Ruin problems for processes in a changing environment

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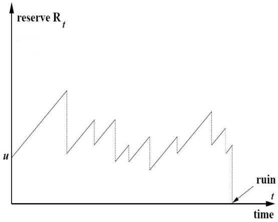
Outline

Potential changes in the insurance industry

- Property and casualty insurance
  - Flooding - increased precipitation, rise in sea level etc.
  - Increase in frequency and severity of floods and storms
  - Pests - forestry and agriculture
- Life and health insurance
  - Change in tropical disease vectors (*new* diseases or resurgence of diseases that were supposed to have disappeared, e.g. dengue fever, malaria, cholera in North America)
  - Mortality rate changes (e.g. - increase of respiratory diseases and allergies)
- Potential financial areas of vulnerability?
  - Reserves (investments and surplus)
  - Ratings and solvency

The classical model



$$R_t = u + ct - \sum_{i=1}^{N_t} X_i \qquad t \geq 0$$

$u \geq 0$ initial capital

$c > 0$ premium intensity

X_i the size of the i-th claim (i.i.d. r.v.s of d.f. F)

$N(t)$ homogeneous Poisson process (λ)

A time dependent Framework

$$R_t = u + p(t) - \sum_{i=1}^{N_t} X_{\mathcal{T}_i} \qquad t \geq 0$$

- $\mathcal{T}_i \ldots$ time of i-th claim
- $X_t \ldots$ independent random variables with distribution F_t
- $N(t) \ldots$ inhomogeneous Poisson process with intensity $\lambda(t)$
- $p(t) = (1+\rho)\int_0^t \lambda(s)\mathbb{E}\left[X_s\right] \mathrm{d}s \ldots$ collected premiums

Let $\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s$ and $R^h(t) = R(\Lambda^{-1}(t))$

- $p(\Lambda^{-1}(t)) = (1+\rho)\int_0^t \mathbb{E}\left[X_{\Lambda^{-1}(s)}\right] \mathrm{d}s$
- $N_{\Lambda^{-1}(s)} \ldots$ homogenous Poisson process
- w.l.o.g. $\lambda(s) = \lambda$

What are we interested in

$\tau(u) = \inf\{t : R(t) < 0\} \ldots$ time of ruin

- The ruin probability

$$\psi(u) = \mathbb{P}(\tau(u) < \infty).$$

- Finite time ruin probability up to time T.

$$\psi(u, T) = \mathbb{P}(\tau(u) < T)$$

- How and when does ruin occur?
- How to choose premiums $(p(t))$ in practice?
- What is a realistic model for the claims X_t?

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We concentrate on asymptotic results for $\psi(u)$.

Classes of distributions for X_t

Definition (Light-tailed distributions)

A random variable X with values in $\mathbb{R}$ and distribution F is light-tailed if there exists an $s > 0$ with

$$\hat{F}(s) := \mathbb{E}\left[e^{sX}\right] < \infty, \quad \hat{F} \ldots \text{ moment generating function}$$

MDA(Λ): exponential, gamma, normal; MDA(Ψ)

Definition (Heavy-tailed distributions)

A random variable X with values in $\mathbb{R}$ and distribution F is heavy-tailed if for all $s > 0$

$$\hat{F}(s) = \infty$$

MDA(Φ): Pareto, Burr; MDA(Λ): lognormal, weibull;

Regularly varying function and distributions

Definition

A function $g(x)$ is regularly varying with index α if for all $y > 0$

$$\lim_{x \to \infty} \frac{g(yx)}{g(x)} = y^{\alpha}.$$

Definition

A random variable X with distribution function F is regularly varying with index α if $\overline{F}(x) = 1 - F(x)$ is regularly varying with index $-\alpha$

- Pareto $\overline{F}(x) = (1 + x/d)^{-\alpha}$
- Burr $\overline{F}(x) = (1 + (x/d)^{\gamma})^{-\alpha}$
- log-Gamma distribution

Two concrete models for heavy-tailed claims

Assume X_t is Pareto distributed and

$$\mathbb{E}\left[X_t\right] = \frac{d}{\alpha_0 - 1}\left(1 + c_\alpha t\right), \quad p(t) = \frac{(1+\rho)\lambda d(1+c_\alpha t)^2}{2c_\alpha(\alpha_0 - 1)}$$

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1. Change in shape parameter: $\overline{F}_t^{(1)}(x) = (1 + x/d)^{-\alpha_t}$ where

$$\alpha_t = \frac{\alpha_0 - 1}{1 + c_\alpha t} + 1.$$

2. Change in scale parameter: $\overline{F}_t^{(2)}(x) = (1 + x/d_t)^{-\alpha_0}$ where

$$d_t = d(1 + c_\alpha t).$$

- Which model has more risk?

