Leave Pima Indians alone

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Outline

1. Introduction
2. Fast approximations
3. Sampling-based methods
4. Numerical study
5. Variable selection
6. Conclusions

Binary regression models

Models wih data $y_i \in \{-1, 1\}$, predictors $\mathbf{x}_i \in \mathbb{R}^p$, and likelihood

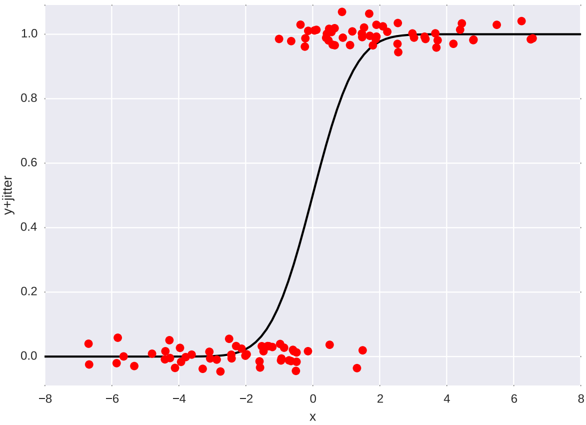
$$p(\mathcal{D}|\boldsymbol{\beta}) = \prod_{i=1}^{n_{\mathcal{D}}} F(y_i \boldsymbol{\beta}^T \mathbf{x}_i)$$

where $F : \mathbb{R} \to [0, 1]$ is a CDF.

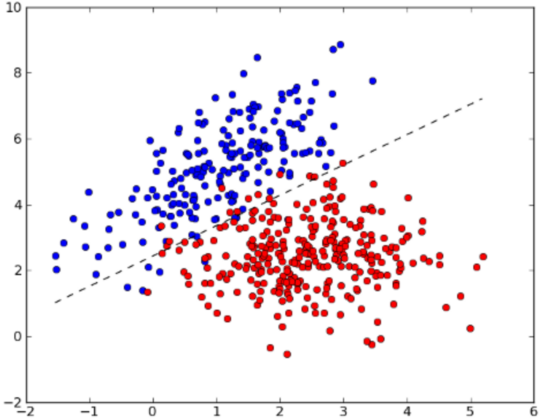
Common examples:

- $F = \Phi$ (probit),
- $F = L$ (logit), where $L(z) = 1/(1 + e^{-z})$.

When $p = 1$



Connection with classification



Properties

- Unless there is **complete separation** in the data, the log-likelihood is concave: MLE is uniquely defined.
- One nice way to deal with complete seperation is to add a proper prior, e.g. Gaussian or Cauchy. (Under Gaussian prior, log-post is concave.)
- Good practice is to standardise the predictors before eliciting the prior (Gelman et al, 2008).

Binary regression in Bayesian Computation papers

- a long chain of papers on Gibbs sampling for different variants of binary regression models (Albert & Chib, 1993; Holmes & Held, 2006; Fruwirth-Schnatter (2009); Gramacy and Polson, 2012; Polson et al, 2013)
- nearly any paper introducing any new **generic** way to compute a posterior includes a binary regression example:
  - SMC: C (2002), Del Moral et al (2006)
  - HMC and variants: Neal (2010), Shahbaba & Neal (2011), Girolami & Calderhead (2011)
  - NUTS: Hoffman and Gelman (2013)
  - nested sampling: C & Robert (2007)

Questions

1. Does it make sense to promote binary regression as a **benchmark** for Bayesian computation? (see similar practice in optimisation)
2. In practice, which method one should use???

Plan

1. review of fast approximation schemes:
    - Laplace (and variants)
    - EP
    - Variational Bayes? (see Consonni & Marin, 2007)
2. review of sampling-based approaches:
    - importance sampling
    - MCMC (Gibbs, RWHM)
    - HMC (and variants)
    - SMC
3. Discussion and comparison

Considered scenarios

- Model: probit and logit.
- prior: Gaussian and Cauchy (predictors are standardised).

