On tail dependence coefficients of transformed multivariate Archimedean copulas

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Contents

Tail behaviour of a copula

- Depending upon targeted applications, understanding the tail behaviour of a copula is of great importance.
- In many practical problems, like hydrology, finance, insurance, etc. one needs to understand the **risk of simultaneous threshold crossing** for the considered random variables.
- Tail dependence measurements have been proposed in the literature to explain the **asymptotic probability that all random variables in a given set become large, given that random variables of another set are also large**. For example, the probability that losses of some financial derivatives are large, given that losses of other derivatives are also large.
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Bivariate tail dependence coefficients

The classical bivariate upper and lower tail dependence coefficients, λ_U and λ_L, are defined as

$$\lambda_L = \lim_{u \to 0^+} \mathbb{P}[V \leq u \,|\, U \leq u] \quad \text{and} \quad \lambda_U = \lim_{u \to 1^-} \mathbb{P}[V > u \,|\, U > u],$$

see **Sibuya (1960)**.

- For the bivariate independence copula $\Pi(u, v) = u\, v$ we have $\lambda_L = \lambda_U = 0$ (tail independence)
- For the bivariate comonotonic copula $M(u, v) = \min\{u,\ v\}$ we have $\lambda_L = \lambda_U = 1$ (perfect tail dependence).
- These tail dependence coefficients are a copula-based dependence measures.

Some drawback..

- They evaluate the copula C solely on its diagonal section.
- The limiting behaviour may be very different if we tend to the copula's lower left (resp. copula's upper right) corner on a different route than on the main diagonal (see **Schlather (2001)**, **Frahm et al. (2005)**).

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Bivariate tail dependence coefficients

Moreover, it is well known that the bivariate lower and upper tail coefficients can also be defined as (see, for example **Coles et al. (1999)**):

$$\lambda_L = \lim_{u\to 0^+} \frac{C(u,u)}{u} = 2 - \lim_{u\to 0^+} \frac{\ln(1-2u+C(u,u))}{\ln(1-u)},$$

$$\lambda_U = \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u} = 2 - \lim_{u\to 1^-} \frac{\ln C(u,u)}{\ln u}.$$

Furthermore, the bivariate lower and upper tail coefficients λ_U and λ_L can be written using the diagonal section $\delta_C(u) = C(u,\, u)$ of the associated copula C (see, e.g., **Nelsen (1998)**):

$$\lambda_L = \lim_{u\to 0^+} \frac{\mathrm{d}}{\mathrm{d}u}\delta(u) = \delta_C'(0^+),$$

$$\lambda_U = 2 - \lim_{u\to 1^-} \frac{\mathrm{d}}{\mathrm{d}u}\delta(u) = 2 - \delta_C'(1^-).$$

Multivariate tail dependence coefficients

The tail dependence coefficients in the general multivariate case can be expressed as follows (see **De Luca and Rivieccio (2012)**, **Li (2009)**).

Definition (Multivariate tail dependence coefficients)

Let the copula C be the distribution of some random vector $\mathbf{U} := (U_1, \ldots, U_d)$ with uniform marginals. Denote $I = \{1, \ldots, d\}$ and consider two non-empty subsets $I_h \subset I$ and $\bar{I}_h = I \setminus I_h$ of respective cardinal $h \geq 1$ and $d - h \geq 1$. Provided that the limits exist, a multivariate version of classical bivariate tail dependence coefficients is given by

$$\begin{cases} \lambda_L^{I_h,\bar{I}_h} & = \lim_{u \to 0^+} \mathbb{P}\left[U_i \leq u, i \in I_h \mid U_i \leq u, i \in \bar{I}_h\right], \\ \lambda_U^{I_h,\bar{I}_h} & = \lim_{u \to 1^-} \mathbb{P}\left[U_i \geq u, i \in I_h \mid U_i \geq u, i \in \bar{I}_h\right]. \end{cases}$$

If for all $I_h \subset I$, $\lambda_L^{I_h,\bar{I}_h} = 0$, (*resp.* $\lambda_U^{I_h,\bar{I}_h} = 0$) then we say $\mathbf{U}$ is lower tail independent (*resp.* upper tail independent).

