Statistical inverse problems and geometric "wavelet"construction.

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These lectures are dedicated to the memory of Laurent Cavalier.

Summary of the talk

• In the first part of the talk, we will look to some statistical inverse problems for which the natural framework is no more an Euclidian one.

• In the second part we will try to give the initial construction of (not orthogonal) wavelets -of the 80- by Frazier,Jawerth,Weiss, before the Yves Meyer ORTHOGONAL wavelets theory.

• In the third part we will propose a construction of a geometric wavelet theory. In the Euclidian case, Fourier transform plays a fundamental role.In the geometric situation this role is given to some"Laplacian operator" with some properties.

• In the last part we will show that the previous theory could help to revisit the topic of regularity of Gaussian processes, and to give a criterium only based on the regularity of the covariance operator.

PART I-Inverse Problem: Toy Model

Inverse problem in statistic could be described as the following simplified (White noise) model : We have two Hilbert spaces : $\mathcal{A}$, $\mathcal{B}$ and $K : \mathcal{A} \mapsto \mathcal{B}$ a continuous injective operator. Let

$$Y = Kf + \epsilon W$$

This equation has the following meaning : $\forall b \in \mathcal{B}$ we can observe

$$\langle Y, b \rangle = \langle Kf, b \rangle + \epsilon \langle W, b \rangle$$

where $W_b = (\langle W, b \rangle)_{b \in \mathcal{B}}$ is a Gaussian centered process of covariance

$$\mathbb{E}(W_b W_{b'}) = \langle b, b' \rangle$$

The statistical challange is : how one recover $f \in \mathcal{A}$? .

Inverse Problem:

A more realistic model would be : $\mathcal{A}$,$\mathcal{B}$ two function spaces and we observe

$$Y_i = K(f)(X_i) + \epsilon_i, \quad i = 1, 2, ..n. \quad K(f) \in \mathcal{B}.$$

where $X_1..X_n$ are fixed or random design, ϵ_i are iid gaussian noise.
The statistical challange is : how one recover $f \in \mathcal{A}$? . This has to be done as quickly as possible when the number n of observation goes to ∞.
We will focus on the toy model. It could be proved that the two models are in some sens equivalent with $\epsilon = \frac{1}{\sqrt{n}}$

Projection estimator

The most popular estimator use Hilbertian technics : Let us look to the simplest one :
If e_n is an orthornormal basis and $f = \sum \alpha_n e_n$. Is it possible to build estimators $\hat{\alpha_n}$, at least for $n \leq N$ such that

$$\mathbb{E}\|f - \sum_{n \leq N} \hat{\alpha_n} e_n\|_{\mathcal{A}}^2) = \sum_{n \leq N} \mathbb{E}(\alpha_n - \hat{\alpha_n})^2 + \sum_{k > N} \alpha_k^2$$

is as small as possible when $\epsilon \longrightarrow 0$?

The Singular Value Decomposition basis

Let us suppose K an injective, compact operator. So K^*K is a positive compact operator and we have the classical singular value decomposition : $\exists e_n$ an orthonormal basis of $\mathcal{A}$ and h_n an orthonormal family of $\mathcal{B}$, $\exists \mu_n > 0$ $\mu_n \longrightarrow 0$ such that :

$$K^*K(e_n) = \mu_n^2 e_n, \quad K(e_n) = \mu_n h_n,\ K^*(h_n) = \mu_n e_n.$$

$$f = \sum \alpha_n e_n, \quad \alpha_n = \langle f, e_n \rangle$$

$$\langle Y, h_n \rangle = \frac{1}{\mu_n} \langle Kf, K(e_n) \rangle + \epsilon \langle W, h_n \rangle =$$

$$\frac{1}{\mu_n} \langle f, K^*K(e_n) \rangle + \epsilon \langle W, h_n \rangle = \mu_n \langle f, e_n \rangle + \epsilon \langle W, h_n \rangle$$

So we have the following natural estimator of α_n :

$$\hat{\alpha_n} = \frac{1}{\mu_n}\langle Y, h_n\rangle = \frac{1}{\mu_n}\langle Kf, h_n\rangle + \frac{1}{\mu_n}\epsilon\langle W, h_n\rangle$$

