Hölderian weak invariance principle for strictly stationary sequences

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Contents

Plan

Context

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T\colon \Omega \to \Omega$ be a bijective bi-measurable *measure-preserving* function.

- Let $f\colon \Omega \to \mathbb{R}$. The sequence $\left(f \circ T^j\right)_{j \geqslant 0}$ is a *strictly stationary sequence*, that is, the sequences $\left(f \circ T^j\right)_{j \geqslant 0}$ and $\left(f \circ T^{j+1}\right)_{j \geqslant 0}$ have the same distribution.

- We define $S_N(f) := \sum_{j=0}^{N-1} f \circ T^j$. In probability theory, an important problem is the understanding of the asymptotic behaviour of the sequence $(S_N(f))_{N \geqslant 1}$.

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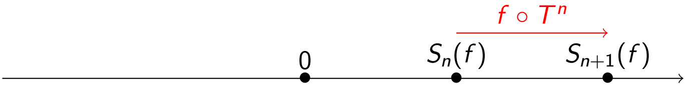

Figure : Illustration of $S_n(f)$

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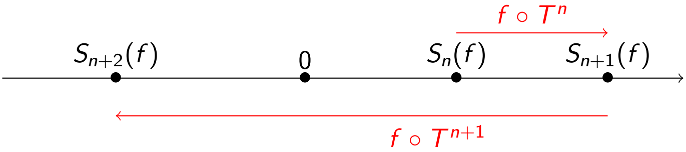


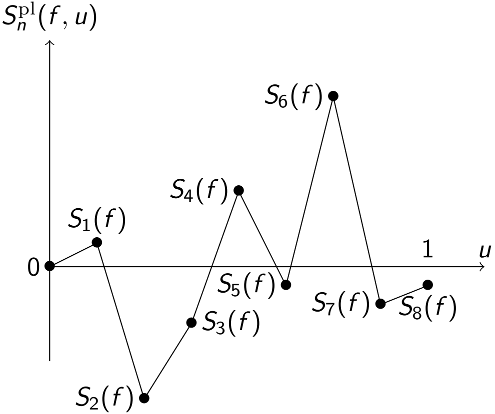
Figure : Illustration of $S_n(f)$

Partial sum process

Let $f\colon \Omega \to \mathbb{R}$. The random function $S_n^{\mathrm{pl}}(f, \cdot)$ is defined by

$$S_n^{\mathrm{pl}}(f,t) = \begin{cases} S_k(f) & \text{if } t = k/n, 0 \leqslant k \leqslant n; \\ \text{linear interpolation} & \text{if } t \in (k/n, (k+1)/n)\,. \end{cases}$$

Figure : The function $u \mapsto S_n^{\mathrm{pl}}(f,u)$ for $n = 8$



The invariance principle

We investigate the weak convergence of the sequence $S_n^{\mathrm{pl}}(f, \cdot)$ in some functional spaces.

- Let $C[0,1]$ denote the *space of continuous functions* on the unit interval endowed with the uniform norm. The random function $t \mapsto S_n^{\mathrm{pl}}(f, t)$ belongs to this space.

- **Donsker** (1952) showed that if $\left(f \circ T^j\right)_{j \geqslant 0}$ is independent, centered and $\mathbb{E}\left[f^2\right] = \sigma^2$, then for each $F\colon C[0,1] \to \mathbb{R}$ continuous and bounded,

$$\lim_{n \to +\infty} \mathbb{E}\left[F\left(n^{-1/2} S_n^{\mathrm{pl}}(f, \cdot)\right)\right] = \mathbb{E}\left[F\left(\sigma W\right)\right],$$

  where W a standard Brownian motion. When this convergence holds, we say that f satisfies the invariance principle in $C[0,1]$ or f satisfies the functional central limit theorem (FCLT) in $C[0,1]$.

Plan

Hölder spaces

In view of statistical applications, one may try to prove the convergence

$$\mathbb{E}\left[F\left(S_n^{\mathrm{pl}}(f,\cdot)/\sqrt{n}\right)\right] \to \mathbb{E}\left[F(W)\right] \tag{CF}$$

for the largest possible class of functionals F.

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$\mathcal{H}_\alpha[0,1]$	$\lVert x\rVert_\alpha := \sup_{s\neq t} \frac{\lvert x(s)-x(t)\rvert}{\lvert s-t\rvert^\alpha} + \lvert x(0)\rvert < +\infty$	No
$\mathcal{H}_\alpha^o[0,1]$		

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The paths of a standard Brownian motion belong to $\mathcal{H}_\alpha[0,1]$ for each $\alpha \in (0,1/2)$.

