Spatial dependence issues for extremes

Gwladys Toulemonde[1], **Jean-Noel Bacro**[1], **Carlo Gaetan**[2]
[1] *IMAG, University of Montpellier, France.*
[2] *DAIS, Università Ca' Foscari, Italy.*

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Outline

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Multivariate max-stable distributions : characterization

Theorem

Let $(X_i, Y_i) \sim F$ be independent random vectors with w.l.g. unit Fréchet margins $K(x) = \exp(-1/x)$, $x > 0$. A limit distribution for $(M_{x,n}, M_{y,n}) = (\max_{i=1,\dots,n} X_i, \max_{i=1,\dots,n} Y_i)$ is said to exist $(F \in D(G))$ if

$$\lim_{n\to\infty} \mathbb{P}\left(M_{x,n} \leq nx, M_{y,n} \leq ny\right) = G(x,y)$$

with G a non degenerate distribution.

- Limit distributions G are **max-stable :** $G^k(kx_1, kx_2) = G(x,y)$
- If

$$G(x,y) = K(x)\,K(y) = \exp\left(-\frac{1}{x}\right)\exp\left(-\frac{1}{y}\right)$$

$\hookrightarrow$ *ultimately, normalized maxima of X and Y are independent.*

(X, Y) are said to be **Asymptotically Independent (AI)**

⚠ (X, Y) AI $\not\Rightarrow$ (X, Y) independent, *only the converse is true . . .*

⚠ (X, Y) may exhibit non-negligible dependence at all observable levels even if AI ! Example : the Gaussian case.

Dependence measures χ and $\overline{\chi}$

Let $(X,Y) \sim F \in D(G)$, with F_X and F_Y margins.

The χ parameter

$$\chi = \lim_{u\to 1} \mathbb{P}\left(F_Y(Y) > u | F_X(X) > u\right)$$
$$= \lim_{u\to 1} 2 - \frac{\log \mathbb{P}(F_X(X)\leq u, F_Y(Y)\leq u)}{\log \mathbb{P}(F_X(X)\leq u)}$$
$$\equiv \lim_{u\to 1} \chi(u)$$

- For max-stable distributions, $\chi(u) = \chi$

$\rightsquigarrow$ *same dependence structure* $\forall u$ *!*

- $\chi = 0 \Rightarrow X$ and Y are AI.
- $\chi > 0 \Rightarrow X$ and Y are **AD** ; moreover **the value of χ quantifies the strength of the extremal dependence.**

$\hookrightarrow$ **χ unable to provide dependence information for AI case !**

The $\overline{\chi}$ parameter

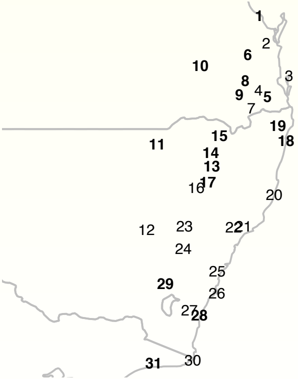
$$\overline{\chi} = \lim_{u\to 1} \frac{2\log \mathbb{P}(F_X(X)>u)}{\log \mathbb{P}(F_X(X)>u, F_Y(Y)>u)} - 1$$
$$\equiv \lim_{u\to 1} \overline{\chi}(u)$$

- $\overline{\chi} = 1 \Rightarrow X$ and Y are AD.
- $-1 \leq \overline{\chi} < 1 \Rightarrow X$ and Y are **AI** ; moreover $\overline{\chi}$ **provides a measure that increases with dependence strength.**

$\rightsquigarrow$ Example : Gaussian vectors with correlation parameter $\rho \neq 1$: $\overline{\chi} = \rho$.

Motivation : a spatial data set where daily precipitation data from an observational network covering a region S of East-Australia, are analysed for the period 1955-2003.

- 31 sites observed from East-Australia during 49 winters (April-September), (Lavery, Joung and Nicholls 1996).