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Note that the existence of such coefficients is not guaranteed

$\hookrightarrow$ As an example, assume $d \geq 3$ and $d - h = \mathrm{card}(\bar{I}_h) \geq 2$. Consider a $d-$dimensional copula with generator ϕ_θ as the generator (4.2.2) in Nelsen (1999), with $\theta > 1$. For any $u < (1 - (\frac{1}{d-h})^{1/\theta})^\theta$, $\mathbb{P}\left[U_i \leq u, i \in \bar{I}_h\right] = 0$, so that the above limit does not exist. The limit however exists in the case where $d - h = 1$.

Multivariate tail dependence coefficients

Assumption (Considered Archimedean generators)

Consider a valid initial Archimedean generator ϕ, thus $d-$monotone on $[0,\infty)$ with $\phi(0)=1$. One assumes that ϕ is differentiable and that ϕ is a strict generator, i.e. strictly positive with $\lim_{t\to+\infty}\phi(t)=0$. Then ϕ has a proper inverse $\psi(t)=\phi^{-1}(t)$, $t\in(0,1]$.

Definition (Multivariate TDC for Archimedean copulas)

For Archimedean copulas

$$C(u_1,\ldots,u_d)=\phi\left(\phi^{-1}(u_1)+\ldots+\phi^{-1}(u_d)\right),\ \text{for}\ u_1,\ldots,u_d\in(0,1],$$

the multivariate lower and upper TDC are respectively:

$$\begin{aligned}\lambda_L^{(h,d-h)} &= \lim_{u\to0^+}\frac{\psi^{-1}\left(d\,\psi(u)\right)}{\psi^{-1}\left((d-h)\,\psi(u)\right)},\\ \lambda_U^{(h,d-h)} &= \lim_{u\to1^-}\frac{\sum_{i=0}^{d}(-1)^i\,C_d^i\,\psi^{-1}\left(i\,\psi(u)\right)}{\sum_{i=0}^{d-h}(-1)^i\,C_{d-h}^i\,\psi^{-1}\left(i\,\psi(u)\right)}.\end{aligned}$$

Transformations of copulas

Transformations of copulas are based on

↪ an initial copula C_0

↪ a transformation function T

The main interest of transformations is that, under suitable conditions, it becomes easy to get for instance

- analytical expressions for level curves of the copula;
- some properties that are useful for estimating parameters of the transformation.

(see for example in **Di Bernardino and Rullière (2013, 2014)**)

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Transformations of copulas

Different types of transformations can be found in the literature:

- **Valdez and Xiao (2011)** and **Michiels and De Schepper (2012)** for a review of some existing transforms;
- **Durante and Sempi (2005)**, **Klement et al. (2005)** and **Klement et al. (2013)**, for transformations in the bivariate case;
- **Morillas (2005)** for transformations based on mixtures;
- Class containing both the Archimedean and the extreme-value copulas : *Archimax* copulas (see **Capéraà, P., Fougères, A.-L., and Genest, C. (2000)** for the bivariate case, **Charpentier, A., Fougères, A.-L., Genest, C., and Nešlehová, J. (2014)** for the multivariate case).

Archimedean copulas and (λ_L, λ_U)

- Commonly used families of Archimedean copulas have the property that either $\lambda_U = \lambda_L$ or that only one of the coefficients is nonzero.
- **Larsson and Nešlehová (2011)** provided an example in order to overcome this restrictive behaviour of classical Archimedean copulas.
- There exist some two-parameter copulas that have both upper and lower tail dependence; see the BB1, BB4 and BB7 copulas in **Joe (1997)**.
- Another method to generate copulas that meet the requirement is to use compositions of two Laplace transforms, with one from a one-parameter family and the other from another one-parameter family.
- As stated in **Larsson and Nešlehová (2011)**: *"It is rather difficult to construct new Archimedean copula models that exhibit specific tail behaviour"*.

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Transformations of copulas

We aim to propose a flexible generator of an Archimedean copula with given tail dependence coefficients $\lambda_L^{(h,d-h)}$ and $\lambda_U^{(h,d-h)}$.

