III-Construction of needlet .

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Part III :Framework of Dirichlet spaces (Th.Coulhon,P.Petrushev,G.K.)

Let (M, μ) be a connected, locally compact space with μ a borelian measure with support M.
We will try to show that a positive self-adjoint operator rich enough would help us to build a theory of regularity spaces on M. Let us first review some fact on positive self-adjoint operators, closed quadratic forms, semigroup and Markov semi-group.

Semi-group, positive operator and quadratic form.

Let $E = \mathbb{L}^(M, \mu)$.

- Self-adjoint contraction semigroup:

$$\forall t > 0, P_t = P_t^*,\ P_t \circ P_s = P_{t+s}, \|P_t(f)\| \le \|f\|,$$

$$\forall f \in E, \lim_{t \mapsto 0} \|P_t(f) - f\| = 0$$

Then the infinitesimal generator A with domain

$$D(A) = \{f \in E, \lim_{t \mapsto 0} \frac{P_t(f) - f}{t} = A(f)\} \text{ exists}$$

then $A = A^*,\ \forall f \in D(A),\ -\langle A(f), f\rangle \ge 0.$

• If A is a self-adjoint operator , with dense domain $D(A)$, and

$$\forall f \in D(A),\ -\langle A(f), f\rangle \geq 0.$$

Then by Hille-Yoshida theorem A is the generator of a self-adjoint contraction semigroup. There exists a spectral decomposition of identity associated to $-A$

$$Id = \int_0^\infty dE_\lambda,\ -A = \int_0^\infty \lambda dE_\lambda$$

$$\text{and} \quad P_t = e^{tA} = \int_0^\infty e^{-t\lambda} dE_\lambda$$

• If $-A$ is a positive self-adjoint operator , with dense domain $D(A)$ then we can define a quadratic form $\mathcal{E}(f,g)$:

$$D(\mathcal{E}) = D(\sqrt{-A}),\ \forall f,g \in D(\mathcal{E}),\ \mathcal{E}(f,g) = \langle \sqrt{-A}(f), \sqrt{-A}(g)\rangle$$

then $D(\mathcal{E})$ with norm $\|f\|_{\mathcal{E}}^2 = \|f\|^2 + \mathcal{E}(f,f)$ is complete. $\mathcal{E}$ is a closed positive quadratic form.

From quadratic form to positive self adjoint operator.

If $\mathcal{E}$ a closed positive quadratic form, with dense domain $D(\mathcal{E}) \subset E$. Then one can associate a positive self-adjoint operator L:

$$D(L) = \{f \in D(\mathcal{E}), \exists C < \infty \text{ such that } \forall g \in D(\mathcal{E}), \quad |\mathcal{E}(f,g)| \leq C\|g\|\}$$

For such f one define $L(f)$ as

$$\forall g \in D(\mathcal{E}), \quad \mathcal{E}(f,g) = \langle L(f), g \rangle$$

Then L is a positive self adjoint operator.

Practical definition of a positive self-adjoint operator: Friedrich extension.

Actually what one has usually a positive symmetric operator with dense domain : $A, D(A)$. Not self-adjoint. Then this operator could always extended to a positive self-adjoint operator

$$\overline{A} = (\overline{A})^*, D(A) \subset D(\overline{A}),\ \overline{A}|D(A) = A$$

(Friedrich extension). This is due to the fact that the quadratic form:

$$D(\mathcal{E}) = D(A),\ \mathcal{E}(f, g) = \langle A(f), g \rangle$$

is actually closable to $\overline{\mathcal{E}}, D(\overline{\mathcal{E}})$ and one use the previous procedure to build a self adjoint extension to A.

Beurling-Deny conditions and Markov semi-group.

For a positive operator $-A$ on $\mathbb{L}^2(\mu, M)$ with $\mathcal{E}$ and P_t the associate quadratique form and semi-group we have the equivalence (Beurling-Deny conditions):

1. If $u \in D(\mathcal{E})$ then,

$$(u \wedge 1)_+ \in D(\mathcal{E}), \text{ and } \mathcal{E}((u \wedge 1)_+, (u \wedge 1)_+) \leq \mathcal{E}(u, u)$$

2. $$\forall f \in D(A),\ (f-1)_+ \in D(A),\ \langle A(f), (f-1)_+ \rangle \leq 0$$

3. P_t is a SUBMARKOVIAN operator :

$$0 \leq f \leq 1, f \in \mathbb{L}^2 \Longrightarrow 0 \leq P_t f \leq 1.$$

Then P_t could be extended as a semigroup on $\mathbb{L}^p, 1 \leq p \leq \infty$ (with some precautions for $p = \infty$) and L_l, as the infinitesimal generator is defined for each p on some dense domain $D(L_{(p)}) \subset \mathbb{L}^p$.

"Gradient".

Under some further regularity, we can define (under the hypothesis :

$$\Gamma(f,f) = \frac{1}{2}A(f^2) - fA(f)$$

(for instance if $A(f) = f''$ then $\Gamma(f,f) = |f'|^2$. ($\Gamma(f,f)$ is the "square of the gradient " of f,) $\Gamma(f,f) \geq 0$ and one can check :

$$\int_M A(f)(x)f(x)d\mu(x) = -\int_M \Gamma(f,f)(u)d\mu(u)$$

(For a Laplacian on a Riemannian manifold : $\Gamma(f,f) = |\nabla(f)|^2$)
Then we define

$$\rho(x,y) = \sup_{\Gamma(\psi,\psi)\leq 1} (\psi(x) - \psi(y))$$

We suppose that ρ is a complete metric compatible with the original topology.
In the case of Riemannian geometry,$\rho(x,y)$ is the Riemannian distance.

Main hypothesis.

We will assume the following properties

1. We suppose that (M, ρ, μ) has the doubling property : $\exists d > 0$ (which plays the role of an upper dimension) such that:

$$\forall x \in M, r > 0,\ 0 < |B(x, 2r)| \leq 2^d |B(x, r)| < \infty$$

2. POINCARE INEQUALITY : $\exists C$ such that, $\forall\ B(x, r)$:

$$\int_{B(x,r)} |f(u) - f_{B(x,r)}|^2 d\mu(u) \leq C r^2 \int_{B(x,2r)} \Gamma(f, f)(u) d\mu(u)$$

$(f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(x) d\mu(x))$

Heat kernel bounds.

These two previous properties are equivalent to the following one :
The semi-group P_t is a positive symmetric kernel operator

$$P_t(f)(x) = \int_M P_t(x,y) f(y) d\mu(y)$$

and we have : $\exists C_1 > 0, C_2 > 0, c_1 > 0, c_2 > 0$, such that $\forall (x,y) \in M \times M, \quad \forall 0 < t \leq 1,$

$$\frac{C_1 e^{-c_1 \frac{\rho^2(x,y)}{t}}}{\sqrt{|B(x,\sqrt{t})||B(y,\sqrt{t}))}} \leq P_t(x,y) \leq \frac{C_2 e^{-c_2 \frac{\rho^2(x,y)}{t}}}{\sqrt{|B(x,\sqrt{t})||B(y,\sqrt{t})|}}$$

Moreover $\exists\ 0 < \alpha \leq 1$ such that $(x,y) \mapsto P_t(x,y)$ is lip-α.

Exemples

Compact Riemannian manifold
Riemannian manifold with Positive Ricci curvature....
Nilpotent Lie Group, compact Lie group,homogeneous spaces G/K, associated to sublaplacian.