Extremal behaviour ?

Spatial context : strength of the dependence related to the distance h between two points in $\mathbb{R}^2$ s and $s+h$
$\hookrightarrow$ bivariate extremal dependence tools as a function of the distance :
$\chi_h(u)$, $\chi(h)$, $\overline{\chi}_h(u)$, $\overline{\chi}(h)$, $\eta(h)$ ...

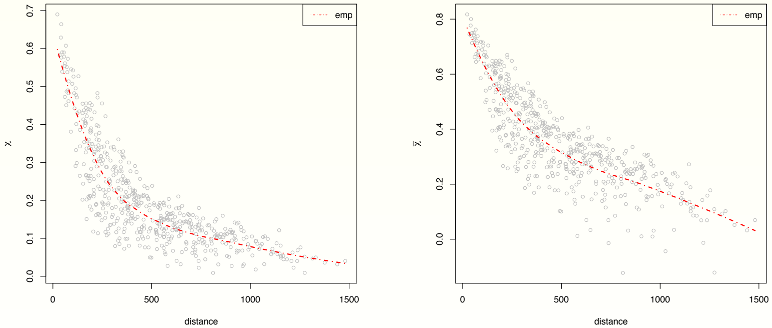


Figure 1 : Smoothed values of the empirical estimates of the functions $\widehat{\chi}(h,u)$ (left) and $\widehat{\overline{\chi}}(h,u)$ (right) with $u=0.975$.

Spatial context : strength of the dependence related to the distance h between two points in $\mathbb{R}^2$ s and $s+h$
$\hookrightarrow \chi_h \equiv \lim_{u\to 1} \chi_h(u)$ and $\overline{\chi}_h \equiv \lim_{u\to 1} \overline{\chi}_h(u)$

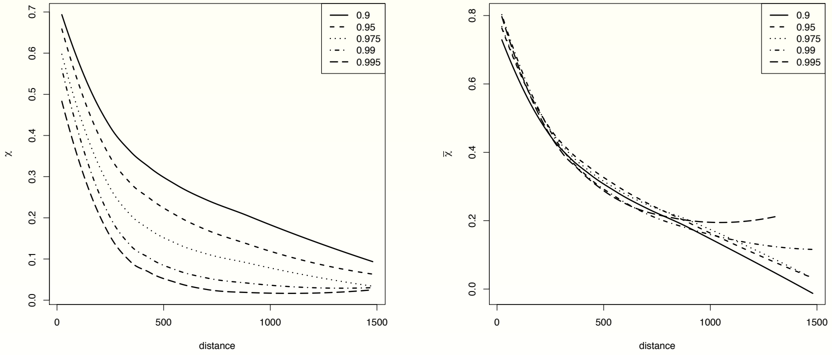


Figure 2 : Smoothed values of the empirical estimates of the functions $\widehat{\chi}(h,u)$ (left) and $\widehat{\overline{\chi}}(h,u)$ (right) at different values of the threshold u.

Our Goal

Our goal is to propose an asymptotically justified model for spatial extremes that is able to model a pairwise :

- extremal dependence for sites which are spatially close ;
- extremal independence for sites which are spatially distant ;
- asymptotic independence for sites which are at intermediate distances.

$\hookrightarrow$ *any potential sub-asymptotic pairwise extremal dependence is taken into account whatever the considered distance . . .*

Outline

Max-stable processes : the Truncated Extremal Gaussian (TEG) process

Representation of a max stable process with unit Fréchet margins

$$Z(s) = \max_{k\geq 1} \xi_k W_k(s)$$

with $\{\xi_k, k \geq 1\}$ points of a Poisson process on $(0, \infty)$ with intensity measure $\xi^{-2} d\xi$ and W_k i.i.d. copies of a positive process $\{W_k(s)\}$ such that $\mathbb{E}(W(s)) = 1$ for all $s \in \mathscr{S}$.