Laplace

Based on a second order Taylor expansion of the log posterior:

$$\log p(\beta|\mathcal{D}) \approx \log p(\boldsymbol{\beta}_{\mathrm{MAP}}|\mathcal{D}) - \frac{1}{2}\left(\boldsymbol{\beta} - \boldsymbol{\beta}_{\mathrm{MAP}}\right)^T \mathbf{Q}\left(\boldsymbol{\beta} - \boldsymbol{\beta}_{\mathrm{MAP}}\right)$$

where $\mathbf{Q}$ is minus the Hessian of $\log p(\beta|\mathcal{D})$ at $\boldsymbol{\beta} = \boldsymbol{\beta}_{\mathrm{MAP}}$.

Exponentiate to get a Gaussian approximation of the posterior. In practice, use Newton-Raphson to obtain $\boldsymbol{\beta}_{\mathrm{MAP}}$ and $\mathbf{Q}$.

Very fast. May not converge if p is very large.

Impoved Laplace

For each marginal:

$$p(\beta_j|\mathcal{D}) \propto \frac{p(\boldsymbol{\beta})p(\mathcal{D}|\boldsymbol{\beta})}{p(\boldsymbol{\beta}_{-j}|\beta_j, \mathcal{D})}$$

Choose a fine grid of β_j values; for each β_j value, compute a Laplace approximation of $p(\boldsymbol{\beta}_{-j}|\beta_j, \mathcal{D})$.

Note: more expensive, connection with INLA.

EM-Laplace

For a Student prior, Gelman et al (2008) derive an approximate EM scheme based on

$$\beta_j|\sigma_j^2 \sim \mathrm{N}_1(0,\sigma_j^2), \quad \sigma_j^2 \sim \mathrm{Inv}-\mathrm{Gamma}(\nu/2, s_j\nu/2)$$

However, we will observe in our simulations that Laplace still works well for such a prior.

Expectation Propagation

From the following decomposition:

$$p(\boldsymbol{\beta}|\mathcal{D}) = \frac{1}{p(\mathcal{D})}\prod_{i=0}^{n_{\mathcal{D}}} l_i(\boldsymbol{\beta}), \quad l_i(\boldsymbol{\beta}) = F(y_i\boldsymbol{\beta}^T\mathbf{x}_i) \text{ for } i \geq 1,$$

and l_0 is prior, EP computes iteratively a parametric approximation of the posterior with the same structure:

$$q_{\mathrm{EP}}(\boldsymbol{\beta}) = \prod_{i=0}^{n_{\mathcal{D}}} \frac{1}{Z_i} q_i(\boldsymbol{\beta}).$$

Taking q_i to be an unnormalised Gaussian density

$$q_i(\boldsymbol{\beta}) = \exp\left\{-\frac{1}{2}\boldsymbol{\beta}^T\mathbf{Q}_i\boldsymbol{\beta} + \boldsymbol{\beta}^T\mathbf{r}_i\right\},$$

q_{EP} is a Gaussian with parameters $\mathbf{Q} = \sum_{i=0}^{n}\mathbf{Q}_i$, $\mathbf{r} = \sum_{i=0}^{n}\mathbf{r}_i$

EP site update

Update each 'site' in turn: update q_i, while keeping q_j, $j \neq i$ fixed, by minimising the Kullback-Leibler divergence between

$$h(\boldsymbol{\beta}) \propto l_i(\boldsymbol{\beta}) \prod_{j \neq i} q_j(\boldsymbol{\beta})$$

and $q(\boldsymbol{\beta}) \propto \prod_j q_j$.

Thanks to nice properties of exponential families, this boils to match the moments of h and q.

In binary regression, these site updates lead to explicit expressions (probit) or one-dimensional integrals that are easy to approximate accurately (logit).