Exemple :Jacobi

$$M = [-1, 1]; \quad d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta} dx = W(x)dx,$$

with $_1 < \alpha, \beta$. Let $\mathcal{P}$ the vector space of polynomials restricted to $[-1, 1]$. This vector space is dense in $\mathbb{L}^2(M, \mu)$. Let us define $\forall f \in \mathcal{P}$,

$$L(f) = \frac{1}{W(x)} \frac{d}{dx}((1-x^2)W(x)\frac{d}{dx}(f))$$

$$= (1-x^2)f" + [\beta - \alpha - (2+\beta-\alpha)x]f'$$

$$\int L(f)(x)f(x)W(x)dx = -\int (1-x^2)|f'(x)|^2 W(x)dx \leq 0$$

$$\mathcal{E}(f,g) = \int (1-x^2)f'(x)g'(x)W(x)dx$$

So : $\Gamma(f,f) = (1-x^2)|f'(x)|^2 = \frac{1}{2}\{L(f^2) - 2fL(f)\}$

On can verify the Beurling-Deny conditions so that e^{tL} is a subMarkovian semigroup.

Moreover $L(1) = 0$ so $P_t(1) = 1.$

So P_t is Markovian. Moreover the other regularity could be checked and :

$$\rho(x,y) = \arccos(x.y + \sqrt{1-x^2}\sqrt{1-y^2}) = |\arccos x - \arccos y|$$

$$= \int_x^y \frac{du}{\sqrt{1-u^2}}$$

Then we have the following behavior of the balls :

$$\mu(B(x,r)) \sim r(r^2+1-x)^{\alpha+\frac{1}{2}}(r^2+1+x)^{\beta+\frac{1}{2}}$$

So the measure of the balls is not the same when x is close to 0 or close to the boundary $1, -1$. Moreover

$$\frac{\mu(B(x,2r))}{\mu(B(x,r))} \leq 3.$$

So the doubling property is verified.

The eigenvectors are the Jacobi polynomials $P_k^{\alpha,\beta}(x) \in \mathcal{P}$.

$$A(P_k^{\alpha,\beta}) = -k(k+\alpha+\beta+1)P_k^{\alpha,\beta} = \lambda_k p_k$$

So if $q_k = q_k^{\alpha,\beta}$ are the normalized Jacobi polynomials, one can see that the operator is closable in the following way :

$$D(\overline{L}) = \{f = \sum \alpha_k q_k, \quad \sum |\alpha_k|^2 \lambda_k^2 < \infty\}$$

$$\overline{L}(f) = \sum \lambda_k \alpha_k q_k$$

One can check that $\overline{L}$ is selfadjoint, so in this case the Friedrich extension is the closure of the operator (L us essentially self-adjoint. In the general case it could happen that there is different non comparable self-adjoint extensions. The semigroup P_t is actually a kernel operator with a positive kernel:

$$P_t(x,y) = \sum_k e^{-\lambda_k t} q_k(x) q_k(y)$$

(As it is written it is not obvious that it positive.)

With some work one can verify the Poincare inequality.
So we have the Gaussian behavior of the heat kernel.

$$P_t(x,y) \leq C \frac{1}{\sqrt{\mu(B(x,\sqrt{t})\mu(B(y,\sqrt{t})}} e^{-c\frac{d(x,y)^2}{t}}$$

$$P_t(x,y) \geq C' \frac{1}{\sqrt{\mu(B(x,\sqrt{t})\mu(B(y,\sqrt{t})}} e^{-c'\frac{d(x,y)^2}{t}}$$

with the previous distance and measure of the balls.

Main result : Functional calculus.

Let Θ be an even function in $\mathcal{D}(\mathbb{R})$, and $\delta > 0$, the operator:

$$\Theta(\delta\sqrt{L}) = \int_0^\infty \Theta(\delta\sqrt{\lambda})dE_\lambda$$

is actually a kernel operator, and the kernel $\Theta(\delta\sqrt{L})(x,y)$ verifies REGULARITY property:

$$\bullet(x,y) \in M \times M \mapsto \Theta(\delta\sqrt{L})(x,y) \quad \text{is} \;\; Lip - \alpha$$

CONCENTRATION on the diagonal properties :

$$\bullet\ \forall s > 0, \delta > 0, |\Theta(\delta\sqrt{L})(x,y)| \leq C(\Theta, s)\frac{1}{\sqrt{|B(x,\delta)||B(y,\delta)|}}\frac{1}{(1+\frac{\rho(x,y)}{\delta})^s}$$

As a consequence, by Young Lemma :

$$\bullet\exists C,\ \forall\delta > 0, \forall f \in \mathbb{L}^p, \|\Theta(\delta\sqrt{L})f\|_p \leq C\|f\|_p$$

Spectral decomposition, Spectral space.

Let

$$L = \int_0^\infty \lambda dE_\lambda; \quad \sqrt{L} = \int_0^\infty \lambda dF_\lambda; \quad F_\lambda = E_{\lambda^2}$$

The operator F_λ is a kernel operator, with a real symetric non negative kernel, but NOT localised. Let us define :

$$\Sigma_\lambda = \{f \in \mathbb{L}^2, \quad F_\lambda(f) = f\}$$

And more we can extend this definition and we can define $\Sigma_\lambda^p,\ 1 \leq p \leq \infty$ and

$$1 \leq p \leq q \leq \infty \Longrightarrow \Sigma_\lambda^1 \subset \Sigma_\lambda^p \subset \Sigma_\lambda^q \subset \Sigma_\lambda^\infty;$$

These are the "low frequencies" spaces or Shannon spaces.

Σ_λ^p as a space of analytic vectors.

We have the following equivalence :

1. $f \in \Sigma_\lambda^p$
2. $f \in \cap_{k=1}^{\infty} D(L_{(p)}^k)$ and

$$\forall \nu > \lambda,\ \exists C_\nu > 0, \quad \forall k \in \mathbb{N},\ \|L^k(f)\|_p \leq C_\nu \nu^{2k} \|f\|_p$$

$(z \in \mathbb{C} \mapsto e^{-zL}(f) = \sum_{k\in\mathbb{N}} (-1)^k \frac{z^k L^k(f)}{k!}$ is a ($\mathbb{L}^p$ value) entire funtion of type exponential 2λ.)

Definition of spaces of distribution

Let us fix some $a \in M$.

$$\mathcal{S}(M) = \{\phi \in \cap_m D(L^m);$$

$$\forall l, n \in \mathbb{N},\ \mathcal{P}_{l,n}(\phi) = \sup_{x \in M} (1 + \rho(x,a))^l |L^n(\phi)(x)| < \infty\}$$

(This coincides, in the $\mathbb{R}^d$ case with the usual definition)
One can see :

$$\forall f \in \mathcal{D}(\mathbb{R}), f \text{ even}, \ \forall y \in M, \quad x \mapsto f(\sqrt{L})(x,y) \in \mathcal{S}.$$

The dual of $\mathcal{S}$ is the space of distribution $\mathcal{S}'$.

Littlewood-Paley decomposition

Let us define, for $1 < b < \infty$ the b- Littlewood-Paley functions :

$$\Phi_0 \geq 0,\ \Phi_0 \in \mathcal{D}(\mathbb{R}), \Phi \text{ even}$$

$$|u| \leq 1 \Longrightarrow \Phi_0(u) = 1,\ supp(\Phi_0 \subset \{|u| \leq b\}.$$

Moreover let us take Φ non increasing on $\mathbb{R}_+$.

$$\forall j \geq 1,\ \Phi_j(u) = \Phi_0(\frac{u}{b^j}) - \Phi_0(\frac{u}{b^{j-1}}) = \Phi_1(\frac{u}{b^{j-1}}).$$

So

$$\Phi_j \geq 0,\ \Phi_j \in \mathcal{D}(\mathbb{R}),\ supp(\Phi_j \subset \{b^{j-1} \leq |u| \leq b^{j+1}\}.$$

$$1 = \sum_j \Phi_j(u)$$

Then, due to the concentration properties :

$$\forall f \in \mathcal{S}', \ f = \sum_{j=0}^{\infty} \Phi_j(\sqrt{L}) f$$

The convergence is in the $\mathbb{L}^p$ sense if $1 \leq p < \infty$ if $f \in \mathbb{L}^p$ and uniform if f is unformly continuous and bounded (U.C.B.)

