $(1/2, 1/50)$-dense.

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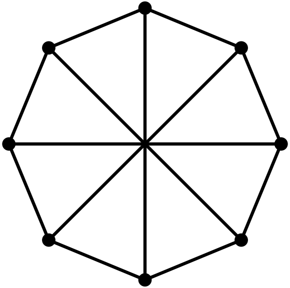

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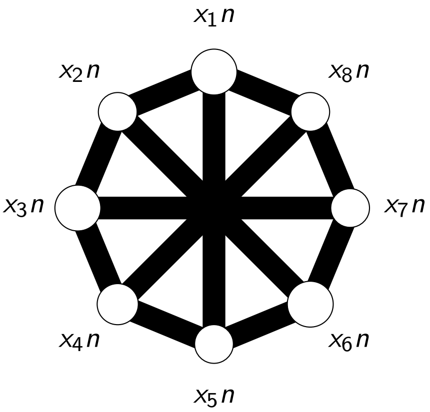
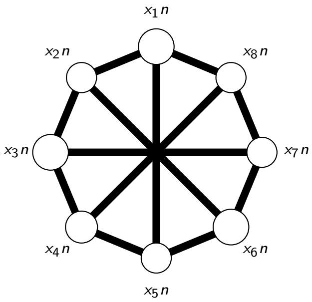


Proof for blow-ups of F_3



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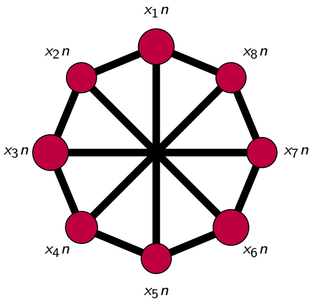
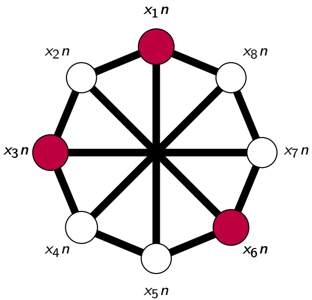


$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 1.$$

Proof for blow-ups of F_3



$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 1.$
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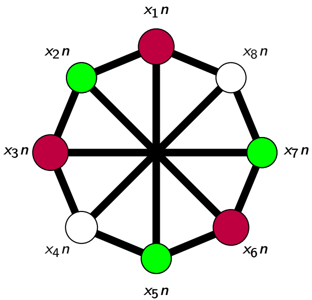
Proof for blow-ups of F_3



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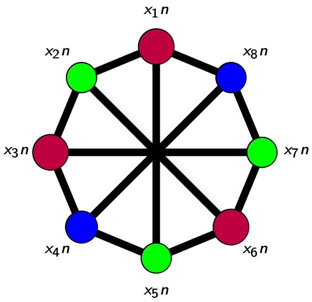
Proof for blow-ups of F_3



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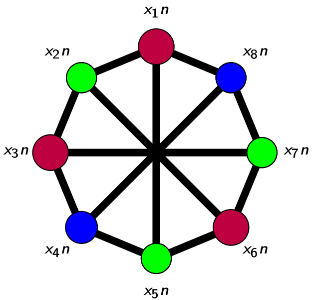

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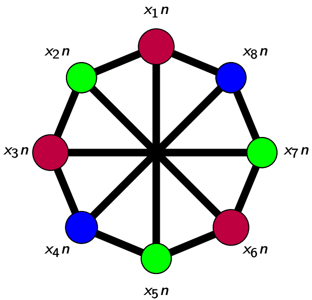

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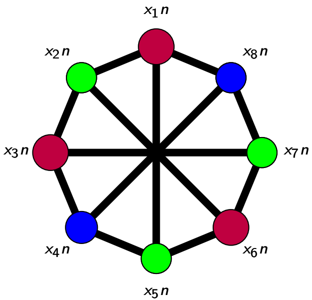

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Suppose $x_4 \geq x_8$.