General remarks

- Since the approximation methods covered in the previous section are faster by orders of magnitude than sampling-based methods, we will assume that a Gaussian approximation $q(\boldsymbol{\beta})$ (from Laplace or EP) has been computed in a preliminary step.
- Complexity: Laplace is $O(n_{\mathcal{D}} + p^3)$, EP is $O(n_{\mathcal{D}} p^3)$.

Importance sampling

Proposal q set to some Gaussian approx of the posterior. Then to approximate $p(\mathcal{D})$, generate $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_N \sim q$, compute

$$Z_N = \frac{1}{N} \sum_{n=1}^{N} w(\boldsymbol{\beta}_n), \quad w(\boldsymbol{\beta}) := \frac{p(\boldsymbol{\beta}) p(\mathcal{D}|\boldsymbol{\beta})}{q(\boldsymbol{\beta})}$$

and to approximate the posterior expectation of φ, compute

$$\varphi_N = \frac{\sum_{n=1}^{N} w(\boldsymbol{\beta}_n) \varphi(\boldsymbol{\beta}_n)}{\sum_{n=1}^{N} w(\boldsymbol{\beta}_n)}.$$

IS pros and cons

Pros:

- simple, generic
- embarassingly parallel
- approximates the marginal likelihood at no extra cost
- IID sampling: MC error is easy to assess
- can plug in QMC points

Cons:

- ESS may collapse when p is large.

MCMC general remarks

The following points

- choice of starting point
- MCMC convergence assessment

are not big issues for binary regression models.

More important issues for us are:

- chain autocorrelations
- difficulty to parallelise

Gibbs

Well-known, based on data augmentation:

$$
\begin{aligned}
z_i &= \boldsymbol{\beta}^T \mathbf{x}_i + \epsilon_i \\
y_i &= \operatorname{sgn}(z_i)
\end{aligned}
$$

then sample iteratively (probit/Gaussian case):

1. $\boldsymbol{\beta}|\mathbf{z}$ (regression posterior, tractable)
2. $\mathbf{z}|\boldsymbol{\beta}, \mathbf{y}$ (product of truncated Gaussians)

Gibbs is particularly **not generic**: any change in the prior of F requires deriving a new algorithm. This can also change the complexity (e.g. from $\mathcal{O}(p^2)$ to $\mathcal{O}(p^3)$ when using a Student prior).

Random walk Metropolis-Hastings

One iteration of RWMH

```
Input: $\boldsymbol{\beta}$
Output: $\boldsymbol{\beta}'$
1. Sample $\boldsymbol{\beta}^\star \sim \mathrm{N}_p(\boldsymbol{\beta}, \boldsymbol{\Sigma})$
2. With probability $1 \wedge r$,
   $r = \frac{p(\boldsymbol{\beta}^\star)p(\mathcal{D}|\boldsymbol{\beta}^\star)}{p(\boldsymbol{\beta})p(\mathcal{D}|\boldsymbol{\beta})}$,
set $\boldsymbol{\beta}' = \boldsymbol{\beta}^\star$; otherwise set $\boldsymbol{\beta}' = \boldsymbol{\beta}$
```

In practice, choose $\boldsymbol{\Sigma}$ as some fraction of $\boldsymbol{\Sigma}_q$.

HMC

Consider $(\boldsymbol{\beta}, \boldsymbol{\alpha})$, $\boldsymbol{\beta} \sim p(\boldsymbol{\beta}|\mathcal{D})$, $\boldsymbol{\alpha} \sim N_p(0, M^{-1})$, with joint un-normalised density $\exp\{-H(\boldsymbol{\beta}, \boldsymbol{\alpha})\}$,

$$H(\boldsymbol{\beta}, \boldsymbol{\alpha}) = E(\boldsymbol{\beta}) + \frac{1}{2}\boldsymbol{\alpha}^T \mathbf{M}\boldsymbol{\alpha}, \quad E(\boldsymbol{\beta}) = -\log\{p(\boldsymbol{\beta})p(\mathcal{D}|\boldsymbol{\beta})\}.$$

The physical interpretation of HMC is that of a particle at position $\boldsymbol{\beta}$, with velocity $\boldsymbol{\alpha}$, potential energy $E(\boldsymbol{\beta})$, kinetic energy $\frac{1}{2}\boldsymbol{\alpha}^T M\boldsymbol{\alpha}$, and thus total energy given by $H(\boldsymbol{\beta}, \boldsymbol{\alpha})$. The particle is expected to follow a trajectory such that $H(\boldsymbol{\beta}, \boldsymbol{\alpha})$ remains constant over time.