Spaces of low-frequency approximation.

For $f \in \mathbb{L}^p(M),\ 1 \le p \le \infty$, we define:

$$\sigma(t, f, p) = \inf_{g \in \Sigma_t^p} \|f - g\|_p$$

Then for $1 \le p \le \infty,\ 0 < q \le \infty,\ 0 < s < \infty$,

$$\|f\|_{B^s_{p,q}} \sim \|f\|_p + (\int_1^\infty (t^s \sigma(t, f, p))^q \frac{dt}{t})^{1/q} < \infty\}$$

Clearly, for $b > 1$, fixed, we have the discretized caracterisation :

$$\|f\|_{B^s_{p,q}} \sim \|f\|_p + \|b^{-js} \sigma(b^j, f, p)\|_{l_q(j)}$$

Littlewood-Paley definition of Besov and Triebel-Lizorkin spaces

Let Φ_j be a $b-$Littlewood Paley family of functions, and $f \in \mathcal{S}'$. Let $s \in \mathbb{R},\ 0 < q, p \leq \infty :$

$$f \in B^s_{p,q} \quad : \{f \in \mathcal{S}',\ (\sum_j (b^{js} \|\Phi_j(\sqrt{L})f\|_p)^q)^{1/q} = \|f\|_{B^s_{p,q}} < \infty\}$$

(usual modification for $q = \infty$) This is due to

$$\exists C, \forall 1 \leq p \leq \infty, \forall \delta > 0,\ \|\Phi(\delta\sqrt{L})f\|_p \leq C\|f\|_p$$

Triebel-Lizorkin spaces.

Let us define now:
Triebel-Lizorkin $F^s_{p,q}$: Let $s \in \mathbb{R},\ 0 < q \leq \infty,\ 0 < p < \infty :$

$$f \in F^s_{p,q} : \{f \in \mathcal{S}',\ \|(\sum_j |b^{js}\Phi_j(\sqrt{L})f(x)|^q)^{1/q}\|_p = \|f\|_{F^s_{p,q}} < \infty$$

(usual modification for $q = \infty$)
These definitions are independant of $b > 1$ and any related Littlewood-Paley family. All the related norms are equivalent.

Sobolev and Triebel-Lizorkin spaces $F^s_{p,q}$

Let us recall the definition of Sobolev space :
$s \in R,\ 1 \leq p \leq \infty$:

$$\|f\|_{H^p_s} = \|(I_d + L)^{s/2}(f)\|_p$$

Then

$$\forall s \in \mathbb{R}, \quad \forall 1 < p < \infty, \quad H^s_p = F^s_{p,2}$$

For $\quad s = 0, \quad \forall 1 < p < \infty, \quad H^s_0 = \mathbb{L}^p = F^0_{p,2}$

Besov Spaces as interpolation spaces..

Let $1 \leq p \leq \infty$, and $0 < s < k \in \mathbb{N}$

$$\|f\|_{H_p^k} = \|f\|_p + \|L_{(p)}^{k/2}(f)\|_p$$

$$B_{p,q}^s = [\mathbb{L}^p, H_p^k]_{\theta,q}, \quad s = \theta k$$

$$\|f\|_{[\mathbb{L}^p, D(L_{(p)}^k)]_{\theta,q}} \sim \|f\|_p + (\int_0^1 (t^{-\theta k}\|(tL)^k e^{tL}(f)\|_p)^q \frac{dt}{t})^{1/q}$$

(Jackson and Bernstein properties)

Injections

1.

$$\forall s \in \mathbb{R}, \quad \forall q \leq p, \quad B^s_{p,q} \subset F^s_{p,q}$$

$$\forall s \in \mathbb{R}, \quad \forall p \leq q, \quad F^s_{p,q} \subset B^s_{p,q}$$

2. $\forall s \in \mathbb{R}, \quad \forall 1 \leq p \leq p' \leq \infty,$

$$B^s_{p,q} \subset B^{s'}_{p',q}, \quad s - \frac{d}{p} = s' - \frac{d}{p'}.$$

$B^s_{\infty,\infty}$ and Lipschitz spaces

Let us recall : $\forall s > 0$,

$$Lip(s) = \{f, \quad \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^s} = \|f\|_{Lip(s)} < \infty\}$$

Then:

$$\forall 0 < s < \alpha, \; Lip(s) = B^s_{\infty,\infty}$$

Semi-group caracterization

1. Let $1 \leq p \leq \infty,\ 0 < s < \infty$. Let $m \in \mathbb{N}$ such that $0 < s < m$. Then

$$\|f\|_{B^s_{p,q}} \sim \|f\|_p + (\int_0^1 [t^{-s/2}\|(tL)^m e^{-tL} f\|_p]^q \frac{dt}{t})^{1/q}$$

2. Let $1 < p < \infty,\ 0 < s < \infty$. Let $m \in \mathbb{N}$ such that $0 < s < m$. Then

$$\|f\|_{F^s_{p,q}} \sim \|f\|_p + \|(\int_0^1 [t^{-s/2}|(tL)^m e^{-tL} f(x)|]^q \frac{dt}{t})^{1/q}\|_{\mathbb{L}_p}$$

With the usual modification for $q = \infty$.

Spectral space and sampling

$\delta-$**net.**
Let us recall that a $\delta-$net of a metric space (M, ρ) is a set $\mathcal{A} \subset M$ such that $\forall x \neq y,\ x, y \in \mathcal{A}$, we have $\rho(x, y) \geq \delta$.

Maximal $\delta-$**net.** Let $\mathcal{A}$ be a δ-net . If there is no δ-net $\mathcal{B}$, $\mathcal{B} \neq \mathcal{A}$, $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{A}$ is said maximal δ-net.
If $\mathcal{A}$ is a maximal δ-net, then :

$$\cup_{x \in \mathcal{A}} B(x, \delta) = M;$$

$$x, y \in \mathcal{A}, x \neq y \Longrightarrow B(x, \delta/2) \cap B(y, \delta/2) = \emptyset.$$

Sampling theorem

THEOREM :There exists $\gamma > 0$, only depending of the structural constant, such that $\forall \lambda > 0$ and and for any $\mathcal{A}_\delta$, a maximal $\delta-$net with $\delta = \frac{\gamma}{\lambda}$, we have :

$$\forall 1 \leq p \leq \infty, \quad \forall f \in \Sigma_\lambda^p,$$

$$(\sum_{\xi \in \mathcal{A}_\delta} |f(\xi)|^p |B(\xi, \delta|)^{1/p} \simeq \|f\|_p$$

(usual modification for $p = \infty$.)

Spectral spaces and cubature formula.

THEOREM : There exists $\gamma > 0$, only depending of the structural constant, such that $\forall \lambda > 0$ and $\mathcal{A}_\delta$, a maximal $\delta-$net with $\lambda\delta = \gamma$, it exist $(\mu_\xi^\lambda)_{\xi \in \mathcal{A}_\delta}$ positive weights such that :

$$\forall f \in \Sigma_\lambda^1, \quad \int_M f(x)dx = \sum_{\xi \in \mathcal{A}_\delta} \mu_\xi^\lambda f(\xi)$$

$$\frac{2}{3}|A_\xi| \leq \mu_\xi^\lambda \leq 2|A_\xi|$$

Where A_ξ is a partition associated to $\mathcal{A}_\delta$.