Ruin probabilities

Theorem

$$\psi^{(1)}(u) \sim \frac{\pi d u^{-0.5}}{2} \sqrt{\frac{2(\alpha_0 - 1)}{(1+\rho)\lambda c_\alpha}}.$$

$$\psi^{(2)}(u) \sim \frac{\lambda}{c_\alpha} d^{\alpha_0} u^{-\frac{\alpha_0 - 1}{2}} \int_0^\infty \left(\frac{1}{t} + \frac{\rho\lambda\mu}{2c_\alpha} t \right)^{-\alpha_0} dt.$$

- For $\alpha_0 > 2$ Model 1 is more dangerous
- For $\alpha_0 < 2$ Model 2 is more dangerous

The principle of the single big jump heuristic

- The process behaves in a normal way until a single big jump happens.
- Heuristic:

$$\begin{aligned}\psi(u) &= \lambda \int_0^\infty \mathbb{E}\left[\overline{F}_t\left(u + p(t) - \sum_{i=1}^{N_t} X_{\mathcal{T}_i}\right) 1_{\{\tau(u)>t\}}\right] \mathrm{d}t \\ &\approx \lambda \int_0^\infty \overline{F}_t\left(u + p(t) - \mathbb{E}\left[\sum_{i=1}^{N_t} X_{\mathcal{T}_i}\right]\right) \mathrm{d}t\end{aligned}$$

- D. Denisov, A. B. Dieker, and V. Shneer. Large deviations for random walks under subexponentiality: the big-jump domain. *Ann. Probab.*, 36(5):1946–1991, 2008.

A remark on Model 1 or a Model with infinite mean

We will consider

$$R_t = u + p(t) - \sum_{i=1}^{N_t} Y_i$$

where the Y_i are i.i.d. with distribution function F

Theorem

If $X_1, X_2, \ldots$ are i.i.d. random variables with distribution $F(x)$ that is regularly varying with index $0 < \alpha \leq 1$, and regularly varying density $f(x)$. If further $p(T)$ is regularly varying with index $\beta > 1/\alpha$ (continuous and strict monotonic increasing) then

$$\psi(u) \sim \lambda \int_0^\infty \overline{F}(u + p(T))dT \sim \lambda p^{-1}(u)\overline{F}(u) \int_0^\infty (1 + t^\beta)^{-\alpha} dt$$

$$p(t) - \sum_{i=1}^{N_t} X_i \sim p(t) \quad \text{a.s.}.$$

Bounds

Theorem

- *F ... distribution with regularly varying density $f(x)$ (index α)*
- *F_t ... distribution with $F_t(x) \geq F(x)$, $\forall x > 0, t > 0$*
- *$p(t)$ ... regularly varying with index $\beta > 1/\alpha$*
- $\exists \gamma > 0 : \forall \delta > \exists, x_\delta > 0$ *and all* $y > x > x_\delta$
  $\overline{F}_t(y) \geq (1-\delta)(x/y)^\gamma \overline{F}_t(x)$

Then for the risk process

$$R_t = u + p(t) - \sum_{i=1}^{N_t} X_{T_i}$$

we have that

$$\lambda \int_0^\infty \overline{F}_t(u+p(t))dt \lesssim \psi(u) \lesssim \lambda \int_0^\infty \overline{F}(u+p(t))dt.$$

A note on Model 2

Theorem

If $X_1, X_2, \ldots$ are i.i.d. random variables with distribution $F(x)$ that is regularly varying with index $\alpha > 1$, mean μ, and regularly varying density $f(x)$ then

$$\psi(u) \sim \lambda \int_0^\infty \overline{F}\left(\frac{u}{t} + \frac{\rho}{2}t\right) dt \sim \lambda\sqrt{u}\,\overline{F}(\sqrt{u}) \int_0^\infty \left(\frac{1}{t} + \frac{\rho}{2}t\right)^{-\alpha} dt \quad (1)$$

Light tailed case

- For Exponential distribution Model 1 and 2 equivalent.
- We will consider the process

$$S_t = \sum_{i=1}^{N_t} \mu(T_i) X_i - (1+\rho) \int_0^t \mu(s) ds$$

- $X_i \dots$ iid random variables
- $\mu(t) \dots$ multiplicative change over time
- N_t homogeneous Poisson process with rate 1

- $\varphi(t) = \mathbb{E}\left[e^{tX_1}\right]$,
- $\kappa(t, \theta) = \log\left(\mathbb{E}\left[e^{\theta S_t}\right]\right)$
- $p(t) = (1+\rho)\int_0^t \mu(s) ds$.