Thus we may try to prove the convergence (CF) for functionals which are continuous on Hölder spaces (approach followed by Lamperti).

It has potential statistical applications like change point detection.

The i.i.d. case

Let $\alpha \in (0, 1/2)$ and $p(\alpha) := (1/2 - \alpha)^{-1} \in (2, +\infty)$.

Lamperti (1962) showed that if $(f \circ T^j)_{j \geqslant 0}$ is i.i.d., centered and for each t, $c_1 \leqslant t^{p(\alpha)} \mu \{|f| > t\} \leqslant c_2$, then the sequence $(n^{-1/2} S_n^{\mathrm{pl}}(f))_{n \geqslant 1}$ is not tight in $\mathcal{H}_\alpha[0, 1]$.

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Theorem (Račkauskas, Suquet, 2003)

Let $\alpha \in (0, 1/2)$ and let $\left(f \circ T^j\right)_{j \geqslant 0}$ be an i.i.d. centered sequence with unit variance. Then the following conditions are equivalent:

1. $\lim_{t \to \infty} t^{p(\alpha)} \mu\{|f| > t\} = 0$;
2. *the sequence $(n^{-1/2} S_n^{\mathrm{pl}}(f))_{n \geqslant 1}$ converges to a standard Brownian motion in the space $\mathcal{H}_\alpha^o[0, 1]$.*

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Question

What about strictly stationary non-independent sequences?

General strategy

- The finite dimensional distributions characterize probability measures on $\mathcal{H}^o_\alpha$.
- The convergence of the finite dimensional distributions will always hold under our assumptions.
- Therefore, the main difficulty is to establish tightness of the sequence $\left(n^{-1/2}S_n^{\mathrm{pl}}(f)\right)_{n\geqslant 1}$ in $\mathcal{H}^o_\alpha$.
- Quantities like

$$\mu\left\{\sup_{1\leqslant i<j\leqslant n}\frac{|S_j(f)-S_i(f)|}{(j-i)^\alpha}>t\right\}$$

are not easy to handle compared with $\mu\left\{|S_n(f)|>t\right\}$.

An equivalent norm

Define for $j \geqslant 1$, $x\colon [0,1] \to \mathbb{R}$ and $t \in \left[2^{-j}, 1-2^{-j}\right]$,

$$\lambda_j(t,x) := x(t) - \frac{x\left(t+2^{-j}\right) + x\left(t-2^{-j}\right)}{2}.$$

The sequential norm is defined by

$$\|x\|_{\alpha}^{\mathrm{seq}} := \max\left\{ |x(0)|, |x(1)|, \sup_{j\geqslant 1} 2^{j\alpha} \max_{0\leqslant k<2^{j-1}} \left|\lambda_j((2k+1)2^{-j}, x)\right| \right\},$$

and is equivalent to $\|\cdot\|_{\alpha}$ (**Ciesielski**, 1960).

A tightness criterion

Following **Suquet** (1999), we obtain that a sequence of processes $(\xi_n(\cdot))_{n\geqslant 1}$ such that $\xi_n(0) = 0$ for each n is tight in $\mathcal{H}_\alpha^o$ if and only if for each positive ε,

$$\lim_{J\to+\infty} \limsup_{n\to+\infty} \mu\left\{\sup_{j\geqslant J} 2^{j\alpha} \max_{0\leqslant k<2^{j-1}} \left|\lambda_j((2k+1)2^{-j}, \xi_n)\right| > \varepsilon\right\} = 0.$$

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When $\xi_n(t):=n^{-1/2}S_n^{\mathrm{pl}}(f,t)$, we have the following sufficient condition for tightness in $\mathcal{H}_\alpha^o$: for each positive ε,

$$\lim_{J\to+\infty}\limsup_{n\to+\infty}\sum_{j=J}^{\log_2 n}2^j\mu\left\{\max_{1\leqslant i\leqslant n2^{-j}}|S_i(f)|>\varepsilon n^{1/2}2^{-\alpha j}\right\}=0.$$

Plan

Definition of martingales

Definition

Let $\mathcal{M}$ be a sub-σ-algebra of $\mathcal{F}$ such that $T\mathcal{M} \subset \mathcal{M}$ (in this way, $(T^{-i}\mathcal{M})_{i\geqslant 0}$ is a filtration). We say that $\left(m \circ T^j\right)_{j\geqslant 0}$ is a martingale differences sequence if the function m is $\mathcal{M}$-measurable, integrable and $\mathbb{E}\left[m \mid T\mathcal{M}\right] = 0$.