Such result would help

- ↪ adjusting parametrically **both the tail and the central part** of the considered copula.
- ↪ One can imagine starting from a copula exhibiting a good fit on the central part of the multivariate data, and applying a transformation to improve the fit of the tails.
- ↪ Or more generally finding the best transformation T to fit both the tails and the central part of a given multivariate data-set, starting from a given copula C_0.
- ↪ **It is possible to produce Archimedean copulas having tunable regular variation properties, and thus to get specific targeted multivariate lower and upper tail coefficients $\lambda_L^{(h,d-h)}$ and $\lambda_U^{(h,d-h)}$.**

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Transformations of copulas

We take transformations $T : [0,1] \to [0,1]$, where T is an increasing continuous bijection, with $T(0) = 0$ and $T(1) = 1$.

We consider here a transformed copula $\widetilde{C}_{T,C_0}$, which is transformed from an initial copula C_0 using the transformation T, i.e.,

$$\widetilde{C}_{T,C_0}(u_1, \dots, u_d) = T \circ C_0 \left(T^{-1}(u_1), \dots, T^{-1}(u_d)\right).$$

If C_0 is an Archimedean copula with generator ϕ_0, then obviously $\widetilde{\phi} = T \circ \phi_0$.

For a d-times differentiable transformation T, it is possible to use Faa Di Bruno's formula to get more explicit expressions of the d-monotony of $\widetilde{\phi} = T \circ \phi_0$. These expressions are quite complex so omitted here ...

Particular case : the initial copula C_0 is the independent one, tractable necessary and sufficient admissibility conditions are given.

Regular Variation (recall)

At infinity:

$$f \in \mathcal{RV}_\alpha(\infty) \;\Leftrightarrow\; \forall\, s > 0,\ \lim_{x \to +\infty} \frac{f(s\,x)}{f(x)} = s^\alpha.$$

At zero, using $M(x) = 1 - x$ and $I(x) = 1/x$,

$$f \in \mathcal{RV}_\alpha(0) \Leftrightarrow f \circ I \in \mathcal{RV}_{-\alpha}(\infty) \Leftrightarrow \forall\, s > 0,\ \lim_{x \to \mathbf{0}^+} \frac{f(s\,x)}{f(x)} = s^\alpha.$$

At one,

$$f \in \mathcal{RV}_\alpha(1) \Leftrightarrow f \circ M \circ I \in \mathcal{RV}_{-\alpha}(\infty) \Leftrightarrow \forall\, s > 0,\ \lim_{x \to \mathbf{0}^+} \frac{f(1 - s\,x)}{f(1 - x)} = s^\alpha.$$

Valid regularly varying Archimedean generator

Table 4.1 in **Nelsen (1999)** contains several examples of strict Archimedean copulas whose inverse generators ψ are regularly varying at one and zero.

Remark (Valid regularly varying generator)

Consider a valid generator ϕ and its inverse ψ. It holds that:

i) *If $\psi \in \mathcal{RV}_{-r}(0)$, then necessarily $r \in [0, +\infty]$.*

ii) *If $\psi \in \mathcal{RV}_{\rho}(1)$, then necessarily $\rho \in [1, +\infty]$.*