HMC iteration

One iteration of HMC

Input: $\boldsymbol{\beta}$

Output: $\boldsymbol{\beta}'$

1. Sample momentum $\boldsymbol{\alpha} \sim \mathrm{N}_p(0, \mathbf{M})$.

2. Perform L leap-frog steps, starting from $(\boldsymbol{\beta}, \boldsymbol{\alpha})$; call $(\boldsymbol{\beta}^\star, \boldsymbol{\alpha}^\star)$ the final position.

3. With probability $1 \wedge r$, $r = \exp\left\{H(\boldsymbol{\beta}, \boldsymbol{\alpha}) - H(\boldsymbol{\beta}^\star, \boldsymbol{\alpha}^\star)\right\}$ set $\boldsymbol{\beta}^{'} = \boldsymbol{\beta}^\star$; otherwise set $\boldsymbol{\beta}^{'} = \boldsymbol{\beta}$.

Leapfrog step

Leapfrog step

```
Input: $(\boldsymbol{\beta}, \boldsymbol{\alpha})$
Output: $(\boldsymbol{\beta}_1, \boldsymbol{\alpha}_1)$
1. $\boldsymbol{\alpha}_{1/2} \leftarrow \boldsymbol{\alpha} - \frac{\epsilon}{2}\nabla_{\boldsymbol{\beta}} E(\boldsymbol{\beta})$
2. $\boldsymbol{\beta}_1 \leftarrow \boldsymbol{\beta} + \epsilon\boldsymbol{\alpha}_{1/2}$
3. $\boldsymbol{\alpha}_1 \leftarrow \boldsymbol{\alpha}_{1/2} - \frac{\epsilon}{2}\nabla_{\boldsymbol{\beta}} E(\boldsymbol{\beta}_1)$
```

HMC variants

- Riemanian HMC (Girolami and Calderhead, 2011): simply too expensive
- NUTS (No U-Turn Sampler, Hoffman & Gelman, 2013): HMC with on-the-fly calibration of L and ϵ. Included in our comparisons.

SMC

We consider tempering SMC, i.e. SMC for sequence

$$\pi_t(\boldsymbol{\beta}) \propto q(\boldsymbol{\beta})^{1-\delta_t} \left\{p(\boldsymbol{\beta})p(\mathcal{D}|\boldsymbol{\beta})\right\}^{\delta_t}$$

Principle: sequence of importance sampling steps, from π_{t-1} to π_t. When weight degeneracy becomes too high, resample, and move particles through MCMC (e.g. random walk Metropolis).

The algorithm can choose the δ_j on the fly (Jasra et al, 2011).

SMC algorithm

Operations involving index n must be performed for all $n \in 1:N$.

0. Sample $\boldsymbol{\beta}_n \sim q(\boldsymbol{\beta})$ and set $\underline{\delta} \leftarrow 0$.
1. Let, for $\delta \in [\underline{\delta}, 1]$,

$$\mathrm{EF}(\delta) = \frac{1}{N} \frac{\left\{\sum_{n=1}^{N} w_\gamma(\boldsymbol{\beta}_n)\right\}^2}{\left\{\sum_{n=1}^{N} w_\gamma(\boldsymbol{\beta}_n)^2\right\}}, \quad u_\delta(\boldsymbol{\beta}) = \left\{\frac{p(\boldsymbol{\beta})p(\mathcal{D}|\boldsymbol{\beta})}{q(\boldsymbol{\beta})}\right\}^\delta.$$

If $\mathrm{EF}(1) \geq \tau$, stop and return $(\boldsymbol{\beta}_n, w_n)_{n=1:N}$ with $w_n = u_1(\boldsymbol{\beta}_n)$; otherwise, use the bisection method to solve numerically in δ the equation $\mathrm{EF}(\gamma) = \tau$.