The TEG process (Schlather, 2002 ; Davison and Gholamrezaee, 2012) :

$$W_k(s) = c \max(0, \varepsilon_k(s)) I_{B_k}(s - U_k)$$

such that $\varepsilon_k(\cdot)$ a stationary standard Gaussian process with correlation function $\rho(\cdot)$, I_B is the indicator function of a compact random set $B \subset \mathscr{S}$, of which $(B_k)_k$ are independent replicates and $(U_k)_k$ are points of a homogeneous Poisson process of unit rate on $\mathscr{S}$, independent of the $\varepsilon_k(\cdot)$.

We can compute $\chi_Z(h) = \alpha(h) \left\{1 - 2^{-\frac{1}{2}} [1 - \rho(h)]^{\frac{1}{2}}\right\} \in [0, 1]$
where $\alpha(h) = \mathbb{E}\{|B \cap (h + B)|\} / \mathbb{E}(|B|)$ with $h = ||s_1 - s_2||$.

In the sequel, B will be a disc of fixed radius r :
$\alpha(h) \approx 1 - \frac{h}{2r}$ if $h < 2r$ (and 0 otherwise).

A.I. processes (unit Fréchet margins) :

de Oliveira, 1962

A multivariate vector is AI iff all its pairs of components are AI.

As a consequence, if all the bivariate distributions of a stochastic process are AI, the stochastic process is said to be AI.

Bivariate model (Ledford and Tawn, 1996, 1997)

$\mathbb{P}(X > x, Y > x) = \overline{F}(x,x) \sim \mathscr{L}(x) x^{-1/\eta}$ when $x \to \infty$ where $\mathscr{L}$ is a slowly varying function, i.e. satisfying $\mathscr{L}(tx)/\mathscr{L}(x) \to 1$ when $x \to \infty$ for all given $t > 0$.
The η parameter, so-called tail dependence coefficient, determines the decay rate of the joint survival function $\overline{F}(x,x)$ for high values of x.
Under Ledford-Tawn model $\overline{\chi} = 2\eta - 1$. If $0 < \eta < 1$, the variables are AI.

- Example 1 : $Y(s) = -1/\log(\Phi(Y'(s))$ with $Y'(s)$ a stationnary Gaussian process with zero mean, unit variance and correlation function $\rho(h)$.

- Example 2 : $Y(s) = -1/\log(1 - e^{-1/Z(s)})$ with $Z(s)$ a max stable process.

Outline

Mixture of a max-stable process and an A.I. process

From an original construction of Wadsworth and Tawn (2012) :

Bacro, Gaetan and Toulemonde, JSPI, 2016

$$X(s) = \max\big(aZ(s), (1-a)Y(s)\big), \quad a \in [0,1]$$

with Z a TEG process with unit Fréchet margins and Y an asymptotically independent stationary process with unit Fréchet margins.

Only one condition is necessary for the A.I. process $Y(\cdot)$: the bivariate distribution function $F_Y^h(\cdot,\cdot) \equiv F_{Y(s_1),Y(s_2)}(\cdot,\cdot)$ for pairs of sites s_1 and s_2 which are separated by a distance h verifies

$$P(Y(s_1) > y, Y(s_2) > y) \sim y^{-\frac{1}{\eta(h)}} \mathscr{L}_h(y) \text{ as } y \to \infty$$

where $0 \le \eta(h) < 1$, $\mathscr{L}_h(\cdot)$ a slowly varying function, that is $\mathscr{L}_h(\cdot)$ satisfies $\mathscr{L}_h(xt)/\mathscr{L}(t) \to 1$ as $t \to \infty$ for all fixed $x > 0$.