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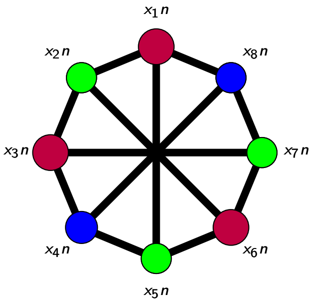

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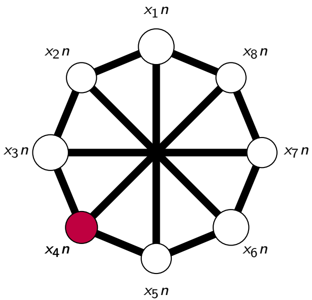

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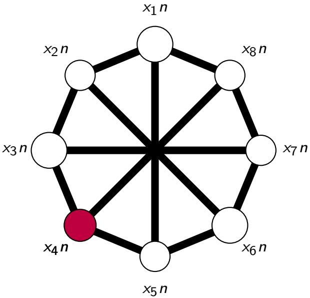
Suppose $x_4 \geq x_8$.
We have $x_4 \geq 1/2 - \alpha$.

Proof for blow-ups of F_3



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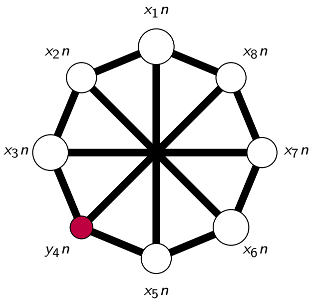
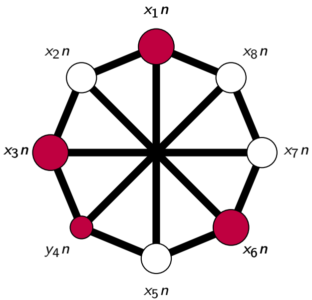


$x_4 \geq 1/2 - \alpha$. Consider $y_4 = 1/2 - \alpha$

Proof for blow-ups of F_3



$x_4 \geq 1/2 - \alpha$. Consider $y_4 = 1/2 - \alpha$ and $x_1 + x_3 + x_6$.

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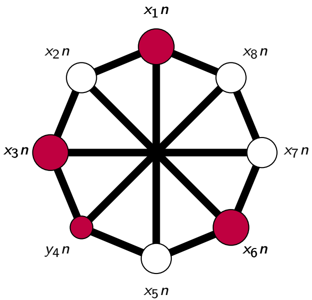
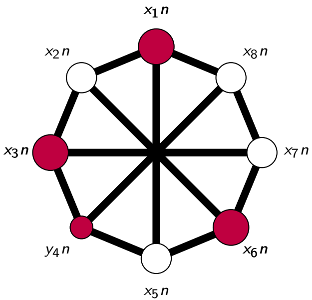


$x_4 \geq 1/2 - \alpha$. Consider $y_4 = 1/2 - \alpha$ and $x_1 + x_3 + x_6$.
Total of $n/2$ vertices.

Proof for blow-ups of F_3



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Case 1: $\alpha \leq 2/5$

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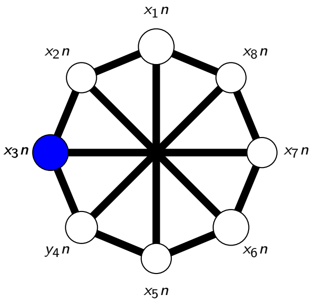

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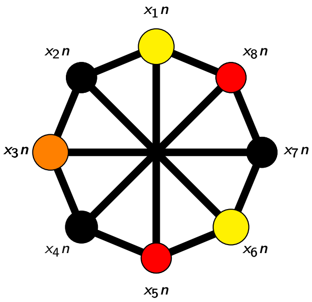
Proof for blow-ups of F_3



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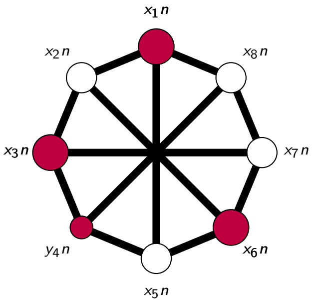

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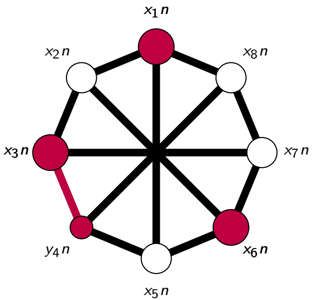
$3\alpha \geq 1 + x_3$.

