# MODEL POINTS AND TVAR IN LIFE INSURANCE

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# Problem of interest

- **Motivation:**
  - Life insurance models are becoming more and more sophisticated under Solvency 2 regulation.
    - Cash-flow projection: based on a **policy-by-policy approach**.
  - However, the use of Monte-Carlo simulations based on a policy-by-policy approach often leads to **large running times**.
  - A **way to address this problem: to rely on grouping methods**.
    - Under certain conditions, the regulator permits the projection of future cash-flows based on suitable **model points**.
    - Basic idea:
      **(1)** To aggregate policies into homogeneous groups;
      **(2)** To replace the group of contracts with a representative insurance policy in order to speed the simulation process.
  - **Goal here**: to homogenize a group of policies by controlling the impact on Tail-Value-at-Risk (TVaR).

# Problem of interest

- **Context:**
  We consider independent risks $X_1, \ldots, X_n$ causing an aggregate loss amount

$$S = X_1 + X_2 + \ldots + X_n.$$

We denote by $F_i$ the distribution function of $X_i$ $(i = 1, \ldots, n)$ and we assume that

$$F_1(x) \geq F_2(x) \geq \cdots \geq F_n(x) \quad \text{for all } x.$$

# Problem of interest

- **Goal:**
  To build two sets of iid random variables $X_1^-, \ldots, X_n^-$ and $X_1^+, \ldots, X_n^+$ such that

  $$TVaR[X_1^- + \ldots + X_n^-] \leq TVaR[S] \leq TVaR[X_1^+ + \ldots + X_n^+].$$

# Upper bound: main result

- **Proposition:**
  *Let us define iid random variables $X_i^+$ ($i = 1, \ldots, n$) with common distribution function*

$$F^+(x) = \frac{1}{n}\sum_{k=1}^{n} F_k(x), \;\; x \in \mathbb{R}.$$

  *Then,*

$$TVaR[S] \leq TVaR[X_1^+ + \ldots + X_n^+].$$

# Upper bound: main result

- **Conclusion:**
  Let $X_1, \ldots, X_n$ be the losses of a group of heterogeneous contracts.

  Replacing the losses by homogeneous $X_1^+, \ldots, X_n^+$ with common distribution function $F^+$ turns out to be a conservative strategy.

  - Indeed, this inflates the Tail-VaR associated to the aggregate loss.

# Upper bound: life insurance applications

- Let a portfolio made of $n$ life insurance contracts.
- **Notations:**
  - $T_i$: the remaining lifetime of policyholder $i$ $(i = 1, \ldots, n)$.
  - $b_i\, g(T_i)$: the random loss associated to policy $i$.

# Upper bound: life insurance applications

- **Examples of specific life insurance contracts:**

*Ex* 1 Let $g(T_i) = \sum_{k=1}^{\lfloor T_i \rfloor} v(k)$, where $v(t)$ is the present value of a unit payment made at time $t$.

$\Rightarrow b_i\, g(T_i) =$ the present value of a **life annuity contract** paying $b_i$ at the end of each year, as long as policyholder $i$ survives.

*Ex* 2 Let $g(T_i) = v(T_i)$.

$\Rightarrow b_i\, g(T_i) =$ the present value of a **whole life insurance contract** with payment $b_i$ at time of death of policyholder $i$.

*Ex* 3 Let $g(T_i) = v(T_i)\mathbb{I}[T_i < m] + v(m)\mathbb{I}[T_i \geq m]$.

$\Rightarrow b_i\, g(T_i) =$ the present value of an **endowment insurance** with benefit $b_i$.