2. Resample according to normalised weights $W_n = w_n / \sum_{m=1}^{N} w_m$; with $w_n = u_\delta(\boldsymbol{\beta}_n)$.
3. Update the $\boldsymbol{\beta}_n$'s through m MCMC steps that leaves invariant

Remarks on SMC

- Completely automatic: we can use the current set of particles to adjust the random walk proposal, the number of MCMC steps, and so on.
- Will often collapse to a **single** IS step (when ESS from q to posterior is not too low)

First set of datasets

Dataset	$n_{\mathcal{D}}$	p
Pima (Indian diabetes)	532	8
German (credit)	999	25
Heart (Statlog)	270	14
Breast (cancer)	683	10
Liver (Indian Liver patient)	579	11
Plasma (blood screening data)	32	3
Australian (credit)	690	15
Elections	2015	52

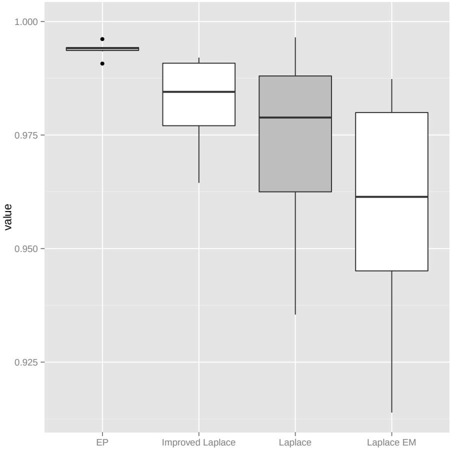
This is a superset of datasets considered in most papers.

Fast approximations

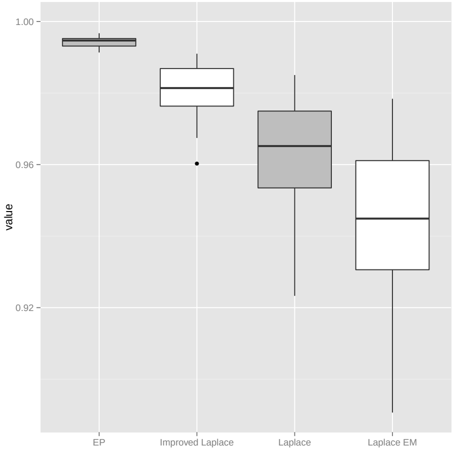
Logit/Cauchy scenario. We compare: Laplace, Improved Laplace, EM-Laplace, and EP, in term of

- marginal accuracies (one minus half the L_1 distance between approximate and true marginals)
- approximation error for marginal likelihood