Theorem

If

- $\mathbb{E}[X_1] = 1$ *and there exists an* $s_0 \le \infty$ *with* $\lim_{t\to s_0}\varphi(t) = \infty$.
- $\mu(x)$ *is strict monotone increasing and is regularly varying with index* $\alpha > 0$
  Further $\mu'(x)/(\alpha\mu(x)) \sim x$ *(i.e.* μ *has a regularly varying density).*
- $F(x)$ *has density* $f(x)$ *and for all* $s < s_0$,
  $\lim_{x\to\infty} e^{sx}\overline{F}(x) = \lim_{x\to\infty} e^{sx} f(x) = 0$.
- *For all* $\theta < s_0$
$$\lim_{x\to\infty} e^{\theta x} \sup_{t>0} \frac{\overline{F}(x+t)}{\overline{F}(t)} = 0.$$

then

$$\mathbb{P}(\tau(u) < \infty) \sim \frac{\rho e^{-\gamma\frac{u}{\mu(t_0)} + \kappa(t_0,\theta^*)}}{\sqrt{\varphi'(\gamma) - (1+\rho)}\sqrt{(\alpha+1)\int_0^1 s^\alpha\left(\varphi'(\gamma s^\alpha) - (1+\rho)\right) ds}}$$

We are following ideas from

- N. G. Duffield and N. O'Connell. Large deviations and overflow probabilities for the general single-server queue, with applications. *Math. Proc. Cambridge Philos. Soc.*, 118(2):363–374, 1995.
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$$S_t = \sum_{i=1}^{N_t} \mu(\mathcal{T}_i) X_i - (1+\rho) \int_0^t \mu(t) \mathrm{d}t$$
- $\kappa(t,\theta) = \log(\mathbb{E}\left[\exp(\theta S_t)\right])$

$$\kappa(t,\theta) = \int_0^t \varphi(\theta\mu(s))\mathrm{d}s - t - \theta(1+\rho)\int_0^t \mu(s)\mathrm{d}s.$$

- $\theta_t \ldots$ the solution to $\kappa_\theta(t,\theta_t) = u$ (θ_t depends on u)

- Define family of processes $S_t^s = \sum_{i=1}^{N_t^s} \mu(\mathcal{T}_i^s) X^s_{\mathcal{T}_i^s} - p(t)$
- $N_t^s \ldots$ inhomogeneous Poisson process (on $[0,T)$ ($\mu(T)\theta_s = s_0$) with intensity

$$\lambda^s(t) = \mathbb{E}\left[e^{\mu(s)\theta_s X_1}\right] = \varphi(\mu(t)\theta_s)$$

- $X_t^s \ldots$ stochastic process independent of N_t^s
  For $(r \neq t)$ X_t^s independent from X_r^s
  X_t^s has distribution function $F_{\theta_s\mu(t)}(x)$ where $(\theta > 0)$

$$F_\theta(x) = \frac{1}{\varphi(\theta)}\mathbb{E}\left[e^{\theta X_1} 1_{\{X_1 \le u\}}\right]$$

- $$\mathrm{d}S_t^s = e^{\theta_s S_t - \kappa(t,\theta_s)}\mathrm{d}S_t \quad t \le T$$
- $\mathbb{E}\left[S_s^s\right] = u$

Denote with $h(t) = h_u(t) = \theta_t u - \kappa(t, \theta_t)$ then

$$\mathbb{P}(S_s > u) = e^{-h_u(s)} \mathbb{E}\left[e^{-\theta_s (S_s^s - u)} 1_{\{S_s^s > u\}} \right] \approx e^{-h_u(s)}$$

$$\mathbb{P}(\sup_{s>0} S_s > u) \approx e^{-\inf_s h_u(s)}.$$

Define $(t_0, \theta^*) = (t_0(u), \theta^*(u))$ as the pair (t_0, θ_{t_0}) where t_0 minimizes $h(t)$. We can show that

$$h'(t_0) = \kappa_t(t_0, \theta^*) = 0.$$

or

$$\varphi(\theta^* \mu(t_0)) - 1 - (1 + \rho)\theta^* \mu(t_0) = 0.$$

$\gamma = \theta^* \mu(t_0)$
$t_0 = t_0(u)$ is regularly varying with index $1/(\alpha + 1)$ and $\theta^* = \theta^*(u)$ depends on u.

Heuristic:

- Ruin occurs around t_0
- Most likely path to ruin is when S_t behaves like $S_t^{t_0}$

The further proof depends on 4 Lemmas

- $\tau(u) = \inf\{t : S_t > u\}$.
- $S_t^* = S_t^{t_0}$ (S_t^* depends on u!)