Proof for blow-ups of F_3



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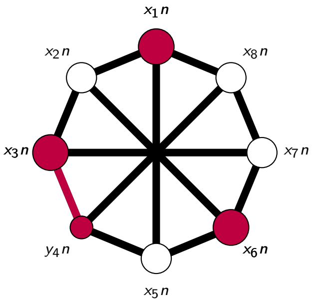

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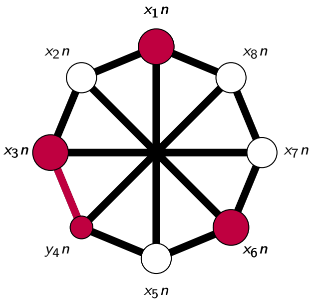

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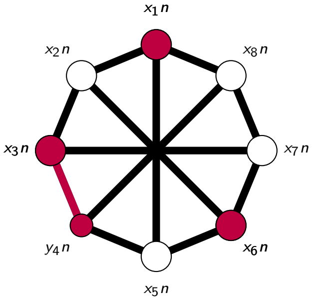
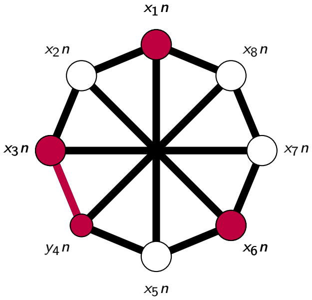
$\#\text{edges} = x_3 y_4 n^2 \leq (3\alpha - 1)(1/2 - \alpha)n^2 \leq n^2/50$.

Proof for blow-ups of F_3



$x_4 \geq 1/2 - \alpha$. Consider $y_4 = 1/2 - \alpha$ and $x_1 + x_3 + x_6$.
Case 2: $\alpha > 2/5$

Proof for blow-ups of F_3



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If $x_3 \leq \alpha/2$, then $x_3 y_4 \leq (\alpha/2)(1/2 - \alpha) \leq 1/50$.

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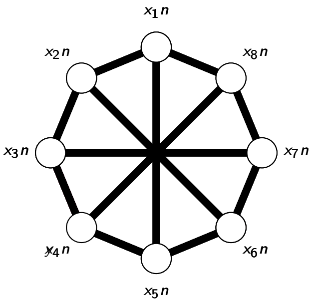

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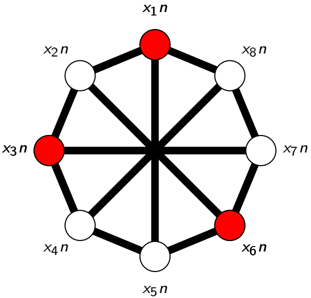

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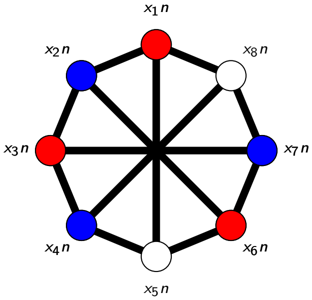

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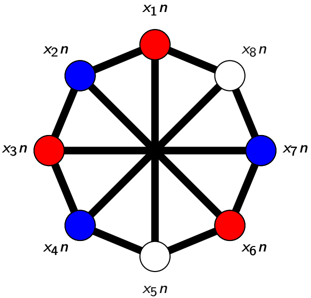
If $x_3 \leq \alpha/2$, then $x_3 y_4 \leq (\alpha/2)(1/2 - \alpha) \leq 1/50$. Thus, $x_3 > \alpha/2$.
Since $x_1 + x_3 + x_6 = \alpha$, we have $x_1 + x_6 < \alpha/2$. Note that
$x_2 + x_4 + x_7 < 1/2$.