In this way, the sequence $\left(S_n(m)\right)_{n\geqslant 1}$ is a martingale with respect to the filtration $(T^{-i}\mathcal{M})_{i\geqslant 0}$.

The invariance principle in $C[0,1]$ and the law of the iterated logarithms hold for square integrable martingale differences sequences.

If $\left(m \circ T^j\right)_{j\geqslant 0}$ is a martingale differences sequence such that $m \in \mathbb{L}^p$, then the sequence $(\mathbb{E}\left|S_n(m)\right|^p / n^{p/2})_{n\geqslant 1}$ is bounded.

Moment inequalities do not suffice

Theorem (G., 2016)

Let $\alpha \in (0,1/2)$, $p(\alpha) := (1/2-\alpha)^{-1}$. There exists a strictly stationary sequence $(f \circ T^j)_{j \geqslant 0}$ such that

- *the finite dimensional distributions of $(S_n^{\mathrm{pl}}(f)/\sqrt{n})_{n \geqslant 1}$ converge to those of a standard Brownian motion,*
- *the sequence $\left(\mathbb{E}\left|S_n(f)\right|^{p(\alpha)}/n^{p(\alpha)/2}\right)_{n \geqslant 1}$ is bounded and*
- *the process $(S_n^{\mathrm{pl}}(f)/\sqrt{n})_{n \geqslant 1}$ is not tight in $\mathcal{H}_\alpha[0,1]$.*

The tail condition does not suffice

Let $\alpha \in (0, 1/2)$, $p(\alpha) := (1/2 - \alpha)^{-1}$.

Theorem (G., 2016)

Let $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system with positive entropy. There exists a function $m \colon \Omega \to \mathbb{R}$ and a σ-algebra $\mathcal{M}$ for which $T\mathcal{M} \subset \mathcal{M}$ such that

- *the sequence $(m \circ T^i)_{i \geqslant 0}$ is a martingale difference sequence with respect to the filtration $(T^{-i}\mathcal{M})_{i \geqslant 0}$;*
- *the convergence $\lim_{t \to +\infty} t^{p(\alpha)} \mu \{|m| > t\} = 0$ takes place;*
- *the sequence $(n^{-1/2} S_n^{\mathrm{pl}}(m))_{n \geqslant 1}$ is not tight in $\mathcal{H}_\alpha^o[0, 1]$.*

Sufficient condition for martingales

Let $\mathcal{M}$ a sub-σ-algebra of $\mathcal{F}$ such that $T\mathcal{M} \subset \mathcal{M}$. Let $\alpha \in (0, 1/2)$, $p(\alpha) := (1/2 - \alpha)^{-1}$.

Theorem (G., 2016)

Let $\left(m \circ T^j, T^{-j}\mathcal{M}\right)_{j \geqslant 0}$ be a strictly stationary martingale difference sequence. Assume that $t^{p(\alpha)}\mu\left\{|m| > t\right\} \to 0$ and $\mathbb{E}[m^2 \mid T\mathcal{M}] \in \mathbb{L}^{p(\alpha)/2}$. Then

$$n^{-1/2} S_n^{\mathrm{pl}}(m) \to \eta \cdot W \text{ in distribution in } \mathcal{H}_\alpha^o[0,1], \tag{HIP}$$

where η is independent of the Brownian motion W and $\eta = \lim_{\substack{n \to +\infty \\ \mathbb{L}^1}} n^{-1/2} \left(\mathbb{E}\left[S_n(m)^2 \mid \mathcal{I}\right]\right)^{1/2}$.

In particular, (HIP) *takes place if m belongs to $\mathbb{L}^{p(\alpha)}$.*

How to check the tightness criterion? (1)

We use a deviation inequality.

Theorem (Nagaev, 2003)

Let $q > 0$ *and let* $(S_n, \mathcal{F}_n)$ *be a martingale. Then*

$$\mu \left\{ \max_{1 \leqslant k \leqslant n} S_k \geqslant t \right\} \leqslant C(q) \int_0^1 Q(tu)\, u^{q-1} \mathrm{d}u,$$

where

$$Q(u) := \mu \left\{ \max_{1 \leqslant k \leqslant n} |X_k| > u \right\} + \mu \left\{ \left(\sum_{k=1}^{n} \mathbb{E}\left[X_k^2 \mid \mathcal{F}_{k-1} \right] \right)^{1/2} > u \right\}.$$

How to check the tightness criterion? (2)