So

$$\hat{\alpha_n} \quad \text{is a Gaussian Variable } N(\alpha_n, \frac{\epsilon^2}{\mu_n^2})$$

and the variance of the estimator $\hat{\alpha_n}$ is becoming bigger and bigger.

$$\mathbb{E}\|f - \sum_{n \le N} \hat{\alpha_n} e_n\|_{\mathcal{A}}^2) =$$

$$\sum_{n \le N} \mathbb{E}(\alpha_n - \hat{\alpha_n})^2 + \sum_{k > N} \alpha_k^2 = \epsilon^2 \sum_{n \le N} \frac{1}{\mu_n^2} + \sum_{k > N} \alpha_k^2$$

Drawback of the SVD basis estimator.

The first problem is of course : How to choose N?
The second problem is that the SVD basis is too much linked to the Hilbert space structure. In all the real situation $\mathcal{A} = L^2(X, \mu)$, where X has a differentiable structure (Riemann manifold,..), μ is a natural measure and it is often more important to have a good performance of

$$\mathbb{E}\|\hat{f} - f\|_p^p,$$

for some $1 \leq p \leq \infty$, specially $p = \infty$ than the "energy" risk.

In this framework, the REGULARITY class of f plays an important role, but this is not suitably described by the SVD basis. We need to build a "wavelet" type basis.

Some statistical examples in non euclidian framework.

These last years, to solve some statistical problem , one has to go from classical $\mathbb{R}^d$ space to more intricate geometry :

$1-$ Framework of Jacobi polynomials:

• Wicksell problem, and Jacobi polynomials.

Needlets algorithms for estimation in inverse problems. G.K., P. Petrushev, D. Picard. and T. Willer, Electronic Journal of Statistics, Vol. 1 (2007)

$2-$ Estimation on the sphere $\mathbb{S}^{d-1}$ (or compact homogeneous space)

"assymptotic for spherical needlet needlets."
P. Baldi, G.K. , D. Marinucci, D. Picard. Annals of Stat. :
Study of the Cosmological Microwave Background.

"Adaptive density estimation for directional data using needlets." P. Baldi, G.K. , D. Marinucci, D. Picard. Annals of Stat.

• Deconvolution on the sphere (interraction of the geometry of $\mathbb{S}^{d-1}$ and $SO(d)$).

"Localized deconvolution on the sphere."
G.K., T.M. Pham Ngoc, D. Picard.

• *Concentration inequalities and confidence bands for needlet density estimators on compact homogeneous manifolds. K.G.; NICKL, R., PICARD, D., Probability Theory and Related Field.*

$3-$ Radon transform on the ball.

• Tomography problem (The framework is the ball with a non euclidian distance.)

"Inversion of noisy Radon transform by SVD based needlets". G. K. , G. Kyriazys, E. Le Pennec, P. Petrushev, D. Picard. - ACHA .

"Radon needlet thresholding". G. K. , E. Le Pennec, D. Picard. Bernouilli .

In this first part we will mainly focus on the tomography problem.

Statistic and Geometry :The sphere and the deconvolution problem.

Let M a compact topological space and G a compact group acting continuously and transitively on M :

$$\forall g \in G,\; g : x \in M :\mapsto g.x \in M \;\text{ is continuous}$$

$$\text{and } \forall x, y \in M,\; \exists g \in G,\; gx = y$$

Let us define:$[\gamma(g).f](x) = f(g^{-1}x)$. Let μ a Haar measure on G. i.e

$$\forall g \in G,\; \int_G F(g^{-1}h)d\mu(h) = \int_G F(hg)d\mu(h) = \int_G F(h)d\mu(h)$$

There exists a measure ν on the Borel sets of X such that :

$$\forall g \in G,\; \int_M \phi(g^{-1}x)d\mu(x) = \int_M \phi(x)d\mu(x)$$

Let Y a $M-$valued random variable with a density $f:\ Y \sim f(x)d\nu(x)$, which is the target function

Let U a $G-$valued random variable with value in G independant of X and $U \sim F(g)d\mu(g)$, where F is known.