Mixture of a TEG process and an A.I. process

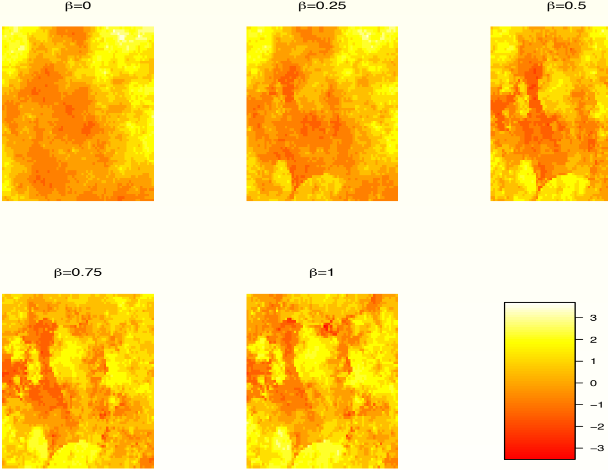


Figure 3 : Simulation of the max-mixture process with $a \in \{0, 0.25, 0.50, 0.75, 1\}$. B is a disc with a fixed radius $r = 0.25$. An exponential correlation function with parameter $\rho_1 = 0.2$ is chosen for the underlying Gaussian process involved in the TEG process. For the AI process, a Gaussian random field is considered with a spherical correlation function with parameter $\rho_2 = 0.8$.

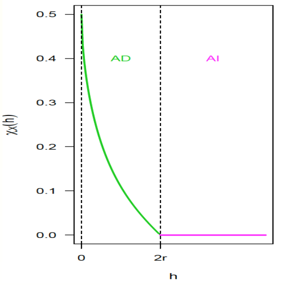
Joint probability of exceedances : information on dependence.

$$P(X(s_1) > z, X(s_2) > z) \quad = \quad \frac{a\chi_Z(h)}{z} + \left(\frac{z}{1-a}\right)^{-\frac{1}{\eta_Y(h)}} \mathscr{L}_h\left(\frac{z}{1-a}\right) + \mathscr{O}\left(\frac{1}{z^2}\right).$$

With the specific choice of the TEG process for $Z(.)$ with fixed radius r, we obtain

$$\chi_X(h) = a\chi_Z(h) = a\left(1 - \frac{h}{2r}\right)\left\{1 - 2^{-\frac{1}{2}}[1-\rho_1(h)]^{\frac{1}{2}}\right\} \text{ if } h < 2r \text{ (and 0 otherwise).}$$

Summing up, pairs of sites separated by a distance h are asymptotically dependent if h is smaller than $2r$ and asymptotically independent otherwise.

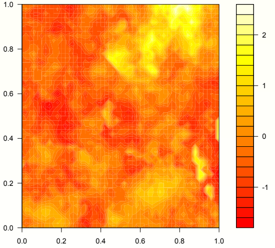


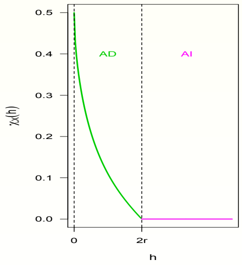
Computation of $\overline{\chi}_X$ to go further...

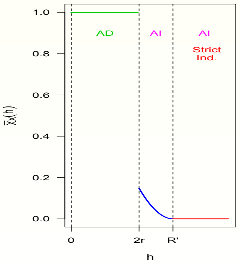
Mixture of a max-stable process and an A.I. one

- If $h < 2r$, $\chi_X(h) > 0$, $\overline{\chi}_X(h) = 1$ A.D.
- If $h \geq 2r$, $\chi_X(h) = 0$, $\overline{\chi}_X(h) = \overline{\chi}_Y(h) < 1$ A.I.
- If $\overline{\chi}_Y(h)$ is such that $\overline{\chi}_Y(h) = 0$ for $h > R' > 2r$, then
  if $h > R' > 2r$, $\chi_X(h) = 0$, $\overline{\chi}_X(h) = \overline{\chi}_Y(h) = 0$ Exact Independence

Here, it corresponds to the case of a process $Y(\cdot)$ with a correlation function $\rho_2(\cdot)$ such that $\rho_2(h) = 0$ when $h > R'$ (spherical correlation function for example).