# Upper bound: life insurance applications

- **Homogeneous lifetimes:**
  - $T_1, \ldots, T_n$ are assumed to be iid.
  - **Proposition:**
    Let $B_1, \ldots, B_n$ be random variables independent of $T_1, \ldots, T_n$ with common distribution

$$B_i = \begin{cases} b_1 \text{ with probability } \frac{1}{n} \\ b_2 \text{ with probability } \frac{1}{n} \\ \vdots \\ b_n \text{ with probability } \frac{1}{n}. \end{cases}$$

Then, we have

$$TVaR\left[\sum_{i=1}^{n} b_i\, g(T_i)\right] \leq TVaR\left[\sum_{i=1}^{n} B_i\, g(T_i)\right].$$

# Upper bound: life insurance applications

- **Homogeneous lifetimes:**
  - **Conclusion:**
    When the lifetimes are homogeneous but the amounts of benefits vary between contracts, it is conservative to replace the deterministic benefits $b_i$ with a stochastic one $B_i$ randomly drawn from $\{b_1, \dots, b_n\}$.

# Upper bound: life insurance applications

- **Heterogeneous lifetimes:**
  - $T_1, \ldots, T_n$ are assumed to be independent such that

    $$\Pr[T_1 > t] \leq \Pr[T_2 > t] \leq \cdots \leq \Pr[T_n > t].$$

    *Example:* when $T_1, \ldots, T_n$ all obey the same life table and correspond to individuals aged $x_1, \ldots, x_n$ with $x_1 > \ldots > x_n$.
  - **Proposition:**
    If $g$ is a monotonic function, then

    $$TVaR\left[b\sum_{i=1}^{n} g(T_i)\right] \leq TVaR\left[b\sum_{i=1}^{n} g(T_i^+)\right]$$

    where $T_1^+, \ldots, T_n^+$ are independent with common survival function

    $$\Pr[T_1^+ > t] = \frac{1}{n}\sum_{i=1}^{n}\Pr[T_i > t].$$

# Upper bound: life insurance applications

- **Heterogeneous lifetimes:**
  - **Conclusion:**
    Homogenizing life tables appears to be a safe strategy when the benefits $b_i$ are the same for all contracts (i.e. when $b_i = b$ for all $i = 1, \ldots, n$).

# Lower bound: main result

- **Proposition:**
  *Let $U_1, \ldots, U_n$ be iid random variables ($U_i \sim Uni(0,1)$) and let us define iid random variables $X_i^-$ as*

$$X_i^- = \frac{1}{n} \sum_{j=1}^{n} F_j^{-1}(U_i), \quad i = 1, 2, \ldots, n.$$

  *Then,*

$$TVaR\left[X_1^- + \ldots + X_n^-\right] \leq TVaR\left[S\right].$$

# Lower bound: main result

- **Conclusion:**
  Averaging $F_1, \ldots, F_n$ to produce $F^+$ provides the actuary with an upper bound on the aggregate loss whereas the lower bound is obtained by averaging VaRs $F_1^{-1}, \ldots, F_n^{-1}$.

# Lower bound: life insurance applications

- **Homogeneous lifetimes:**
  - $T_1, \ldots, T_n$ are assumed to be iid.
  - **Proposition:**
    We have

    $$TVaR\left[\overline{b}\sum_{i=1}^{n} g(T_i)\right] \leq TVaR\left[\sum_{i=1}^{n} b_i g(T_i)\right]$$

    where $\overline{b} = \frac{1}{n}\sum_{i=1}^{n} b_i$.
  - **Conclusion:**
    When the lifetimes are homogeneous but the amounts of benefits vary between contracts, replacing the benefits $b_i$ with their average value $\overline{b}$ decreases the Tail-VaR of the aggregate loss.

# Lower bound: life insurance applications

- **Heterogeneous lifetimes:**
  - $T_1, \ldots, T_n$ are assumed to be heterogeneous with distribution functions $F_{T_1}, \ldots, F_{T_n}$.
  - **Proposition:** If we assume that $g$ is monotonic, then

$$TVaR\left[b\sum_{i=1}^{n}\left\{\frac{1}{n}\sum_{j=1}^{n} g\left(F_{T_j}^{-1}(U_i)\right)\right\}\right] \leq TVaR\left[b\sum_{i=1}^{n} g(T_i)\right].$$

  - **Conclusion:**
    The lower bound is obtained by replacing each policy by a portfolio of $n$ policies differing from the original one by the fact that
    - the remaining lifetimes are comonotonic,
    - the amounts of benefits are $n$ times smaller.

# Short bibliography


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