Let us suppose that we observe $Y = UX$

$$\text{As :}\quad \mathbb{E}(\Phi(UX)) = \int_X \int_G \Phi(ux)f(x)d\nu(x)F(u)d\mu(u)$$

$$= \int_X \Phi(y) \int_G F(u)f(u^{-1}y)d\mu(u)d\nu(y)$$

we have: $Y \sim (\int_G F(u)f(u^{-1}y)d\mu(u))d\nu(y)$

Let us look to a simpler problem:

$$Y = \int_G F(u) f(u^{-1}y) d\mu(u) + \epsilon W = K(f) + \epsilon W$$

Clearly, $\quad K^*(\phi)(x) = \int_G F(u)\phi(ux) d\mu(u);$

$$K^*K(f)(x) = \int_G \int_G F(u)F(u^{-1}w) d\mu(u) f(w^{-1}x) d\mu(w)$$

$$= \int_G \int_G F(u)F(u^{-1}w) d\mu(u) \gamma_w . f(x) d\mu(w)$$

Let us suppose now that on (M,ν) it exists a positive self-adjoint operator L with dense domain $D(L) \subset \mathbb{L}^2(M,\nu)$ such that:

$$\mathbb{L}^2(M) = \bigoplus \mathcal{H}_{\lambda_k}, \quad \mathcal{H}_{\lambda_k} = ker(A - \lambda_k I)$$

$$\forall g \in G,\ \gamma_g.L(f)(x) = L(f)(g^{-1}x) = L(\gamma_g.f)(x)$$

Clearly if L commute with the $\gamma_g, g \in G$,

$$L(K(f)) = L[\int_G F(u)\gamma_u f d\mu(u)] = \int_G F(u)L[\gamma_u f)]d\mu(u)]$$

$$= \int_G F(u)\gamma_u L[f])d\mu(u)] = K(L(f))$$

This implies that :

$$K^*K :\ \mathcal{H}_{\lambda_k} \mapsto \mathcal{H}_{\lambda_k}$$

So, there is an orthonormal basis $e^j_{\lambda_k}$ of $\mathcal{H}_{\lambda_k}$ (so $L(e^j_{\lambda_k}) = \lambda_k e^j_{\lambda_k}$) which are eigenvectors of K^*K :

$$K^*K(e^j_{\lambda_k}) = \nu_{\lambda_k,j} e^j_{\lambda_k}$$

(with some chance $\mathcal{H}_{\lambda_k}$ is also an eigen space of K^*K i.e. $\nu_{\lambda_k,j} = \nu_{\lambda_k}$)

The sphere $\mathbb{S}^d$ and $SO(d+1)$.

Let $M = \mathbb{S}^d \subset \mathbb{R}^{d+1}$. Clearly the group $G = SO(d+1)$ acts continually and transitively on $\mathbb{S}^d$.

• The geodesic distance on $\mathbb{S}^d$ is given by

$$d(x,y) = \cos^{-1}(\langle x,y\rangle), \quad \langle x,y\rangle = \sum_{i=1}^{d+1} x_i y_i$$

• There is a natural measure ν on $\mathbb{S}^d$, invariant by rotation.
• There is a natural Laplacian on $\mathbb{S}^d$, $\Delta_{\mathbb{S}^d}$, invariant by rotation. We have the following spectral decomposition:

spherical harmonics of order k.

$\mathcal{H}_k$ is the restriction to $\mathbb{S}^d$ of polynomials of degree k which are homogeneous and harmonic.

$$\bullet P = \sum_{|\alpha|=k} a_\alpha x^\alpha,$$

$$\bullet \text{ and harmonic} \quad \Delta P = \sum_{i=1}^{d} \frac{\partial^2 P}{\partial x_i^2} = 0.$$

$$\forall P \in \mathcal{H}_k, \quad \Delta_{\mathbb{S}^d} P = -k(k+d-1)P$$

$\mathcal{H}_k$: the space of spherical harmonics of order k.

Projector on $\mathcal{H}_k$.

Let e_k^j any orthonormal basis of $\mathcal{H}_k$.

$$L_k(x,y) = \sum_j^{dim\mathcal{H}_k} e_k^j(x)\overline{e_k^j(x)}$$

$$\text{Moreover} \quad P_{\mathcal{H}_{\lambda_k}}(f)(x) = \int L_k(x,y)f(y)d\nu(y)$$

$$L_k(x,y) = (1+\frac{k}{\nu})G_k^\nu(\langle x,y\rangle); \quad \nu = \frac{d-1}{2}$$

$$dim(\mathcal{H}_k) = C_{k+d}^d - C_{k-1+d}^d$$

G_k^ν : Gegenbauer polynomials.