Valid regularly varying Archimedean generator

	$\phi(t)$	Range θ	Upper tail		Lower tail	
			$-\phi'(1)$	θ_1	$\phi(0)$	θ_0
(1)	$\frac{1}{\theta}(t^{-\theta}-1)$	$[-1,\infty)$	1	1	$\frac{1}{(-\theta)\vee 0}$	$\theta\vee 0$
(2)	$(1-t)^\theta$	$[1,\infty)$	$1(\theta=1)$	θ	1	0
(3)	$\log\frac{1-\theta(1-t)}{t}$	$[-1,1)$	$1-\theta$	1	∞	0
(4)	$(-\log t)^\theta$	$[1,\infty)$	$1(\theta=1)$	θ	∞	0
(5)	$-\log\frac{e^{-\theta t}-1}{e^{-\theta}-1}$	$\mathbb{R}$	$\frac{\theta}{e^\theta-1}$	1	∞	0
(6)	$-\log\{1-(1-t)^\theta\}$	$[1,\infty)$	$1(\theta=1)$	θ	∞	0
(7)	$-\log\{\theta t+(1-\theta)\}$	$(0,1]$	θ	1	$-\log(1-\theta)$	0
(8)	$\frac{1-t}{1+(\theta-1)t}$	$[1,\infty)$	$\frac{1}{\theta}$	1	1	0
(9)	$\log(1-\theta\log t)$	$(0,1]$	θ	1	∞	0
(10)	$\log(2t^{-\theta}-1)$	$(0,1]$	2θ	1	∞	0
(11)	$\log(2-t^\theta)$	$(0,1/2]$	θ	1	$\log 2$	0
(12)	$(\frac{1}{t}-1)^\theta$	$[1,\infty)$	$1(\theta=1)$	θ	∞	θ
(13)	$(1-\log t)^\theta-1$	$(0,\infty)$	θ	0	∞	0
(14)	$(t^{-1/\theta}-1)^\theta$	$[1,\infty)$	$1(\theta=1)$	θ	∞	1
(15)	$(1-t^{1/\theta})^\theta$	$[1,\infty)$	$1(\theta=1)$	θ	1	0
(16)	$(\frac{\theta}{t}+1)(1-t)$	$[0,\infty)$	$1+\theta$	1	∞	1
(17)	$-\log\frac{(1+t)^{-\theta}-1}{2^{-\theta}-1}$	$\mathbb{R}$	$\frac{\theta}{2(2^\theta-1)}$	1	∞	0
(18)	$e^{\theta/(t-1)}$	$[2,\infty)$	0	∞	$e^{-\theta}$	0
(19)	$e^{\theta/t}-e^\theta$	$(0,\infty)$	θe^θ	1	∞	∞
(20)	$e^{t^{-\theta}}-e$	$(0,\infty)$	θe	1	∞	∞
(21)	$1-\{1-(1-t)^\theta\}^{1/\theta}$	$[1,\infty)$	$1(\theta=1)$	θ	1	0
(22)	$\arcsin(1-t^\theta)$	$(0,1]$	θ	1	$\pi/2$	0
(23)	$\frac{1-t}{[-\log(1-t)]^\theta}$	$(0,\infty)$	0	1	∞	θ

Table 1 in **Charpentier and Segers (2009)**, *Tails of multivariate Archimedean copulas*. Journal of Multivariate Analysis, 100(7):1521-1537.

Gumbel generator

$$\psi^{\text{Gumbel}(\theta)} \in \mathcal{RV}_0(0), \quad \text{and} \quad \psi^{\text{Gumbel}(\theta)} \in \mathcal{RV}_\theta(1), \ \text{for} \ \theta \in [1, +\infty).$$

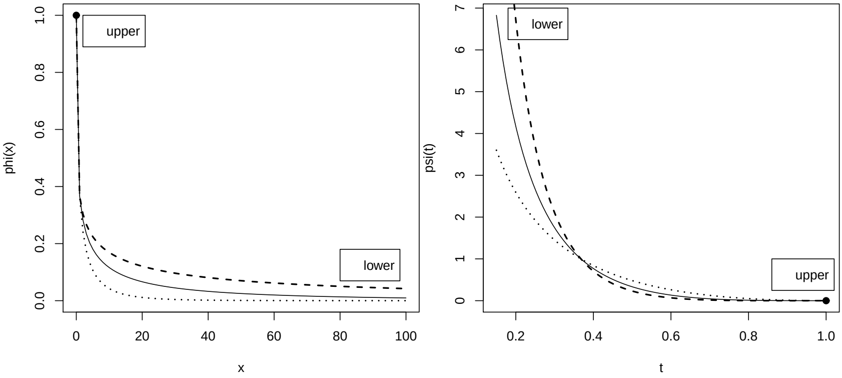

Figure : Generators $\phi^{\text{Gumbel}(\theta)} = \exp(-x^{1/\theta})$ (left) and its inverse $\psi^{\text{Gumbel}(\theta)} = (-\ln t)^\theta$ (right) for a Gumbel copula with parameters $\theta = 4$ (dashed lines), $\theta = 3$ (full lines) and $\theta = 2$ (dotted lines).