Lemma

Let $s > 0$ then for some $u_s > 0$ and all $u > u_s$

$$\mathbb{P}(|\tau(u) - t_0| < s\sqrt{t_0})$$
$$= \mathbb{E}\left[e^{-\theta^* S^*_{\tau^*(u)^*} + \kappa(\tau^*(u),\theta^*)} 1_{\{|\tau^*(u)-t_0|<s\sqrt{t_0}\}}\right]$$
$$= e^{-\gamma\frac{u}{\mu(t_0)}+\kappa(t_0,\theta^*)}\mathbb{E}\left[e^{-\frac{\gamma}{\mu(t_0)}(S^*_{\tau^*(u)}-u)} e^{\kappa(\tau^*(u),\theta^*)-\kappa(t_0,\theta^*)} 1_{\{|\tau^*(u)-t_0|<s\sqrt{t_0}\}}\right]$$
$$= e^{-h_u(t_0)}\mathbb{E}\left[e^{-\frac{\gamma}{\mu(t_0)}(S^*_{\tau^*(u)}-u)} e^{\kappa(\tau^*(u),\theta^*)-\kappa(t_0,\theta^*)} 1_{\{|\tau^*(u)-t_0|<s\sqrt{t_0}\}}\right]$$

Lemma

There exists a function $R(s)$ with $\lim_{s\to\infty} R(s) = 0$ such that

$$\mathbb{P}(|\tau(u) - t_0| > s\sqrt{t_0}, \tau(u) < \infty) \leq R(s)e^{-\gamma\frac{u}{\mu(t_0)}+\kappa(t_0,\theta^*)}.$$

Lemma

$$\limsup_{u\to\infty} \mathbb{E}\left[e^{-\frac{\gamma}{\mu(t_0)}(S^*_{\tau^*(u)}-u)} e^{\kappa(\tau^*(u),\theta^*)-\kappa(t_0,\theta^*)} 1_{\{|\tau^*(u)-t_0|<s\sqrt{t_0}\}}\right]$$
$$= \frac{\rho}{\varphi'(\gamma)-(1+\rho)} \mathbb{E}\left[e^{\kappa(\tau^*(u),\theta^*)-\kappa(t_0,\theta^*)} 1_{\{|\tau^*(u)-t_0|<s\sqrt{t_0}\}}\right] + R_s^u$$
$$\liminf_{u\to\infty} \mathbb{E}\left[e^{-\frac{\gamma}{\mu(t_0)}(S^*_{\tau^*(u)}-u)} e^{\kappa(\tau^*(u),\theta^*)-\kappa(t_0,\theta^*)} 1_{\{|\tau^*(u)-t_0|<s\sqrt{t_0}\}}\right]$$
$$= \frac{\rho}{\varphi'(\gamma)-(1+\rho)} \mathbb{E}\left[e^{\kappa(\tau^*(u),\theta^*)-\kappa(t_0,\theta^*)} 1_{\{|\tau^*(u)-t_0|<s\sqrt{t_0}\}}\right] + R_s^l$$

where $\lim_{s\to\infty} R_s^u = \lim_{s\to\infty} R_s^l = 0$.

Lemma

There exists functions R_s^u *and* R_s^l *with* $\lim_{s\to\infty} R_s^u = \lim_{s\to\infty} R_s^l = 0$ *such that*

$$\limsup_{u\to\infty} \mathbb{E}\left[e^{\kappa(\tau^*(u),\theta^*)-\kappa(t_0,\theta^*)} 1_{\{|\tau^*(u)-t_0|<s\sqrt{t_0}\}}\right]$$

$$\le \frac{\sqrt{\varphi'(\gamma)-(1+\rho)}}{\sqrt{(\alpha+1)\int_0^1 s^\alpha\left(\varphi'(\gamma s^\alpha)-(1+\rho)\right)ds}} + R_s^u.$$

$$\liminf_{u\to\infty} \mathbb{E}\left[e^{\kappa(\tau^*(u),\theta^*)-\kappa(t_0,\theta^*)} 1_{\{|\tau^*(u)-t_0|<s\sqrt{t_0}\}}\right]$$

$$\ge \frac{\sqrt{\varphi'(\gamma)-(1+\rho)}}{\sqrt{(\alpha+1)\int_0^1 s^\alpha\left(\varphi'(\gamma s^\alpha)-(1+\rho)\right)ds}} + R_s^l.$$

Conclusions and Remarks

- Similar heuristics as in the stationary case hold.
- In the heavy tailed case: When change in shape parameter or infinite mean models, net profit condition can be violated.
- Hard to (numerical) check quality of asymptotic results

Further work

- Connection to queueing models
- Include premium rules for risk models
- More realistic models
- Efficient numerical methods

Thank you for the attention