The Radon Transform in the White Noise Model .

- Let B^d be the unit ball of $\mathbb{R}^d$ and $f \in \mathbb{L}^2(B^d, dx)$.
- Let $\theta \in \mathbb{S}^{d-1}$ and $t \in [-1, 1]$. By definition :

$$Rf(\theta, t) = \int_{\langle \theta, x \rangle = t} f(x) dx$$

  is the Radon transform of f

The statistical problem is to recover f from the noisy observation :

$$dY(\theta, t) = R(f) d\mu(\theta, t) + \epsilon dW(\theta, t)$$

This is a typical inverse problem. (We have an **Indirect** noisy observation of f.)

Some Previous Works

- B.F. Logan and L.A.Shepp (1975)
- Korostelev and A. Tsybakov (1991)
- I. Johnstone and B. Silverman (1991)
- D. Donoho (1995)
- L. Cavalier (2001)
- E. Candès and D. Donoho (2000)
- B. Lucier and N. Yong Lee (2001)
- Yuan Xu
- ...

The Radon Transform in the White Noise Model

- Let B^d be the unit ball of $\mathbb{R}^d$ and $f \in \mathbb{L}^2(B^d, dx)$.
- Let $\theta \in \mathbb{S}^{d-1}$ and $t \in [-1, 1]$. By definition :

$$Rf(\theta, t) = \int_{\langle \theta, x \rangle = t} f(x)dx$$

  is the Radon transform of f

- (dx is the $d-1$ Lebesgue measure on the hyperplan $\langle \theta, x \rangle = t$)

Continuity of the Radon Transform

- Let $d\mu(\theta,t) = d\sigma(\theta)\frac{dt}{(\sqrt{1-t^2})^{d-1}}$ on $\mathbb{S}^{d-1}\times[-1,1]$.
- $R : \mathbb{L}^2(B^d, dx) \mapsto Rf(\theta,t) \in \mathbb{L}^2(\mathbb{S}^{d-1}\times[-1,1], d\mu(\theta,t))$

  is continuous
- and if $g(\theta,t) \in \mathbb{L}^2(\mathbb{S}^{d-1}\times[-1,1], d\mu(\theta,t))$

$$R^*(g)(x) = \int_{\mathbb{S}^{d-1}} g(\theta, \langle x,\theta\rangle)\left(\frac{1}{\sqrt{1-|\langle x,\theta\rangle|^2}}\right)^{d-1} d\sigma(\theta)$$

- The statistical problem is to recover f from the noisy observation :

$$dY(\theta, t) = R(f)d\mu(\theta, t) + \epsilon dW(\theta, t)$$

This is a typical inverse problem.

Geometry of the ball.

Let B^d the unit ball of $\mathbb{R}^d$, equipped with the Lebesgue measure

An operator: Let us define the following selfadjoint negative operator:

$$Af = \Delta f - dx.\nabla f - x.\nabla(x.\nabla f) =$$

$$\sum_i \frac{\partial^2 f}{\partial x_i^2} - \sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} - (d+1) \sum_i x_i \frac{\partial f}{\partial x_i}$$

Let $\Pi_k(B^d)$ the space of polynomials of degree $\leq k$ on the unit ball of $\mathbb{R}^d$. It is clear that :

$$A(\Pi_k(B^d)) \subset \Pi_k(B^d)$$

If $f \in \Pi_k(B^d)$ by stokes formula :

$$\int_{B^d} A(f)(x)\overline{f(x)}dx = -\int_{B^d} \{|\nabla(f)(x)|^2 - (x.\nabla(f)(x))^2\}dx$$

A is symmetric and negative operator on $\Pi_k(B^d)$. Let:

$$\Pi_k(B^d) = \mathcal{V}_k(B^d) \bigoplus \Pi_{k-1}(B^d);$$

$$\mathbb{L}^2(B^d) = \overline{\bigoplus_{k=0}^{\infty} \mathcal{V}_k(B^d)}$$

$\mathcal{V}_k(B^d)$ is an eigenspace of A.

$$f \in \mathcal{V}_k(B^d) \Longleftrightarrow A(f) = -k(k+d)f.$$

Orthonormal basis of $\mathcal{V}_k(B^d)$.