Simulation study

Bacro, Gaetan and Toulemonde, JSPI, 2016

$$X(s) = \max\left(aZ(s), (1-a)Y(s)\right), \quad a \in [0,1]$$

with Z a max-stable process with unit Fréchet margins and Y an asymptotically independent stationary process with unit Fréchet margins.

- Parameter of the mixture a $(a \in \{0, 0.25, 0.5, 0.75, 1\})$.
- Parameters of the TEG process.
  - the radius r $(r = 0.25)$.
  - the correlation parameter ρ_1 $(\rho_1 = 0.2)$. Here we have chosen the exponential correlation function : $\exp(-h/\rho_1)$.
- Parameter of the AI process Y where Y is a transformed Gaussian process with unit Fréchet margins.
  - the correlation parameter ρ_2 $(\rho_2 = 0.8)$. Here we have chosen the spherical correlation function : $1-(1.5h)/\rho_2 + (0.5h)/\rho_2^3$ if $h < \rho_2$ and 0 otherwise.

Inference

Censored composite likelihood approach on pairwise sites separated by a distance $h < \delta$.

- Given a high threshold value u : the dependence model for an adequate representation of the data.

  For any (s_i, s_j) such $d(s_i, s_j) < \delta$, pairwise contribution

  $$L(x_{ik}, x_{jk}; \psi) = \begin{cases} \frac{\partial^2}{\partial x_i \partial x_j} G(x_i, x_j; \psi) & \text{if } \max(x_i, x_j) > u \\ G(u, u; \psi) & \text{if } \max(x_i, x_j) \leq u \end{cases}$$

  with $G(\cdot, \cdot)$ the bivariate distribution of the spatial model. The pairwise log-likelihood is defined by

  $$pl(\psi) = \sum_{k=1}^{M} \sum_{i=1}^{N-1} \sum_{j>1}^{N} \omega_{ij} \log L(x_{ik}, x_{jk}; \psi).$$

Simulation study

- 49 random sites.
- 1000 time observations of the process.
  $\Rightarrow$ Estimation of the four parameters by the method of composite likelihood.
- 5 different values of a
- 500 simulations
- Discriminate between asymptotic independence, asymptotic dependence or a mixture of this thanks to the CLIC ?

$$CLIC = -2\left[pl(\widehat{\psi}) - \mathrm{tr}\{\widehat{H}^{-1}\widehat{J}\}\right].$$

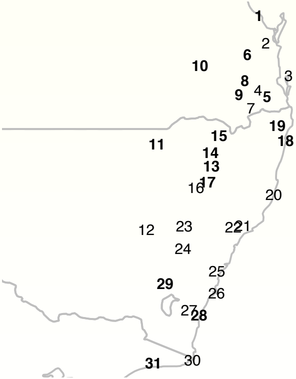
	Gaussian	MM	TEG
MM_0 (Gaussian)	346	154	0
$MM_{0.25}$	0	500	0
$MM_{0.50}$	0	500	0
$MM_{0.75}$	0	498	2
MM_1 (TEG)	0	100	400

Table 1 : Model selection based on the CLIC. The simulation study is based on 500 replications of 1000 independent copies of a MM_a model with $\rho_1 = 0.2$, $\rho_2 = 0.8$, $r = 0.25$ and $a \in \{0, 0.25, 0.50, 0.75, 1\}$.

Application : Coming back to the Australian rainfall data . . .

Motivation : a spatial data set where daily rainfall totals 24h data from an observational network covering a region S of East-Australia, are analysed for the period 1955-2003.

- 31 sites observed from East-Australia during 49 winters (April-September), (Lavery, Joung and Nicholls 1996).



Application : Coming back to the Australian rainfall data ...