Proof for blow-ups of F_3



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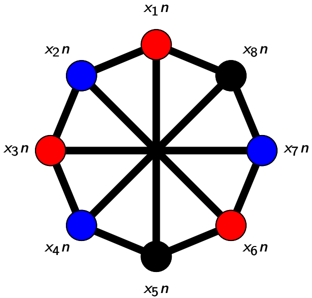
If $x_3 \leq \alpha/2$, then $x_3 y_4 \leq (\alpha/2)(1/2 - \alpha) \leq 1/50$. Thus, $x_3 > \alpha/2$.

Since $x_1 + x_3 + x_6 = \alpha$, we have $x_1 + x_6 < \alpha/2$. Note that

$x_2 + x_4 + x_7 < 1/2$.

$x_5 + x_8 + {\color{red}1/2} + {\color{blue}\alpha} \geq x_5 + x_8 + ({\color{red}x_1 + x_3 + x_6}) + ({\color{blue}x_2 + x_4 + x_7}) = 1$

Proof for blow-ups of F_3



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Since $x_1 + x_3 + x_6 = \alpha$, we have $x_1 + x_6 < \alpha/2$. Note that

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$x_5 + x_8 + 1/2 + \alpha \geq x_5 + x_8 + (x_1 + x_3 + x_6) + (x_2 + x_4 + x_7) = 1$

Therefore, $x_5 + x_8 \geq 1/2 - \alpha$.

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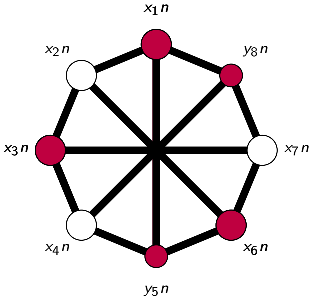

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$x_2 + x_4 + x_7 < 1/2$.

$$x_5 + x_8 + 1/2 + \alpha \geq x_5 + x_8 + (x_1 + x_3 + x_6) + (x_2 + x_4 + x_7) = 1$$

Therefore, $x_5 + x_8 \geq 1/2 - \alpha$.

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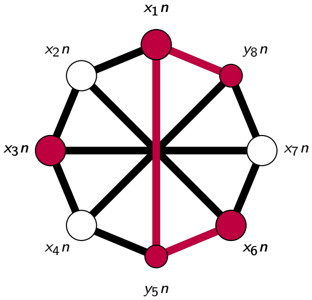

$x_5 + x_8 \geq 1/2 - \alpha$. Consider $y_5 + y_8 = 1/2 - \alpha$ and $x_1 + x_3 + x_6$.

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If $x_3 \leq \alpha/2$, then $x_3 y_4 \leq (\alpha/2)(1/2 - \alpha) \leq 1/50$. Thus, $x_3 > \alpha/2$.

Since $x_1 + x_3 + x_6 = \alpha$, we have $x_1 + x_6 < \alpha/2$. Note that

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$x_5 + x_8 + 1/2 + \alpha \geq x_5 + x_8 + (x_1 + x_3 + x_6) + (x_2 + x_4 + x_7) = 1$

Therefore, $x_5 + x_8 \geq 1/2 - \alpha$.

Open problems / Next steps

- Prove that if $\delta(G_n) > n/3$ and $K_3 \not\subseteq G_n$, then G_n is not $(1/2, 1/50)$-dense.

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- Prove that if $\delta(G_n) > n/3$ and $K_3 \not\subseteq G_n$, then G_n is not $(1/2, 1/50)$-dense.
- Extend the result for (α, β)-dense graphs with general α and β.

Thanks!

Thanks for your attention!