$$g_{k,l,i}(x) = \sqrt{2k+d}\, P_j^{0,\nu+l}(2|x|^2-1)Y_{l,i}(x),$$

$0 \le l \le k,\ k = 2j+l,\ j \in \mathbb{N},\ \nu = \frac{d}{2} - 1$

- $P_j^{0,\nu+l}$ is the Jacobi Polynomial.
- $Y_{l,i}$

is a basis of the spherical harmonics of degree l, $\mathcal{H}_l(\mathbb{S}^{d-1})$ on the sphere, taken as homogeneous polynomials of degree l on B^d.

($Y_{l,i}$ is an homogeneous polynomial of degree l on $\mathbb{R}^d,\ \Delta Y_{l,i} = 0.$)

The kernel projector on $\mathcal{V}_k$ is given by

$$L_k(x,y) = \sum_{i,0\leq l\leq k,k-l\equiv 0(mod 2)} g_{k,l,i}(x)g_{k,l,i}(y)$$

$$L_k(x,y) = \frac{2k+d}{|\mathbb{S}^{d-1}|^2}\int_{\mathbb{S}^{d-1}} C_k^{\nu+1}(\langle x,\xi\rangle)C_k^{\nu+1}(\langle y,\xi\rangle)d\sigma(\xi),$$

$\nu = \frac{d}{2}-1, \quad C_k^{\nu+1}$ is the Gegenbauer polynomial.

Singular Value Decomposition of the Radon Transform

- Let $\Pi_k(B^d)$ the space of polynomials of degree $\leq k$ on the unit ball of $\mathbb{R}^d$.

$$\Pi_k(B^d) = \mathcal{V}_k(B^d) \bigoplus \Pi_{k-1}(B^d)$$

$$\mathbb{L}^2(B^d) = \bigoplus_{k=0}^{\infty} \mathcal{V}_k(B^d)$$

- $\mathcal{V}_k(B^d)$ is an eigenspace of R^*R. So we have a Singular Value Decomposition of R. The corresponding eigenvalue is

$$\mu_k^2 = \frac{\pi^{d-1}2^d}{(k+1)....(k+d)} \sim k^{-(d-1)}$$

- A "natural " basis of $\mathcal{V}_k(B^d)$ is given by

$$g_{k,l,i}(x) = \sqrt{2k+d}P_j^{0,\nu+l}(2|x|^2-1)Y_{l,i}(x),$$

$$k-l=2j\geq 0, \quad \nu=\frac{d}{2}-1$$

- $P_j^{0,\nu+l}$ is the Jacobi Polynomial.
- $Y_{l,i}, i=1...dim(\mathcal{H}_l(\mathbb{S}^{d-1})$ is a basis of the spherical harmonics of degree $l, \mathcal{H}_l(\mathbb{S}^{d-1})$ on the sphere, taken as homogeneous polynomials of degree l on B^d.

$$R(g_{k,l,i}) = \mu_k f_{k,l,i}, \quad R^*(f_{k,l,i}) = \mu_k g_{k,l,i}$$

$$f_{k,l,i}(\theta,t) = \frac{C_k^{d/2}(t)}{\|C_k^{d/2}\|}(1-t^2)^{(d-1)/2}Y_{l,i}(\theta)$$

- The kernel projector on $\mathcal{V}_k$ is given by

$$L_k(x,y) = \sum_{l,ik-l=2j,} g_{k,l,i}(x) g_{k,l,i}(y)$$

$$L_k(x,y) = \frac{2k+d}{|\mathbb{S}^{d-1}|^2} \int_{\mathbb{S}^{d-1}} C_k^{\nu+1}(\langle x,\xi\rangle) C_k^{\nu+1}(\langle y,\xi\rangle) d\sigma(\xi),$$

- $\nu = \frac{d}{2} - 1, \quad C_k^{\nu+1}$ is the Gegenbauer polynomial.

Regularity classes in the previous framework.

P. Petrushev,Y. Xu, F.Narcowich, J. Ward and coauthors have proposed a theory of regularity spaces and wavelet they called "needlet" in the framework of the SPHERE, the INTERVAL $[-1, 1]$ (Jacobi polynomials) the BALL and the SIMPLEX.
We present in the sequel this construction. The proofs use heavily special functions theory.