Clayton generator

$$\psi^{\text{Clayton}(\theta)} \in \mathcal{RV}_{-\theta}(0), \quad \text{and} \quad \psi^{\text{Clayton}(\theta)} \in \mathcal{RV}_{1}(1), \; \forall\, \theta \in (0, +\infty).$$

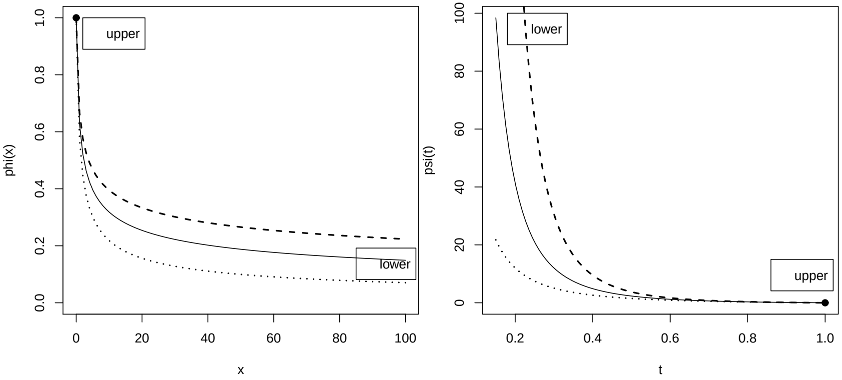

Figure : Generators $\phi^{\text{Clayton}(\theta)}(x) = (1 + \theta\, x)^{-1/\theta}$ (left) and its inverse $\psi^{\text{Clayton}(\theta)}(t) = \frac{1}{\theta}\,(t^{-\theta} - 1)$ (right) for a Clayton copula with parameters $\theta = 4$ (dashed lines), $\theta = 3$ (full lines) and $\theta = 2$ (dotted lines).

Theorem (Regularly varying transformed generator)

Let ϕ_0 be an initial valid generator and T be an admissible transformation.

- *Assume that $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, with $r_0 \in [0, +\infty]$ and that $T \in \mathcal{RV}_{\tilde{a}}(0)$, with $\tilde{a} \in (0, +\infty)$, then*

$$\widetilde{\psi} \in \mathcal{RV}_{-\tilde{r}}(0) \text{ with } \tilde{r} = \frac{r_0}{\tilde{a}}.$$

- *Assume that $\psi_0 \in \mathcal{RV}_{\rho_0}(1)$, with $\rho_0 \in [1, +\infty]$ and that $1 - T \in \mathcal{RV}_{\tilde{\alpha}}(1)$ with $\tilde{\alpha} \in (0, \rho_0]$, then*

$$\widetilde{\psi} \in \mathcal{RV}_{\tilde{\rho}}(1) \text{ with } \tilde{\rho} = \frac{\rho_0}{\tilde{\alpha}}, \quad \text{and } \tilde{\rho} \in [1, +\infty].$$

Proof is based on the composition properties of regularly varying functions.

Theorem (Multivariate lower TDC of transformed Archimedean copula)

Let ϕ_0 be an initial valid generator and T be an admissible transformation.

Assume that

- $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, *with* $r_0 \in [0, +\infty]$;
- $T \in \mathcal{RV}_{\tilde{a}}(0)$, *with* $\tilde{a} \in (0, +\infty)$.

Let $\widetilde{r} = r_0/\tilde{a}$, *then*

$$\widetilde{\lambda}_L^{(h,d-h)} = \begin{cases} \text{see Theorem below}, & \text{if } \widetilde{r} = 0, \\ d^{-\tilde{a}\, r_0{}^{-1}} (d-h)^{\tilde{a}\, r_0{}^{-1}}, & \text{if } \widetilde{r} \in (0, +\infty), \\ 1 & \text{if } \widetilde{r} = +\infty, \end{cases}$$

with $h \geq 1$ *and* $d - h \geq 1$.

Theorem (Multivariate upper TDC of transformed Archimedean copula)

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Let $\widetilde{\rho} = \rho_0/\tilde{\alpha}$, *then*

$$\widetilde{\lambda}_U^{(h,d-h)} = \begin{cases} \text{see Theorem below}, & \text{if } \widetilde{\rho} = 1, \\ \dfrac{\sum_{i=1}^{d} C_d^i (-1)^i \cdot i^{\tilde{\alpha}\rho_0}{}^{-1}}{\sum_{i=1}^{d-h} C_{d-h}^i (-1)^i \cdot i^{\tilde{\alpha}\rho_0}{}^{-1}}, & \text{if } \widetilde{\rho} \in (1, +\infty), \\ 1, & \text{if } \widetilde{\rho} = +\infty, \end{cases}$$

where C_d^i is the number of $i-$combinations from a given set of d elements, with $h \geq 1$ and $d - h \geq 1$.