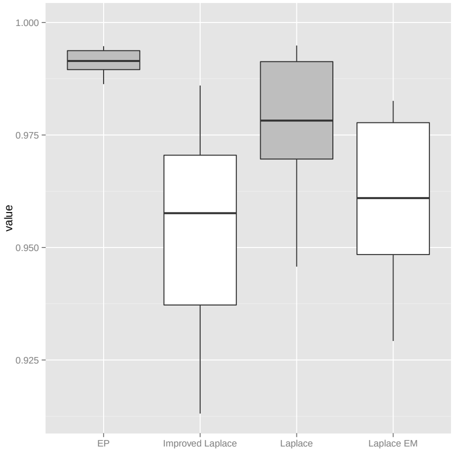
Pima


1.000
0.975
0.950
0.925
value
EP
Improved Laplace
Laplace
Laplace EM

Heart



Breast


1.000
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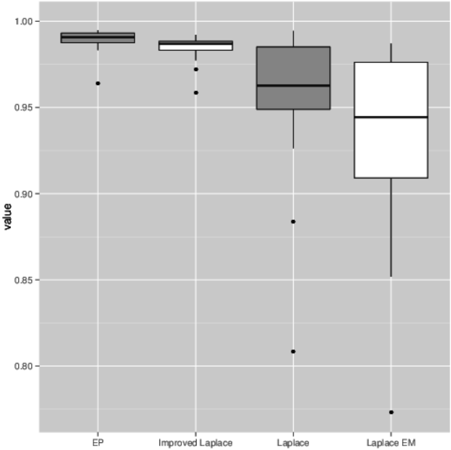
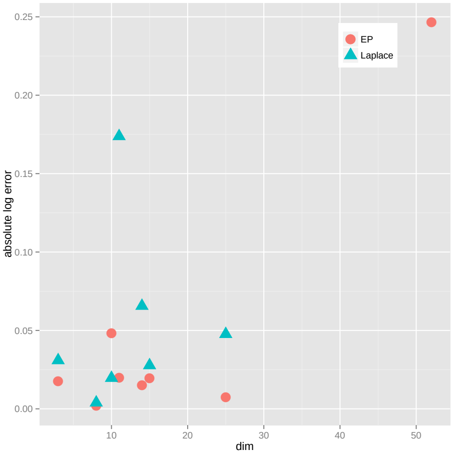
EP
Improved Laplace
Laplace
Laplace EM

German credit



Marginal likelihoods

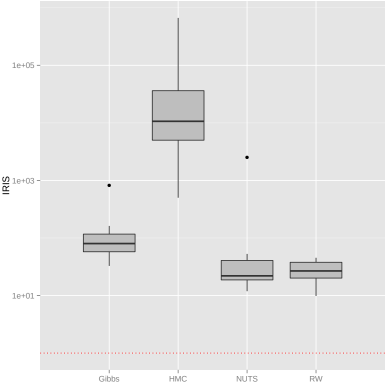


Sampling-based methods: importance sampling

	IS			IS-QMC	
Dataset	EF $= \mathrm{ESS}/N$	CPU time	MT speed-up	MSE x (expect)	MSE x (evid)
Pima	99.5%	37.54 s	4.39	28.9	42.7
German	97.9%	79.65 s	4.51	13.2	8.2
Breast	82.9%	50.91 s	4.45	2.6	6.2
Heart	95.2%	22.34 s	4.53	8.8	9.3
Liver	74.2 %	35.93 s	4.76	7.6	11.3
Plasma	90.0%	2.32 s	4.28	2.2	4.4
Australian	95.6%	53.32 s	4.57	12	20.3
Elections	21.39%	139.48 s	3.87	617.9	3.53

(Probit/Gaussian scenario, to make like easier for Gibbs)

comparison with MCMC



IRIS = Inefficiency relative to IS

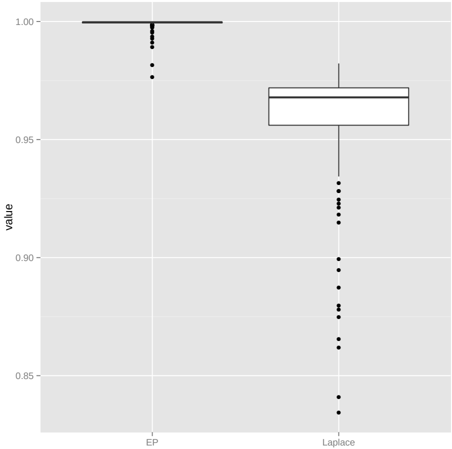
Bigger datasets

Dataset	$n_{\mathcal{D}}$	p
Musk	476	95
Sonar	208	61
DNA	400	180

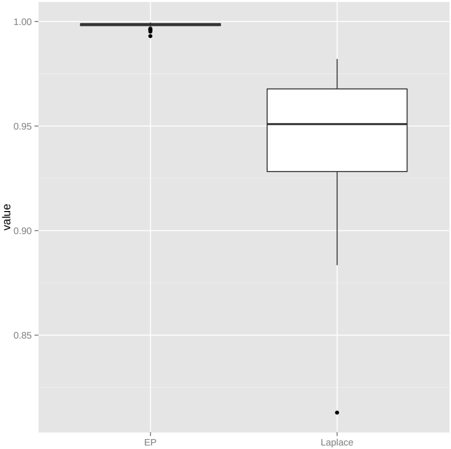
Bigger datasets, but also with higher correlations between predictors. We will look at the probit/Gaussian case.