- **Model A_1** : the MM model $X(.)$

$$X(s) = \max\left(aZ(s), (1-a)Y(s)\right), \quad a \in [0,1]$$

  with
  - a the max-mixture proportion ;
  - $Z(\cdot)$ a TEG process based on a gaussian process with an exponential correlation function $\exp(-h/\rho_1)$ and compact random set B choosen as a disc with a fixed radius r ;
  - $Y(\cdot)$ a Gaussian random field with unit Fréchet margins and a spherical correlation function $1 - \frac{1}{2}\frac{h}{\rho_2} + \frac{1}{2}\left(\frac{h}{\rho_2}\right)^3 I_{\{h \le \rho_2\}}$

- **Model A_2** : the $X(.)$ process specified in A_1 but with exponential correlation function $\exp(-h/\rho_2)$.
- **Model A_3** : a max-mixture model as in A_1 but in which $Y(.)$ is an inverse max-stable process.
- **Model B** : the $Z(.)$ process specified in A_i, $i = 1, 2$.
- **Model C_1** : the $Y(.)$ process specified in A_1
- **Model C_2** : the $Y(.)$ process specified in A_2
- **Model C_3** : the $Y(.)$ process specified in A_3

Application : Coming back to the Australian rainfall data . . .

Model	$\widehat{\rho}_1$	$\widehat{r}_1$	$\widehat{\rho}_2$	$\widehat{r}_2$	$\widehat{a}$	CLIC
A_1	78.71 (9.80)	833.76 (77.70)	1448.52 (57.72)	-	0.38 (0.02)	575518.3
A_2	101.03 (13.93)	658.94 (54.26)	841.08 (51.23)	-	0.38 (0.02)	575515.9
A_3	210.07 (10^{-13})	211.15 (10^{-13})	2164.57 (140.85)	1400.11 (95.08)	0 (10^{-13})	**575183.7**
B	147.09 (6.17)	1706.55 (213.31)	-	-	-	580455
C_1	-	-	814.81 (19.34)	-	-	580351.3
C_2	-	-	429.68 12.38	-	-	578445.3
C_3	-	-	2084.84 (139.76)	1447.33 (106.76)	-	**575188.3**

Table 2 : Summary of the fitted models based on the daily exceedances from the Australian data. Standard errors are reported in parentheses.

Model	$\widehat{\rho}_1$	$\widehat{r}_1$	$\widehat{\rho}_2$	$\widehat{r}_2$	$\widehat{a}$	CLIC
A_1	78.71 (9.80)	833.76 (77.70)	1448.52 (57.72)	-	0.38 (0.02)	330
A_2	101.03 (13.93)	658.94 (54.26)	841.08 (51.23)	-	0.38 (0.02)	**328**
A_3	210.07 (10^{-13})	211.15 (10^{-13})	2164.57 (140.85)	1400.11 (95.08)	0 (10^{-13})	
B	147.09 (6.17)	1706.55 (213.31)	-	-	-	**5267**
C_1	-	-	814.81 (19.34)	-	-	5163
C_2	-	-	429.68 12.38	-	-	3257
C_3	-	-	2084.84 (139.76)	1447.33 (106.76)	-	**0**

Table 3 : Summary of the fitted models based on the daily exceedances from the Australian data. Standard errors are reported in parentheses.