Remark

From these theorems, there are essentially two categories:

- *if* $\widetilde{r} \in (0, +\infty]$ *(resp.* $\widetilde{\rho} \in (1, +\infty]$*), then the lower (resp. upper) transformed tail exhibits asymptotic dependence,*
- *if* $\widetilde{r} = 0$ *(resp.* $\widetilde{\rho} = 1$*), then there is asymptotic independence.*

↪ However in **asymptotic independent cases**, under some regular conditions, it is possible to quantify the rate of convergence toward 0 of $\widetilde{\lambda}_L^{(h,d-h)}(u)$ and $\widetilde{\lambda}_U^{(h,d-h)}(u)$ (see below).

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Theorem ($\widetilde{r}=0$; Transformed lower asymptotic independent case)

Let ϕ_0 be an initial valid generator and T be an admissible transformation.

Assume that

- $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, *with* $r_0 = 0$;
- $T \in \mathcal{RV}_{\tilde{a}}(0)$, *for some index* $\tilde{a} \in (0, +\infty)$.

Then one gets

$$\widetilde{\lambda}_L^{(h,d-h)} = \lim_{u \to 0^+} \widetilde{\lambda}_L^{(h,d-h)}(u) = \lim_{u \to 0^+} \frac{\tilde{\delta}^{(d)}(u)}{\tilde{\delta}^{(d-h)}(u)} = 0.$$

Assume that

- $\mu_{\phi_0} = -\phi_0'/\phi_0 \in \mathcal{RV}_{k_0-1}(\infty)$, *with* $k_0 \in [0, +\infty)$.

Then

$$\widetilde{\lambda}_L^{(h,d-h)}(u) = \frac{\tilde{\delta}^{(d)}(u)}{\tilde{\delta}^{(d-h)}(u)} \in \mathcal{RV}_{\widetilde{z}}(0), \qquad \text{with } \widetilde{z} = d^{k_0} - (d-h)^{k_0},$$

where $\tilde{\delta}^{(d)}$ denote the transformed diagonal section associated to the $d-$dimensional copula $\tilde{C}$.

Lower and upper asymptotic independent cases

Remark

Results about regular variation ($\mathcal{RV}(0)$) of transformed copula diagonal sections $\tilde{\delta}$ is useful to describe the behaviour of $\widetilde{\lambda}_L^{(h,d-h)}$ (in the case $\widetilde{r} = 0$).

***Conversely** results about regular variation ($\mathcal{RV}(1)$) of transformed copula diagonal sections $\tilde{\delta}$ will **not be sufficient** to characterize the regular variation of $\widetilde{\lambda}_U^{(h,d-h)}$ (in the case $\widetilde{\rho} = 1$).*

Embrechts and Hofert (2011) Alsina et al. (2006)
Fernández-Sánchez et al. (2015) Erdely, et al. (2014)

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$\hookrightarrow$ *For this reason, we need to introduce some different assumptions to obtain the desired result in the case $\widetilde{\rho} = 1$.*

Theorem ($\widetilde{\rho} = 1$; Transformed upper asymptotic independent case)

Let ϕ_0 be an initial generator and T be an admissible transformation.

Assume that

- $\tilde{\psi} \in \mathcal{RV}_{\tilde{\rho}}(1)$, *with* $\tilde{\rho} = 1$;
- *the associated* $\widetilde{\phi}$ *is a* d *times continuously differentiable generator.*

Then one gets

$$\widetilde{\lambda}_U^{(h,d-h)} = \lim_{u \to 1^-} \widetilde{\lambda}_U^{(h,d-h)}(u) = \lim_{u \to 1^-} \frac{r_d(u)}{r_{d-h}(u)} = 0,$$

with $r_d(u) = \sum_{i=1}^{d} (-1)^i \, C_d^i \, \tilde{\delta}^{(i)}(u)$ *and* $\tilde{\delta}^{(i)}$ *the transformed diagonal section.*