Frame

$$\text{As} \quad \frac{1}{2} \leq \sum_{j \geq 0} \Phi_j^2(x) \leq 1$$

by spectral theorem

$$\frac{1}{2}\|f\|_2^2 \leq \sum_{j \geq 0} \|\Phi_j(\sqrt{L})(f)\|_2^2 \leq \|f\|_2^2$$

So using the sampling theorem, we get , for $\mathcal{A}_j = \mathcal{A}_{\gamma b^{-j}}$,

$$\frac{1}{4}\|f\|^2 \leq \sum_j \sum_{\mathcal{A}_j} |\langle f, \psi_{j,\xi}\rangle|^2 \leq 2\|f\|_2^2$$

$$\text{where :} \quad \psi_{j,\xi}(x) = \sqrt{|B(\xi, b^{-j})|}\Phi_j(\sqrt{L})(x,\xi).$$

The previous result means exactly that : $(\phi_{j,\xi})_{j\in\mathbb{N},\xi\in\mathcal{A}_j}$ is a frame

Properties of $\psi_{j,\xi}(x)$.

For a suitable choice of Φ and b :

1. $\mathbb{L}^p-$norm control : $\forall 0 < p \leq \infty, \forall j \in \mathbb{N},\ \forall \xi \in \mathcal{A}_j,$

$$\|\psi_{j,\xi}\|_p \simeq |B(\xi, b^{-j})|^{\frac{1}{p}-\frac{1}{2}}$$

2. $\psi_{j,\xi}$ is "almost" supported by $B(\xi, b^{-j})$:

   For $0 < \beta < 1$ $\exists C, \kappa > 0,$ such that $\forall j \in \mathbb{N},\ \xi \in \mathcal{A}_j$

$$|\psi_{j,\xi}(x)| \leq C \frac{1}{\sqrt{|B(\xi, b^{-j})|}} e^{-\kappa(b^j \rho(x,\xi))^\beta}$$

   (Exponential concentration).

3. Spectral localisation : $\psi_{j,\xi} \in \Sigma_{b^{j-1}, b^{j+1}}$

Second main result :Existence of a good dual frame

One can built a family $(\tilde{\psi}_{j,\xi})_{\xi\in\mathcal{A}_j}$, which is a dual frame (not THE dual frame!) to the previous one, with the same properties:

- Splitting property: $\forall j \in \mathbb{N}$,

$$\Phi_j(\sqrt{L})(x,y) = \sum_{\mathcal{A}_j} \overline{\psi_{j,\xi}(y)}\tilde{\psi}_{j,\xi}(x) = \sum_{\mathcal{A}_j} \psi_{j,\xi}(x)\overline{\tilde{\psi}_{j,\xi}(y)})$$

- $\|\tilde{\psi}_{j,\xi}\|_p \simeq |B(\xi, 2^{-j})|^{\frac{1}{p}-\frac{1}{2}}$
- $|\tilde{\psi}_{j,\xi}(x)| \leq C\dfrac{1}{\sqrt{|B(\xi, b^{-j})|}} e^{-\kappa(b^j\rho(x,\xi))^\beta}$
- Spectral localisation $\quad \tilde{\psi}_{j,\xi} \in \Sigma_{b^{j-2},b^{j+2}}$

Frame characterization of Besov and Triebel spaces.

Let b suitably choosen.

- Littlewood-Paley .

$$\forall f \in \mathcal{S}', \quad f = \sum_{j=0}^{\infty} \Phi_j(\sqrt{L}) f$$

- Frame decomposition

$$f = \sum_j \sum_{\xi \in \mathcal{A}_j} \langle f, \psi_{j,\xi} \rangle \tilde{\psi}_{j,\xi}(x)$$

We can exchange ψ and $\tilde{\psi}$

Concentration property.

Due to the concentration properties of the $\psi_{j,\xi}$ and $\tilde{\psi}_{j,\xi}$ we have :

$$\exists C < \infty,\ \forall j \in \mathbb{N}, \quad \sum_{\xi \in \mathcal{A}_j} \|\psi_{j,\xi}\|_1 |\tilde{\psi}_{j,\xi}(y)| \leq C$$

We can exchange ψ and $\tilde{\psi}$

Sparse caracterization of Besov space.

So: using

- $\forall s \in \mathbb{R},\ 0 < p, q \leq \infty$

$$[\sum_j (b^{js} (\sum_{\xi \in \mathcal{A}_j} |\langle f, \psi_{j,\xi} \rangle|^p \|\tilde{\psi}_{j,\xi}\|_p^p)^{\frac{1}{p}})^q]^{1/q} \sim \|f\|_{B^s_{p,q}}$$

We can exchange ψ and $\tilde{\psi}$

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Caracterization of $F^s_{p,q}$. $\forall s \in \mathbb{R},\ 0 < q \leq \infty,\ 0 < p < \infty$:

$$\|\{\sum_j [b^{js} \sum_{\xi \in \mathcal{A}_j} |\langle f, \psi_{j,\xi} \rangle||\tilde{\psi}_{j,\xi}(x)|]^q\}^{1/q}\|_p \sim \|f\|_{F^s_{p,q}}$$

We can exchange ψ and $\tilde{\psi}$

Compact case.

The following properties are equivalent :

- $Diam(M) < \infty$ ($\iff \mu(M) < \infty \iff M$ is compact)
- $\mathbb{L}^2(M) = \bigoplus_k \mathcal{H}_{\lambda_k}, \quad \mathcal{H}_{\lambda_k} = ker(L - \lambda_k I_d); \quad dim(\mathcal{H}_{\lambda_k}) < \infty$

$$\bullet \quad \forall r > 0 \quad \int \frac{1}{|(B(x,r)|} d\mu(x) < \infty$$

$$\bullet \forall \lambda > 0, \ \forall 1 \leq p \leq \infty \quad \Sigma_\lambda^1 = \Sigma_\lambda^p = \Sigma_\lambda^\infty = \bigoplus_{\sqrt{\lambda_k} \leq \lambda} \mathcal{H}_{\lambda_k}$$

$\bullet \forall t > 0, \ e^{-tL}$ is an Hilber-Schmidt operator

$\bullet \forall t > 0, \ e^{-tL}$ is a trace class operator

If this is realized , and if $N(\delta, M)$ is the covering number of M (or the cardinal of a maximal $\delta-$net):

$$dim(\bigoplus_{\lambda_k \le t^{-1}} \mathcal{H}_{\lambda_k}) = dim(\Sigma_{\frac{1}{\sqrt{t}}}) \sim \int \frac{1}{|B(x,\sqrt{t})|} d\mu(x) \sim N(\sqrt{t}, M)$$

$$\sim \int_M P_t(x,x) d\mu(x) = \int_M \int_M P_{t/2}(x,y)^2 d\mu(x) d\mu(y)$$
$$= \sum_k e^{-\lambda_k t} dim(\mathcal{H}_{\lambda_k}) = Tr(e^{-tL}) = \|e^{-t/2L}\|_{HS}^2$$

In particular, in the compact Riemannian case of dim n, without boundary :

$$dim(\bigoplus_{\sqrt{\lambda_k} \le \lambda} \mathcal{H}_{\lambda_k}) \sim \int \frac{1}{|B(x,\lambda^{-1})|} d\mu(x) \sim N(\lambda^{-1}, M) \sim \lambda^n$$

Regularity of Gaussian Process in geometrical framework.