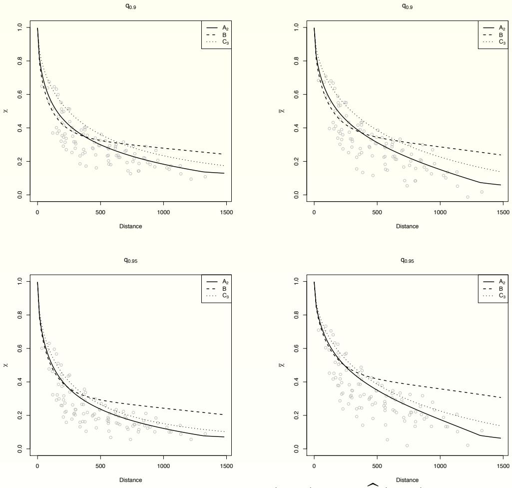
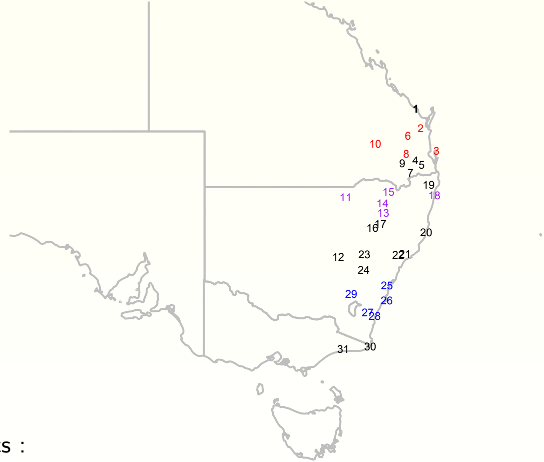


Figure 4 : Empirical and fitted values for $\widehat{\chi}(h,u)$ and $\widehat{\overline{\chi}}(h,u)$. Empirical values are computed using the validation data set and models are fitted using the q_u quantile exceedances. Top row : $u = 0.9$; bottom row : $u = 0.95$.

Empirical and fitted values for the conditional probabilities

Conditional probabilities $\Pr(Z(s) > z, s \in \mathscr{S} \mid Z(s_1) > z)$ with z such that $\Pr(Z(s_1) > z) = 1 - p$ for different values of p.



Three sites sets :

- $\mathscr{S} = \{s_2, s_3, s_6, s_8, s_{10}\}$ (near sites data set) ;
- $\mathscr{S} = \{s_{11}, s_{13}, s_{14}, s_{15}, s_{18}\}$ (medium sites data set) ;
- $\mathscr{S} = \{s_{25}, s_{26}, s_{27}, s_{28}, s_{29}\}$ (far sites data set).

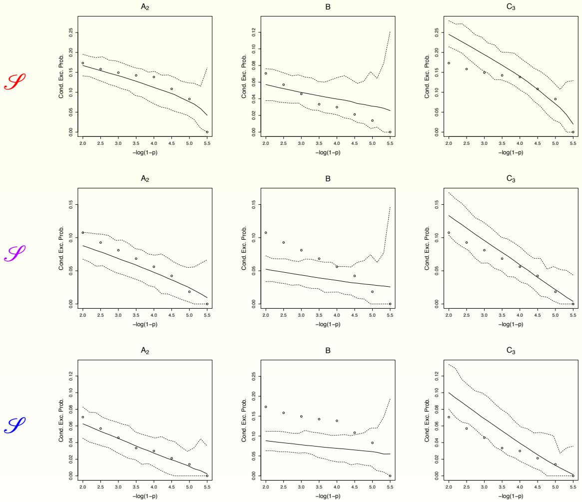


Figure 5 : Empirical and fitted values for the conditional probabilities $\Pr(Z(s) > z, s \in \mathscr{S} \mid Z(s_1) > z)$. Top row : $\mathscr{S} = \{s_2, s_3, s_6, s_8, s_{10}\}$ (near sites data set) ; middle row $\mathscr{S} = \{s_{11}, s_{13}, s_{14}, s_{15}, s_{18}\}$ (medium sites data set) ; bottom row : $\mathscr{S} = \{s_{25}, s_{26}, s_{27}, s_{28}, s_{29}\}$ (far sites data set).

To sum up this first part

- Difficulty to detect the kind of extremal dependence in data
- The kind of extremal dependence may evolve with distances
- We propose a flexible model for spatial extreme analysis (AD, AI according to distances)
- Inference by censored composite likelihood
- Good results on simulation data and on the real data set

Pursuing the same goal...