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$$\widetilde{\lambda}_U^{(h,d-h)}(u) \in \mathcal{RV}_h(1);$$

- *if* $\tilde{\psi}'(1) = 0$ *and the function* $\widetilde{L}(s) := s \, \frac{\mathrm{d}}{\mathrm{d}s}\{\frac{\tilde{\psi}(1-s)}{s}\}$ *is positive and* $\widetilde{L} \in \mathcal{RV}_0(0)$, *then we get*

$$\widetilde{\lambda}_U^{(h,d-h)}(u) \in \mathcal{RV}_0(1).$$

$$\widetilde{\lambda}_U^{(h,d-h)}(u) \in \mathcal{RV}_h(1)$$

- In this case the $\widetilde{\lambda}_U^{(h,d-h)}$ goes to zero as a regular variation function of index $h \geq 1$.
- Upper asymptotic independence for the transformed copula in a rather strong sense, a case which is called *near independence* in **Ledford and Tawn (1997)**.

– –

$$\widetilde{\lambda}_U^{(h,d-h)}(u) \in \mathcal{RV}_0(1)$$

- In this case $\widetilde{\lambda}_U^{(h,d-h)}$ goes to zero as a slowly variation function.
- we are on the boundary between asymptotic independence and asymptotic dependence. **Charpentier and Segers (2009)** called this second case *near asymptotic dependence*.

Transformed copulas using conversion function

Our goal:

$$\widetilde{C}_{T,C_0}(u_1,\ldots,u_d) = T \circ C_0 \left(T^{-1}(u_1),\ldots,T^{-1}(u_d)\right).$$

We consider transformations $T_{f,G} : [0,1] \to [0,1]$ such that

$$T_{f,G}(u) = \begin{cases} 0 & \text{if } u = 0, \\ G \circ f \circ G^{-1}(u) & \text{if } 0 < u < 1, \\ 1 & \text{if } u = 1, \end{cases}$$

where

- f is any continuous bijective increasing function, $f : \mathbb{R} \to \mathbb{R}$, and is said to be a *conversion function*
- the function G is thus chosen as a continuous and invertible c.d.f with support $\mathbb{R}$, i.e. such that $\forall x \in \mathbb{R}$, $G(x) \in (0,1)$.

Consider a transformed Archimedean copula, having generator

$$\tilde{\phi} = T_{f,G} \circ \phi_0 \text{ where } \phi_0 \text{ is an initial given generator of } C_0.$$

Transformed copulas using conversion function

- We choose here Hyperbolic conversion functions $f(x) = H_{m,h,p_1,p_2,\eta}(x)$ with

$$H_{m,\,h,\,p_1,\,p_2,\,\eta}(x) = m - h + (e^{p_1} + e^{p_2})\frac{x-m-h}{2} - (e^{p_1} - e^{p_2})\sqrt{\left(\frac{x-m-h}{2}\right)^2 + e^{\eta - \frac{p_1+p_2}{2}}},$$

with m, h, p_1, $p_2 \in \mathbb{R}$, and one smoothing parameter $\eta \in \mathbb{R}$.

↪ In particular, f has an asymptote $ax + b$ at $-\infty$ with $a = e^{p_1}$, and an asymptote $\alpha x + \beta$ at $+\infty$ with $\alpha = e^{p_2}$.

- Assume that $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, with $r_0 \in (0, +\infty)$, and $\psi_0 \in \mathcal{RV}_{\rho_0}(1)$, with $\rho_0 \in [1, +\infty)$.

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Illustration

Bivariate case:

$$\widetilde{\lambda}_L = 2^{-a^g\, r_0^{-1}}, \quad \widetilde{\lambda}_U = 2 - 2^{\alpha^\gamma \cdot \rho_0^{-1}}$$

So if these tail coefficients are given, then we can easily find $a = \mathrm{e}^{p_1}$ and $\alpha = \mathrm{e}^{p_2}$ as functions of $\widetilde{\lambda}_L$ and $\widetilde{\lambda}_U$.