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Gaussian process.

Let (M, d) a compact metric space.
Let (Ω, P) a probability space.
$(Z_x(\omega))_{x \in X}$ a centered Gaussian process:

$$\forall x_1, .., x_n \in X; \quad \lambda_1, .., \lambda_n \in \mathbb{R},$$

$$\sum \lambda_i Z_{x_i}(\omega) \sim N(0, \sum_{i,j} \lambda_i \lambda_j K(x_i, x_j))$$

where $\qquad K(x, y) = \mathbb{E}(Z_x Z_y)$ is the covariance kernel

$K(x, y)$ is a real, positive definite function. i.e.

$$K(x, y) = K(y, x) \in \mathbb{R}, \text{ and}$$

$$\forall x_1, .., x_n \in X; \quad \lambda_1, .., \lambda_n \in \mathbb{R}, \sum_{i,j} \lambda_i \lambda_j K(x_i, x_j) \geq 0$$

Reciprocally, if $K(x,y)$ is real continuous, positive definite :

$$K(x,y) = \sum_k \nu_k \phi_k(x)\phi_k(y)$$

where $$\int K(x,y)\phi_k(y)d\mu(y) = \nu_k\phi_k(x)$$

Then : $$Z_x(\omega) = \sum_k \sqrt{\nu_k}\phi_k(x)B_k(\omega)$$

where B_k is sequence of iid $N(01)$ $R.V.$. Then $Z_x(\omega)$ is Gaussian Process with covariance K.

Question : How one can decide the kind of regularity of the "trajectory: $x \mapsto Z_x(\omega)$ for almost all ω at least for a suitable version of $Z_x(\omega)$?
Let us recall the famous result of Kolmogoroff :

THEOREM: The process $Z_x(\omega)$ had a continuous modification if it exists $0 < p$, $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+, \psi(0) = 0$ continuous non decreasing, and $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that :

$\mathbb{E}|Z_x - Z_y|^p \leq \psi(d(x, y))$, and if $D(t, X)$ is the covering number

$$\int_0^1 \frac{D(t, X)\psi(2t)}{f(t)^p} dt < \infty, \quad \int_0^1 \frac{f(x)}{x} dx < \infty.$$

Actually we want to describe the regularity of the process $Z_x(\omega)$ directly from the covariance function $K(x, y)$

Random field and geometry.

We suppose that we are in the previous framework.
•(X, μ, d) is a compact metric space, and we suppose that the regularity spaces : Sobolev, Besov, Lipschitz are related to some Positive operator L with all the properties:

• : L is a positive self-adjoint operator determine associate to a regular, and local Dirichlet space with an associated "gradient square" $\Gamma(f,g) : \forall f, g \in D(L)\ \int L(f) g d\mu = \int \Gamma(f,g) d\mu$.

• : $d(x,y) = \sup_{\Gamma(f,f)\leq 1} \psi(x) - \psi(y)$

• Doubling condition: $\mu(B(x,2r)) \leq 2^d \mu(B(x,r))$.

• Poincare Inequality :

$$\inf_{\lambda} \int_{B(x,r)} (f-\lambda)^2 d\mu \leq C r^2 \int_{B(x,r)} \Gamma(f,f) d\mu$$

Compact case.

The following properties are equivalent :

- $Diam(M) < \infty$ ($\Longleftrightarrow \mu(M) < \infty \Longleftrightarrow M$ is compact)

- $\mathbb{L}^2(M) = \bigoplus_k \mathcal{H}_{\lambda_k},\ \mathcal{H}_{\lambda_k} = ker(L - \lambda_k I_d);\ dim(\mathcal{H}_{\lambda_k}) < \infty$

$$\bullet \forall \lambda > 0,\ \forall 1 \leq p \leq \infty \quad \Sigma^1_\lambda = \Sigma^p_\lambda = \Sigma^\infty_\lambda = \bigoplus_{\sqrt{\lambda_k} \leq \lambda} \mathcal{H}_{\lambda_k}$$

•If, $N(\delta, M)$ is the covering number of M (or the cardinal of a maximal $\delta-$net): (Peter-Weyl type result)

$$dim(\bigoplus_{\lambda_k \leq t^{-1}} \mathcal{H}_{\lambda_k}) \sim \int \frac{d\mu(x)}{|B(x, \sqrt{t})|} \sim N(\sqrt{t}, M) \lesssim t^{\frac{d}{2}}$$

$$\mathbb{L}^2 = \bigoplus \mathcal{H}_{\lambda_k}, \quad P_t(x,y) = \sum_k e^{-\lambda_k t} P_{\lambda_k}(x,y).$$

$$P_{\lambda_k}(x,y) = \sum_{i=1}^{dim(\mathcal{H}_{\lambda_k})} e_i^k(x) e_i^k(y),$$

$P_t(x,y)$ has a Gaussian behavior. Moreover $e^{-tL}1 = 1$ which is equivalent to $L1 = 0$.

So we have Sobolev spaces, Besov spaces, Lipschitz spaces. Let us recall

$$\forall 0 < s \leq 1, \; Lip_s \subset B^s_{\infty,\infty}$$

and for some $0 < \alpha, \forall 0 < s < \alpha, \; B^s_{\infty,\infty} \subset Lip_s$. (Actully, in the Riemannian case $\alpha = 1$.

Subordination to the geometry.

Here is the main hypothesis:
We focus on continuous definite positive kernel, subordinate to the spectral decomposition :

$$K(x,y) = \sum_k \sum_{j=1}^{dim(\mathcal{H}_{\lambda_k})} \nu_k^j e_k^j(x) e_k^j(y)$$

i.e. the eigenfunctions of L are eigenfunctions of K. Actually, this is equivalent to $KL = LK$. where K is the kernel operator:

$$f \mapsto Kf(x) = \int K(x,y) f(y) d\mu(y)$$

Let us study the regularity of:

$$Z_x(\omega) = \sum_k \sum_{i=1}^{dim(\mathcal{H}_{\lambda_k})} \sqrt{\nu_k^j} e_k^j(x) B_k^j(\omega); \quad B_k^j, \ i.i.d. \quad N(0,1).$$

Regularity theorem.

THEOREM

1. Let us suppose that $\exists 0 < s$ such that :

$$\sup_{x\in M} \|K(x,.)\|_{B^s_{\infty,\infty}} \leq C < \infty.$$

   Then for almost all $\omega \in \Omega,\ x \mapsto Z_x(\omega) \in B^\alpha_{\infty,1},\ \alpha < \frac{s}{2}$

   (Let us observe that $B^\alpha_{\infty,1} \subseteq B^\alpha_{\infty,\infty},\ B^\alpha_{\infty,1}$ is separable and $B^\alpha_{\infty,\infty}$ is not separable.)

2. Conversely If $\exists \alpha > 0$ such that $Z_x(\omega) \in B^\alpha_{\infty,\infty}$ for almost all ω, then

$$\sup_{x\in M} \|K(x,.)\|_{B^{2\alpha}_{\infty,\infty}} \leq C < \infty.$$

Exemple:

Let us suppose that

$$|K(x,y) - K(x,y')| \le Cd(y,y')^s, \text{ for some } \ 0 < s \le 1$$

So, as $Lip_s \subset B^s_{\infty,\infty}$, the theorem implies

$$\text{for almost all} \quad \omega \in \Omega, \ x \mapsto Z_x(\omega) \in B^\alpha_{\infty,1}, \ \alpha < \frac{s}{2}$$

But $B^\alpha_{\infty,\infty} = Lip_\alpha$, if $\alpha < \alpha_0$. Actually $\alpha_0 = 1$ if X is a compact Riemannian manifold.