Outline

Wadsworth and Tawn (2013)

Model

$$P(X_P > n^{\gamma}, Y_P > n^{\beta}) = L(n;\gamma,\beta)\, n^{-\kappa(\gamma,\beta)}$$

where L is a univariate slowly varying function in n, $n \to \infty$, for all $(\beta,\gamma) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$, and the function $\kappa(\beta,\gamma) > 0$ maps the different marginal growth rates to the joint tail decay rate.

Using $\alpha = \dfrac{\beta}{\beta+\gamma}$, under the assumption that κ is differentiable and that

$$\lim_{n\to\infty} \frac{L(n;\alpha+\log x/\log n, 1-\alpha+\log y/\log n)}{L(n;\alpha,1-\alpha)} = 1,$$

they deduced the tail representation :

$$P(X_P > n^{\alpha}x, Y_P > n^{1-\alpha}y) = n^{-\kappa(\alpha,1-\alpha)}\, x^{-\kappa_1(\alpha)}\, y^{-\kappa_2(\alpha)} \\ L\left(n;\alpha+\frac{\log(x)}{\log(n)}, 1-\alpha+\frac{\log(y)}{\log(n)}\right)$$

where $\{\kappa_1(\alpha),\kappa_2(\alpha)\} = \left\{\frac{\partial\kappa}{\partial\beta},\frac{\partial\kappa}{\partial\gamma}\right\}|_{(\alpha,1-\alpha)}$.

Advantages and weaknesses of the WT model

- Allowing the components to grow at different rates
- Permiting extrapolation into regions where not all components are simultaneously extreme
- Ray independence
- Non parametric approach.

Intented improvements

- Allowing the components to grow at different rates
- Permiting extrapolation into regions where not all components are simultaneously extreme
- Ray independence and ray dependence
- Non parametric approach and semi parametric approach.

Proposed tail model - Joint work with Bacro and Dalhoumi

Let (X_P, Y_P) be a random vector with standard Pareto marginal distributions and assume that for $(\beta,\gamma) \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$ and $(x,y) \in [1,\infty)^2$:

$$Pr(X_P > n^\beta x, Y_P > n^\gamma y) = \mathcal{L}(n^\beta x, n^\gamma y) n^{-\kappa(\beta,\gamma)} x^{\frac{-\kappa(\beta,\gamma)}{2\beta}} y^{\frac{-\kappa(\beta,\gamma)}{2\gamma}}$$

where κ is the function from the Wadsworth-Tawn model and $\mathcal{L}$ is a bivariate slowly varying function,i.e for $(x,y) \in [1,\infty)^2$
for any $(\beta,\gamma) \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$ we have

$$\lim_{\min(n^\beta,n^\gamma)\to\infty} \frac{\mathcal{L}(n^\beta x, n^\gamma y)}{\mathcal{L}(n^\beta, n^\gamma)} = \lim_{n\to\infty} \frac{\mathcal{L}_(n^\beta x, n^\gamma y)}{\mathcal{L}(n^\beta, n^\gamma)} = g_{(\beta,\gamma)}(x,y)$$

with $g_{(\beta,\gamma)}$ verifies a non-standard zero-order homogeneity : for any $c > 0$ and $(x,y) \in (0,\infty)^2$, $g_{(\beta,\gamma)}(c^\beta x, c^\gamma y) = g_{(\beta,\gamma)}(x,y)$.

Connections to existing theory for asymptotic independence and sum up

- Assuming the ray independence condition, and setting $x = y = 1$, our model corresponds to the Wadsworth and Tawn model (2013).
- For $(\beta, \gamma) = (1, 1)$, as $\kappa(1,1) = \frac{1}{\eta}$ we recognize the Ledford and Tawn (1996, 1997) model and the Ramos and Ledford model (2009).

To sum up this work in progress

- Allowing the components to grow at different rates
- Permiting extrapolation into regions where not all components are simultaneously extreme
- Ray independence and ray dependence
- Non parametric approach and semi parametric approach.

To a spatial approach ? With different kinds of dependence according to distances ?

THANK YOU FOR YOUR ATTENTION !

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