IS no longer an option.

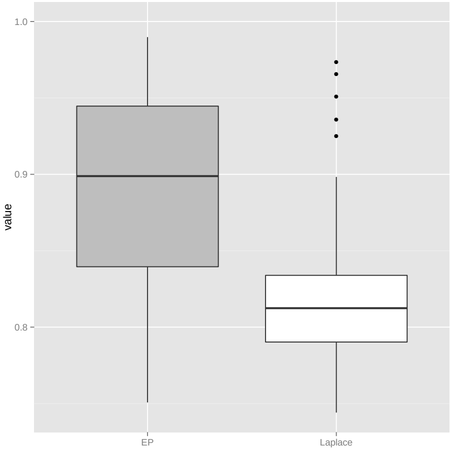
Approximations: Musk



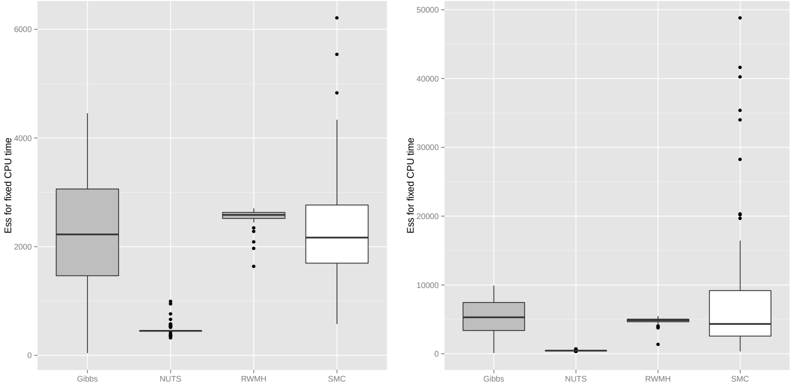
Approximations: Sonar



Approximations: DNA

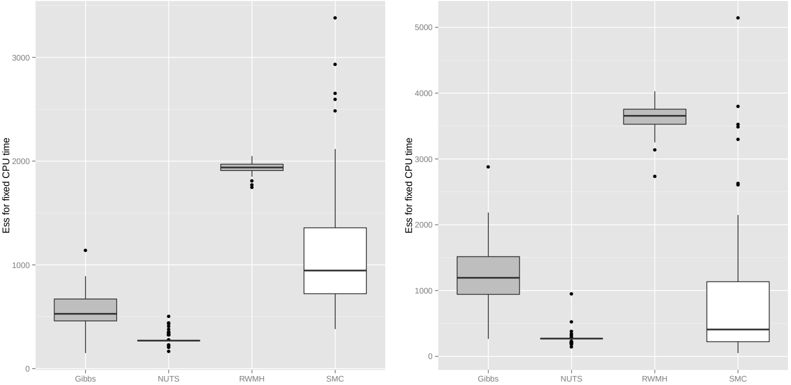


Sampling-based methods: Musk



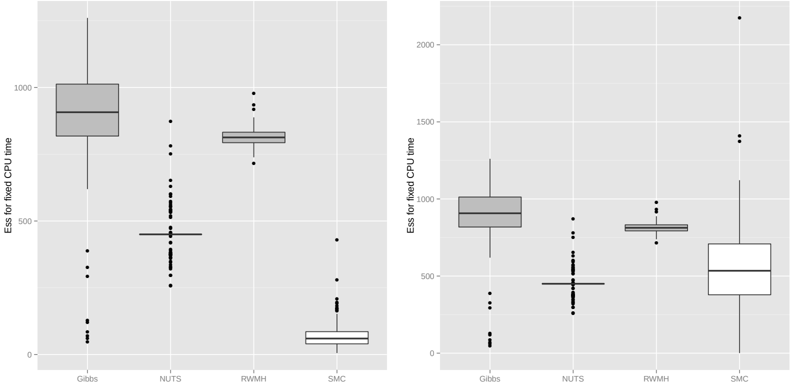
Left: posterior expectations, Right: posterior variances

Sampling-based methods: Sonar



Left: posterior expectations, Right: posterior variances

Sampling-based methods: DNA



Left: posterior expectations, Right: posterior variances

Variable selection

Add for each predictor β_j an indicator $\gamma_j \in \{0, 1\}$; prior for $\boldsymbol{\gamma}$ is Uniform over $\{0, 1\}^p$.

The posterior mixes discrete and continuous components; $p(\boldsymbol{\gamma}|\mathcal{D})$ is severely multimodal.

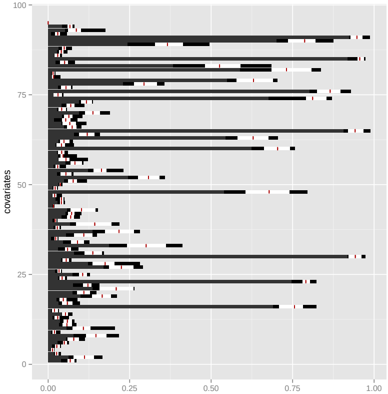
VS: proposed approach

To compute $p(\mathcal{D}|\boldsymbol{\gamma}) = \int p(\mathcal{D}|\boldsymbol{\gamma}, \boldsymbol{\beta}) p(\boldsymbol{\beta}|\gamma)\, d\boldsymbol{\beta}$, use:

1. either Laplace
2. or IS based on Laplace

To simulate from $p(\gamma|\mathcal{D})$, adapt the tempering SMC sampler of Schafer and Chopin (2013), for sampling binary vectors.

Results


100
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covariates
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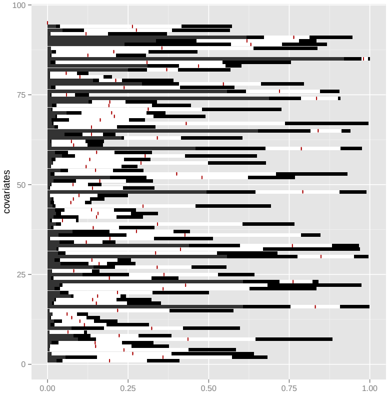
0.50
0.75
1.00


100
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covariates
0.00
0.25
0.50
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1.00

Recommendations to end users (who wish to fit a binary regression model)

- EP is fast and accurate even in difficult cases.
- to improve on EP, one might run SMC; often this will collapse to IS and outperforms everything else significantly.
- That said, for large p, RWHM performs surprising well.
- HMC algorithms seem very difficult to calibrate.

Benchmarks for specialised algorithms

For specialised algorithms (Gibbs), benchmark=dataset.

It is not very clear that the Gibbs samplers developped for binary regression are very useful: corresponding papers tend to showcase these algorithms on datasets with $p < 50$, for which more generic methods fare much better.

Benchmarks for generic algorithms

For generic algorithms (e.g. RWHM), benchmark=posterior.

A binary regression posterior of dimension < 50 is very close to a Gaussian; i.e. it does not represent a very challenging benchmark. However, it is an useful **sanity check**.

More challenging benchmarks: $p \geq 100$, hierarchical regression, spike and slab prior, . . .

More general remarks

Beware ML fast approximation schemes; they are fast and getting better and better. . .

Always compare new methods to well calibrated simple algorithms, like IS and RWHM.

Final word

Comments most welcome!