In this case, hyperbola asymptotes' slopes are:

$$p_1 = \frac{1}{g} \ln\left(-r_0 \frac{\ln \widetilde{\lambda}_L}{\ln 2}\right) \quad \text{and} \quad p_2 = \frac{1}{\gamma} \ln\left(\rho_0 \frac{\ln(2 - \widetilde{\lambda}_U)}{\ln 2}\right).$$

Illustration

- C_0 is a Clayton copula with parameter $\theta > 0$.
- Assume that we want to obtain an arbitrarily chosen couple of target tail coefficients:
$$\widetilde{\lambda}_L = 0.25 \quad \widetilde{\lambda}_U = 0.75$$
.
- We choose here Hyperbolic conversion functions $f(x) = H_{m,h,p_1,p_2,\eta}(x)$.
- We use $\tilde{\psi} = \psi_0 \circ T^{-1}_{f,G}$, where $T^{-1}_{f,G} = G \circ f^{-1} \circ G^{-1}$.
- We consider $G = \mathrm{logit}^{-1}$, then $g = \gamma = 1$.

	chosen parameters				deduced parameters		tails coefficients			
	m	h	η	θ_0	p_1	p_2	$\lambda_{0,L}$	$\lambda_{0,U}$	$\widetilde{\lambda}_L$	$\widetilde{\lambda}_U$
A	0.5	0.9	-1	2	1.386	-1.133	0.707	0	0.25	0.75
B	0.5	-0.9	-1	2	1.386	-1.133	0.707	0	0.25	0.75
C	0.5	0.9	-2	4	2.079	-1.133	0.841	0	0.25	0.75
D	2	-0.9	1	0.2	-0.916	-1.133	0.031	0	0.25	0.75

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C	0.5	0.9	-2	4	2.079	-1.133	0.841	0	0.25	0.75
D	2	-0.9	1	0.2	-0.916	-1.133	0.031	0	0.25	0.75

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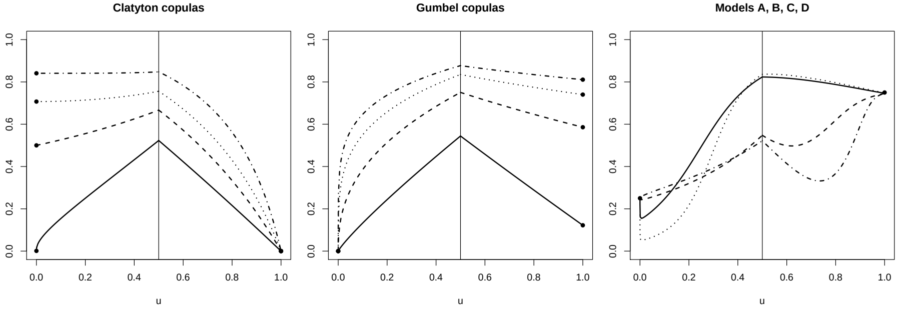
Illustration

We consider the **Tail concentration function**:

$$\lambda_{LU}(u) = \mathbf{1}_{\{u\leq 1/2\}}\lambda_L(u) + \mathbf{1}_{\{u>1/2\}}\lambda_U(u), \quad \text{for } u \in (0,1).$$



Illustration

As noticed in the bivariate case by Avéerous and Dortet-Bernadet (2004),

"*. . . many of the most commonly used parametric families of Archimedean copulas (such as the Clayton, Gumbel, Frank or Ali-Mikhail-Haq systems) possess strong dependence properties:* ***they have the SI or the SD property and are ordered at least by LTD order***."

(Stochastic Increasingness (SI), Stochastically Decreasingness (SD) and Left-Tail Decreasingness (LTD)).

It implies in particular that for many classical Archimedean copulas λ_{LU} **is non-decreasing on** $(0, 1/2)$ **and non-increasing on** $(1/2, 1)$.

This surprising shape of λ_{LU} function emphasis the large diversity of tail behaviours that can be reached by proposed transformed Archimedean copulas $\widetilde{C}$.

Some perspectives

- Derivation of a complete estimation procedure that uses these generators with given tail coefficients.
- Construction of a patched generator.

E. Di Bernardino, D. Rullière, ***On tail dependence coefficients of transformed multivariate Archimedean copulas*** (2016), Fuzzy Sets and Systems, Pages 89-112, Volume 284.

Thank you for your attention