Sketch of the proof.

To simplify the notations we write:

$$K(x,y) = \sum \nu_k u_k(x) u_k(y); \quad L(u_k) = \lambda_k u_k;$$

$$Z_x(\omega) = \sum \sqrt{\nu_k} B_k(\omega) u_k(x)$$

u_k is an orthonormal basis of eigenfunctions of L.

$$\text{Let a Littlewood-Paley decomposition :} \quad 1 = \sum_{j \geq 0} \psi_j(x),$$

$$Supp(\psi_0) \subset \{|\xi| \leq 2\}; \; \forall j \geq 1, \; Supp(\psi_j) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$$

We have to prove:

$$\|Z_.(\omega)\|_{B^\alpha_{\infty,1}} \sim \sum_{j \geq 0} 2^{j\alpha} \|\psi_j(\sqrt{L})(Z_.(\omega))\|_\infty < \infty, \; a.e.$$

It is enough to prove

$$\mathbb{E}[\sum_{j \geq 0} 2^{j\alpha} \|\psi_j(\sqrt{L})(Z_.(\omega))\|_\infty] = \sum_{j \geq 0} 2^{j\alpha} \mathbb{E}[\|\psi_j(\sqrt{L})(Z_.(\omega))\|_\infty] < \infty.$$

If $\mathcal{A}_j$ a maximal $\gamma 2^{-j-1}-$net. We have $Card(\mathcal{A}_j) \lesssim 2^{jd}$.
We have (as $\Psi_j(\sqrt{L})(f) \in \Sigma_{2^{j+1}}$):

$$\mathbb{E}[\|\Psi_j\sqrt{L})(Z_.(\omega)\|_\infty] \sim \mathbb{E}[\sup_{\xi\in\mathcal{A}_j} |\Psi(2^{-j}\sqrt{L})(Z_.(\omega)(\xi)|] =$$

$$\mathbb{E}[\sup_{\xi\in\mathcal{A}_j} |\sum_{2^{j-1}\leq\sqrt{\lambda_k}\leq 2^{j+1}} \Psi_j(\sqrt{\lambda_k})\sqrt{\nu_k}u_k(\xi)B_k(\omega)|]$$

Let us recall:

Pisier inequality If $(X_i)_{i\in\mathcal{A}}$ are centered Gaussian R.V.and $\sigma^2 \geq \mathbb{E}(X_i^2)$, $\forall i$ then

$$\mathbb{E}(\sup |X_i|) \leq \sigma\sqrt{2\log(2card(\mathcal{A})}$$

But

$$\mathbb{E}[\sum_{2^{j-1}\leq\sqrt{\lambda_k}\leq 2^{j+1}} \Psi_j(\sqrt{\lambda_k})\sqrt{\nu_k}u_k(\xi)B_k(\omega)]^2 =$$

$$\sum_{2^{j-1}\leq\sqrt{\lambda_k}\leq 2^{j+1}} \Psi_j^2(\sqrt{\lambda_k})\nu_k u_k^2(\xi)] \leq \sup_{x\in M} \sum_{2^{j-1}\leq\sqrt{\lambda_k}\leq 2^{j+1}} \nu_k u_k^2(x)]$$

So:

$$\sum_{j\geq 0} 2^{j\alpha}\mathbb{E}[\|\psi_j(\sqrt{L})(Z_.(\omega))\|_\infty] \lesssim \sum_{j\geq 0} 2^{j\alpha}\sqrt{j}\{\sup_{x\in M} \sum_{2^{j-1}\leq\sqrt{\lambda_k}\leq 2^{j+1}} \nu_k u_k^2(x)\}^{\frac{1}{2}}$$

But one can check :

$$\sup_{x\in M} \|K(x,.)\|_{B^s_{\infty,\infty}} < \infty \Longleftrightarrow \exists C' < \infty, \quad \sup_{x\in M} \sum_{2^{j-1}\leq\sqrt{\lambda_k}\leq 2^{j+1}} \nu_k u_k^2(x)] \leq C' 2^{-js}$$

So $Z_x(\omega) \in B^\alpha_{\infty,1}$, a.s. if $\alpha < \frac{s}{2}$.

Some more result.

Under the hypothesis of the theorem :

1. (Wiener measure) On $B^{\alpha}_{\infty,1}$, there exists a unique Borel measure Q_α, such that:

$$\delta_x : \omega \in B^{\alpha}_{\infty,1} \mapsto \omega(x)$$

   is a centered Gaussian process and

$$K(x,y) = \int_{B^{\alpha}_{\infty,1}} \delta_x(\omega)\delta_y(\omega) dQ_\alpha(\omega) = \mathbb{E}(\delta_x \delta_y)$$

2. If $\mathbb{H}_K$ is the RKHS associated to $K(x,y)$. Then

$$\text{Moreover } \mathbb{H}_K \subseteq B^{\frac{s}{2}}_{\infty,\infty} \Longleftrightarrow \sup_{x \in M} \|K(x,.)\|_{B^{s}_{\infty,\infty}} < \infty$$

$$\mathbb{H}_K = \{f : M \to \mathbb{R} : f(x) = \sum_k \alpha_k \sqrt{\nu_k} u_k(x),\ \alpha_. \in l_2\}$$

$$\|f\|^2_{\mathbb{H}_K} = \|\alpha_.\|^2_2$$

Ex : The Brownian motion.

$$M = [0,1],\ K(x,y) = \frac{x+y-|x-y|}{2} = x \wedge y$$

Computing the eigen vector and eigen number of the associate kenel operator:

$$K(x,y) = \sum_k \frac{1}{((k+\frac{1}{2})\pi)^2} 2\sin((k+\frac{1}{2})\pi x)\sin(k+\frac{1}{2})\pi y)$$

$$Z_x(\omega) = \sum_k \frac{1}{(k+\frac{1}{2})\pi}\sqrt{2}\sin((k+\frac{1}{2})\pi x)B_k(\omega); \quad B_k \sim N(0,1),\ i.i.d.$$

Now we have the following "bad" Dirichlet space : (with Neumann-Dirichlet conditions)

$$A(f) = f",\ D(A) = C^2(]0,1[, \cap C^1[0,1],\ f(0) = f'(1) = 0.$$

$$\int_0^1 A(f)(x)f(x)dx = -\int_0^1 f'^2(x)dx$$

$$A(\sin((k+\frac{1}{2})\pi.)(x) = -((k+\frac{1}{2})\pi)^2 \sin((k+\frac{1}{2})\pi x)$$

$$\rho(x,y) = \sup_{|f'|\leq 1} f(x) - f(y) = |x-y|;$$

The Poincare and the doubling properties are obvious.

$$\text{Clearly} \quad |K(x,y) - K(x,y')| \leq |y-y'|$$

$$\text{so} \quad x \mapsto Z_x(\omega) \in Lip(s)([0,1]),\ s < \frac{1}{2}; \omega - a.s.$$

$$\mathbb{E}(|Z_x - Z_y|^2) = \psi(x,y) =$$

$$K(x,x) + K(y,y) - 2K(x,y) = |x-y|$$

But unfortunately 1 Does not belong to $D(A)$. and the semi-group is not Markov

Let us go through the circle.