If $\left(m \circ T^i\right)_{i \geqslant 0}$ is a martingale differences sequence with respect to $(T^{-i}\mathcal{M})_{i\geqslant 0}$, then

$$\mu\left\{\frac{1}{\sqrt{n}}\max_{1\leqslant i\leqslant n}|S_i(m)| > t\right\} \leqslant C(q)n\int_0^1 \mu\left\{|m| > tu\sqrt{n}\right\}u^{q-1}\mathrm{d}u + \\ + C(q)\int_0^{+\infty}\mu\left\{\mathbb{E}\left[m^2 \mid T\mathcal{M}\right] > u^2t^2\right\}\min\left\{u, u^{q-1}\right\}\mathrm{d}u.$$

To sum up

Let $\alpha \in (0, 1/2)$, $p(\alpha) := (1/2 - \alpha)^{-1}$.

Dependence of $(f \circ T^i)_{i \geqslant 0}$	**Integrability**	**Does** f **satisfy the HIP?**
Independent	For each t, $0 < c_1 \leqslant t^{p(\alpha)} \mu \{\lvert f \rvert > t\} \leqslant c_2$	No (Lamperti, 1962)
Independent	$t^{p(\alpha)} \mu \{\lvert f \rvert > t\} \to 0$	Yes (Račkauskas, Suquet, 2003)
Martingale differences	$t^{p(\alpha)} \mu \{\lvert f \rvert > t\} \to 0$	Not necessarily (G., 2016)
Martingale differences	$t^{p(\alpha)} \mu \{\lvert f \rvert > t\} \to 0$ and $\mathbb{E}\left[f^2 \mid T\mathcal{M}\right] \in \mathbb{L}^{p(\alpha)/2}$	Yes (G., 2016)

Martingale approximation

Theorem (G., 2015)

Let $\mathcal{M}$ be a sub-σ-algebra of $\mathcal{F}$ such that $T\mathcal{M} \subset \mathcal{M}$.
Let f be a centered $\mathcal{M}$-measurable random variable, $\alpha \in (0, 1/2)$ and $p(\alpha) := (1/2 - \alpha)^{-1}$. Assume that f satisfies one of the following conditions

- *Hannan type condition:*

$$\sum_{i \geqslant 0} \left\| \mathbb{E}\left[f \mid T^i \mathcal{M}\right] - \mathbb{E}\left[f \mid T^{i+1}\mathcal{M}\right] \right\|_{p(\alpha)} < +\infty$$

- *Maxwell and Woodroofe type condition:*

$$\sum_{n \geqslant 1} \frac{1}{n^{3/2}} \left\| \mathbb{E}\left[S_n(f) \mid \mathcal{M}\right] \right\|_{p(\alpha)} < +\infty.$$

Then

$$n^{-1/2} S_n^{\mathrm{pl}}(f) \to \eta \cdot W \text{ in distribution in } \mathcal{H}_\alpha[0,1],$$

where η is independent of the Brownian motion W.

Ideas of proofs (1)

We do not use deviation inequalities.
For Hannan's condition: we use the inequality

$$\mathbb{E}\left\|\frac{1}{\sqrt{n}}S_n^{\mathrm{pl}}\left(\mathbb{E}\left[f \mid T^i\mathcal{M}\right]-\mathbb{E}\left[f \mid T^{i+1}\mathcal{M}\right]\right)\right\|_{\alpha} \leqslant C(\alpha)\left\|\mathbb{E}\left[f \mid T^i\mathcal{M}\right]-\mathbb{E}\left[f \mid T^{i+1}\mathcal{M}\right]\right\|_{p(\alpha)}$$

and the fact that for each R, $\sum_{i=0}^{R}\mathbb{E}\left[f \mid T^i\mathcal{M}\right]-\mathbb{E}\left[f \mid T^{i+1}\mathcal{M}\right]$ admits a martingale-coboundary decomposition in $\mathbb{L}^{p(\alpha)}$.

Ideas of proofs (2)

For Maxwell and Woodroofe condition: an approximating martingale has been constructed for the invariance principle in $C[0,1]$ (**Peligrad, Utev**, 2005). This approximating martingale also works for the topology of $\mathcal{H}_\alpha$. The verification rests on the inequality

$$\mathbb{E}\left\|\frac{1}{\sqrt{n}}S_n^{\mathrm{pl}}(h)\right\|_\alpha \leqslant C(\alpha)\left\|h-\mathbb{E}\left[h \mid T\mathcal{M}\right]\right\|_{p(\alpha)} + C(\alpha)\sum_{j=0}^{\log_2 n-1} 2^{-j/2}\left\|\mathbb{E}\left[S_{2^j}(h) \mid \mathcal{M}\right]\right\|_{p(\alpha)}.$$