By Fourier serie development :

$$x \in [-1,1], \quad |x| = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n\in\mathbb{N}} \frac{\cos(2n+1)\pi x}{(2n+1)^2}$$

so : $\quad x, y \in [-1,1] \quad |x-y| \wedge (2-|x-y|) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n\in\mathbb{N}} \frac{\cos((2n+1)\pi(x-y))}{(2n+1)^2}$

$$K(x,y) = \frac{1}{2} - |x-y| \wedge (2-|x-y|) = \frac{4}{\pi^2} \sum_{n\in\mathbb{N}} \frac{\cos((2n+1)\pi(x-y))}{(2n+1)^2} =$$

$$\frac{4}{\pi} \sum_{n\in\mathbb{N}} \frac{\cos((2n+1)\pi x)\cos((2n+1)\pi y)}{(2n+1)^2} + \frac{4}{\pi} \sum_{n\in\mathbb{N}} \frac{\sin((2n+1)\pi x)\sin((2n+1)\pi y)}{(2n+1)^2}$$

So obviously $K(x,y)$ is P.D. and

$$\psi(x,y) = K(x,x) + K(y,y) - 2K(x,y) = 2[|x-y| \wedge (2-|x-y|)]$$

Associated Dirichlet space.

Now let let us look to the Dirichlet associated to :

$$f \in C^2(]-1,1[) \cap C^1[-1,1],\ A(f) = f\text{''},$$

$$f(-1) = f(1);\ f'(-1) = f'(1).$$

$$\int_{-1}^{1} A(f)(x)g(x)dx = -\int_{-1}^{1} f'(x)g'(x)dx,$$

$\forall x, y \in [-1,1]$,

$$|x-y| \wedge (2-|x-y|) = \inf_{|f'|\leq 1, f(-1)=f(1), f'(-1)=f'(1)} f(x) - f(y)$$

The eigen-vectors associated are clearly $(\cos k\pi x)_{k\in\mathbb{N}}, (\sin k\pi x)_{k\in\mathbb{N}^*}$.

. With respect to the metric $\rho(x,y) = |x-y| \wedge (2-|x-y|)$, Poincaré and the doubling property are easily obtained. So the Gaussian process $Z_x(\omega))_{x\in[-1,1]}$ associated to $\frac{1}{2}K(x,y)$ is a Brownian field with respect to ρ.

When we restrict $\forall x, y \in [0,1], \rho(x,y) = |x-y|$.

So if we look to $W_x = Z_x - Z_0$ restricted to $x \in [0,1]$ we get the classical Brownian Motion $(W_0 = 0, \mathbb{E}(W_x - W_y)^2 = |x-y|)$ and we get its regularity as a by-product.

Gaussian field, Positive Definite (P.D.) functions.

Let X a set. A gaussian field on X is a family of real random variables (R.V.) $(Z_x(\omega))_{x\in X}$ such that

$$\forall n \in \mathbb{N}^*,\ \forall\, x_1, ., x_n \in X,\ \forall\, \lambda_1, ., \lambda_n \in \mathbb{R},\ \sum_{i=1}^{n} \lambda_i Z_{x_i}$$

is a centered Gaussian R.V.. The "law " of the process is completely determined (because of Gaussianity) by the covariance kernel

$$K(x, y) = \mathbb{E}(Z_x Z_y)$$

$K(x, y)$ is **real positive definite (P.D.)** : $\quad K(x, y) = K(y, x) \in \mathbb{R},$

$$\forall\, x_1, ., x_n \in X,\ \forall\, \lambda_1, ., \lambda_n \in \mathbb{R},\ \sum_{i=1}^{n} \lambda_i \lambda_j K(x_i, x_j) \geq 0$$

Reciprocally to a real P.D. function $K(x, y)$ on $X \times X$ there always exists a Gaussian process Z_x such that $K(x, y) = \mathbb{E}(Z_x Z_y)$.

Gaussian field, Negative Definite (N.D.) functions.

To each D.P $K(x,y)$, (or Gaussian field Z_x) one can associate

$$\Psi(x,y)(=\psi_K(x,y)) = \mathbb{E}(Z_x - Z_y)^2 = K(x,x) + K(y,y) - 2K(x,y)$$

$\psi(x,y)$ is **Real Negative Definite (N.D.)** : $\psi(x,y) = \psi(y,x) \in \mathbb{R}, \ \psi(x,x) \equiv 0$

$$\forall\, x_1,.,x_n \in X, \ \forall\, \lambda_1,.,\lambda_n \in \mathbb{R}, \ \sum_{i=1}^{n} \lambda_i = 0 \Longrightarrow \sum_{i=1}^{n} \lambda_i \lambda_j \psi(x_i,x_j) \leq 0$$

Let $\psi(x,y) = \psi(y,x) \in \mathbb{R}, \ \psi(z,z) \equiv 0$. Let $e \in X$. Let us define :

$$K_e^{\psi}(x,y) = \frac{1}{2}(\psi(x,e) + \psi(y,e) - \psi(x,y)$$

Then $\psi \quad N.D \quad \Longleftrightarrow K_e^{\psi} \quad P.D..$ Moreover if $\psi = \psi_K$, then

$$K_e^{\psi}(x,y) = K(x,y) + K(e,e) - K(x,e) - K(y,e) = \mathbb{E}[(Z_x - Z_e)(Z_y - Z_e)]$$

and $\psi_{K_e^{\psi}} = \psi_K$

The law of Gaussian fields and D.P. fonction are corresponding bijectively.

A N.D. kernel ψ does not determines a precise law of a Gaussian fields, unless we impose the cancelation of the process at a point $e \in X$. In this case the process is associated to the P.D. kernel :

$$K_e^{\psi}(x,y) = \frac{1}{2}(\psi(x,e) + \psi(y,e) - \psi(x,y)$$

But for N.D. kernel there is a functional calculus : If ψ is N.D.

- $F(u) = \int_{\mathbb{R}+} (1 - e^{-su}) d\mu(s) \Longrightarrow F(\psi)$ is N.D..

Ex: $\psi \; N.D \implies \forall 0 < \alpha \leq 1, \quad \psi^{\alpha} \; N.D$

- $G(u) = \int_{\mathbb{R}+} e^{-su} d\mu(s) \Longrightarrow G(\psi)$ is P.D..

Ex: $\psi \; N.D. \iff \forall 0 < t, \quad e^{-t\psi} \; P.D$

Brownian field.Fractional Brownian field.

Definition Let $X, d)$ a metric space. Let $\psi(x, y) = d(x, y)$.
IF ψ is a D.N. function then $\exists$ (several) $(Z_x(\omega))_{x \in X}$ Gaussian fields verifying

$$\mathbb{E}(Z_x - Z_y)^2 = d(x, y).$$

Such field is A Brownian field. If we impose $Z_e = 0$ for some e then there a unique (in law) field : THE brownian field which cancel in e. This process is associated to the P.D. function

$$K_e^{\psi}(x, y) = \frac{1}{2}\{d(x, e) + d(y, e) - d(x, y)\}$$

Fractional Brownian field If $d(x, y)$ is N.D then $\forall 0 < \alpha \leq 1,\ (d(x, y))^{\alpha}$ is N.D. the corresponding processes are Fractional Brownian field, with uniqueness if we impose the cancelation in a fixed point $e \in X$.

Regularity.

Regularity Now let us suppose that X has a metric, (or more sophisticated) structure, so that we can define function spaces. For example if (X, d) is a metric space, the $lip(\alpha)-$ spaces $0 < \alpha \leq 1$ Let $K(x, y)$ $D.P.$ and $Z_x(\omega)$ the associated Gaussian process. Is it possible to say :

$$x \in X \mapsto Z_x(\omega) \in \mathbb{R} \quad \text{(for almost all } \omega\text{)}$$

belongs to some function space, from an analysis of the P.D. associated kernel $K(x, y)$?

Compact homogeneous spaces.

Let now M a compact Riemannian space and G a Lie group of isometry acting transitively on M. So $M \sim G/K$ where K is the subgroup of stabilizer of a fixed point $O \in M$. We are interested by $G-$invariant Gaussian process, or equivalently by continuous real, definite positive invariant functions :

$$\forall g \in G, \forall x, y \in M, \quad K(g.x, g.y) = K(x, y)$$

Two points homogeneous space.
If $\forall (x,y),(x',y') \in M \times M, \rho(x,y) = \rho(x',y'), \exists g \in G, g.x = x',\ g.x' = y'$.
Then continuous real, definite positive invariant functions are :

$$K(x,y) = \sum_k \nu_k P_{\lambda_k}(x,y),\ \nu_k \geq 0,\ \sum_k \nu_k dim(\mathcal{H}_k) < \infty,$$

($P_{\lambda_k}(x,y)$ is the projector on the eigenspace $\mathcal{H}_{\lambda_k}$ of Δ corresponding to λ_k)
(Bochner-Godement theorem and characterization of spherical functions)

Sphere.

Let $\mathbb{S}^d \subset \mathbb{R}^d$. the unit sphere of $\mathbb{R}^{d+1}$. This is the simplest example of two points homogeneous space. The geodesic distance is given by :

$$\forall \xi, \eta \in \mathbb{S}^d, \quad d_{\mathbb{S}^d}(\xi, \eta) = \arccos(\langle \xi, \eta \rangle_{\mathbb{R}^{d+1}})$$

We have the following spectral decomposition of the Laplacian $\Delta_{\mathbb{S}^d}$:

$$\mathbb{L}^2(\mathbb{S}^d) = \bigoplus \mathcal{H}_{\lambda_k}, \ \lambda_k = k(k+d-1) = k(k+2\nu),$$

$$P_{\lambda_k}(\xi, \eta) = L_k^d(\langle \xi, \eta \rangle), \quad L_k^d(x) = \frac{1}{|\mathbb{S}^d|}(1 + \frac{k}{\nu}) C_k^\nu(x); \ \nu = \frac{d-1}{2}$$

C_k^ν Gegenbauer polynomial of degree k : $\quad \dfrac{1}{(1-2xr+r^2)^\nu} = \sum_k r^k C_k^\nu(x)$

The invariant continuous definite positive functions are :

$$K(\xi, \eta) = \sum_k \nu_k L_k^d(\langle \xi, \eta \rangle) = \sum_k \nu_k L_k^d(\cos(d_{S^d}(\xi, \eta))$$

Schoenberg- Bingham result

Let f a continuous function defined on $[-1,1]$. Then : $f(\langle \xi, \eta \rangle)$ is a positive definite function on $\mathbb{S}^d$ FOR ALL $d \in \mathbb{N}^*$, if and only if

$$f(x) = \sum_{n \geq 0} a_n x^n, \quad 0 \leq a_n; \quad \sum_n a_n = f(1) < \infty$$

So, for such a function :

$$f(x) = \sum_{k \geq 0} a_k^d L_k^d(x), \quad 0 \leq a_k^d; \quad \sum_{k \geq 0} a_k^d L_k^d(1) = \sum_k a_k = f(1)$$

$$(0 < L_k^d(1) \sim k^{d-1})$$

So $\quad f(\langle \xi, \eta \rangle) = \sum_{k \geq 0} a_k^d L_k^d(\langle \xi, \eta \rangle) = f(\cos(d_{S^d}(\xi, \eta))$

" Brownian " process on the sphere .

Let $f(x) = \frac{1}{2}(\frac{\pi}{2} - \arccos(x)) = \frac{1}{2}\sum_{j\geq 0} \frac{(\frac{1}{2})_j(\frac{1}{2})_j}{j!}\frac{x^{2j+1}}{(\frac{3}{2})_j}$

where $(a)_j = a(a+1)..(a+j-1) = \frac{\Gamma(j+a)}{\Gamma(a)}$

By Gauss formula $\frac{1}{2}\sum_{j\geq 0} \frac{(\frac{1}{2})_j(\frac{1}{2})_j}{j!(\frac{3}{2})_j} = \frac{\pi}{4}$

$$f(\langle \xi, \eta \rangle) = \frac{1}{2}(\frac{\pi}{2} - \arccos(\langle \xi, \eta \rangle_{\mathbb{R}^{d+1}}) = \frac{\pi}{4} - \frac{1}{2}(d_{\mathbb{S}^d}(\xi, \eta))$$

As, $|f(\langle \xi, \eta \rangle) - f(\langle \xi, \eta' \rangle)| \leq \frac{1}{2} d_{\mathbb{S}^d}(\eta', \eta)$

One can built a probability a Gaussian probability on W on $B^s_{\infty,1}(\mathbb{S}^d),\ s < \frac{1}{2}$ such that

$$\int_{B^s_{\infty,1}(\mathbb{S}^d)} (\omega(\xi) - \omega(\eta))^2 dW(\omega) = d_{\mathbb{S}^d}(\xi, \eta)$$

The associated process $Z_\xi(\omega) = (\delta_\xi(\omega))_{\xi \in \mathbb{S}^d}$ is almost surely in $B^s_{\infty,1}(\mathbb{S}^d) \subset Lip^s(\mathbb{S}^d)$ if $s < \frac{1}{2}$.Moreover

Moreover $\mathbb{E}(|Z_\xi - Z_\eta|^2) = 2f(1) - 2f(\langle \xi, \eta \rangle) = d_{\mathbb{S}^d}(\xi, \eta)$

Fractionnal brownian process

From the previous result we have :

$$\psi(\xi,\eta) = d_{\mathbb{S}^d}(\xi,\eta) \quad \text{is an invariant negative definite kernel.}$$

So from the general theory of definite negative kernel
$\forall 0 < \alpha \leq 1, \quad \psi_\alpha(\xi,\eta) = [d_{\mathbb{S}^d}(\xi,\eta)]^\alpha$ is an invariant negative definite kernel.
Then

$$K(\xi,\eta) = C - \frac{1}{2}(d_{\mathbb{S}^d}(\xi,\eta))^\alpha, \text{ C great enough}$$

Is an invariant definite positive kernel.

$$|K(\xi,\eta) - K(\xi,\eta')| = \frac{1}{2}|d_{\mathbb{S}^d}(\xi,\eta))^\alpha - d_{\mathbb{S}^d}(\xi,\eta'))^\alpha| \leq \frac{1}{2}d_{\mathbb{S}^d}(\eta',\eta))^\alpha$$

The associated process Z^α_ξ is almost surely in $Lip^s(\mathbb{S}^d), \; s < \frac{\alpha}{2}$

$$\mathbb{E}(Z^\alpha_\xi - Z^\alpha_\eta)^2 = d^\alpha_{\mathbb{S}^d}(\xi,\eta)$$

General Gaussian process on the sphere.

THEOREM:

$$\text{If } f(x) = \sum_{n \geq 0} B_n x^n; \quad 0 \leq B_n = O(\frac{1}{n^\alpha}); \ \alpha > 0$$

Then;

$$f(\langle \xi, \eta \rangle) = f(\cos(d_{\mathbb{S}^d}(\xi, \eta)))$$

is an invariant definite positive function on $\mathbb{S}^d$ by Schoenberg-Bingham theorem, and one can prove that: the associated centered Gaussian process $(Z_\xi(\omega))_{\xi \in \mathbb{S}^d}$ is almost surely in $B^\gamma_{\infty,1}$, $\gamma < \alpha$.
For example if $0 < a, 0 < b, \ 0 < \alpha = c - a - b$ the hypergeometric function

$$F_{a,b;c}(x) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

fulfill the condition of the previous theorem.

THANK YOU FOR YOUR ATTENTION !