Some references

P. Petrushev, Y. Xu Localized polynomial frames on the ball, Constr. Approx. 27 (2008), 121–148.

P. Petrushev, Y. Xu, Localized polynomial frames on the interval with Jacobi weights, J. Four. Anal. Appl. 11 (2005), 557–575.

F. Narcowich, P. Petrushev, and J. Ward, Decomposition of Besov and Triebel-Lizorkin spaces on the sphere, J. Funct. Anal. 238 (2006), 530–564.

G. Kyriazis, P. Petrushev, and Y. Xu, Decomposition of weighted Triebel-Lizorkin and Besov spaces on the ball, Proc. London Math. Soc. 97 (2008), 477–513.

Pesenson, Pesenson and Geller have a similar theory for compact homogeneous manifold.
D. Geller, I. Z. Pesenson, Band-limited localized Parseval frames and Besov spaces on compact homogeneous manifolds, J. Geom. Anal. 21 (2011), 334–371. These works

were performed between 2000 and 2010.

Polynomials on compact subset of $\mathbb{R}^d$.

$\bullet M$ a compact subspace of $\mathbb{R}^d$.

Let: $\mathcal{P}(\mathbb{R}^d)$: $\mathcal{P}$ the space of polynomials on $\mathbb{R}^d$, and
$\mathcal{P}_k(\mathbb{R}^d)$: $\mathcal{P}_k$ the polynomials of degree k.

$\bullet\mathcal{P}(M)$ the vector space of restriction of polynomials to M

$\bullet\mu$ a finite measure on M

$\bullet\Pi_k(M)$: The restriction of polynomials of degree less then k

$\bullet\mathcal{V}_k(M)$ $\quad \Pi_k(M) = \mathcal{V}_k(M) \bigoplus \Pi_{k-1}(M),\ \mathcal{V}_0(M) = \Pi_0(M).$

So: $$\mathbb{L}_2(M, \mu) = \overline{\bigoplus_{k=0}^{\infty} \mathcal{V}_k(M)}$$

L_k the orthogonal projection on $\mathcal{V}_k$. Then

$$\forall f \in \mathbb{L}_2(\mathcal{Y}, \mu), \quad L_k(f)(x) = \int_{\mathcal{Y}} f(y) L_k(x, y) d\mu(y)$$

$$L_k(x, y) = \sum_{i=1}^{l_k} e_i^k(x) \bar{e}_i^k(y)$$

l_k is the dimension of $\mathcal{V}_k$ and $(e_i^k)_{i=1,\ldots l_k}$ an arbitrary orthonormal basis of $\mathcal{V}_k$. We have :

$$\int L_k(x, y) L_m(y, z) d\mu(z) = \delta_{k,m} L_k(x, z)$$

Example 1: Jacobi.

$$M = [-1, 1]; d\mu(x) = \omega(x)dx;$$

$$\omega(x) = (1-x)^\alpha(1+x)^\beta; \quad \alpha, \beta > -1$$

$\bullet D(f) = \dfrac{(1-x^2)\omega f')'}{\omega} = (1-x^2)f" - (2+\alpha+\beta)x + \alpha - \beta)f'$

is a symmetric second order differential operator.

$$\int_{-1}^{1} Df\overline{f(x)}\omega(x)dx = -\int_{-1}^{1}(1-x^2)|f'(x)|^2\omega(x)dx$$

The eigenvectors are the Jacobi polynomials.

$$D(P_k^{\alpha,\beta}) = -k(k+\alpha+\beta+1)P_k^{\alpha,\beta}$$

$$\forall k \in \mathbb{N},\ dim(\mathcal{H}_k) = 1, \quad L_k(x,y) = P_k(x)P_k(y)$$

Example 2: Sphere.

$$\bullet, \ \mathbb{S}^d \subset \mathbb{R}^{d+1}.$$

• There is a natural measure on $\mathbb{S}^d$, invariant by rotation.
• There is a natural Laplacian on $\mathbb{S}^d$, $\Delta_{\mathbb{S}^d}$, invariant by rotation.
We have the following spectral decomposition: $\mathcal{H}_k$ is the restriction to $\mathbb{S}^d$ of polynomials of degree k which are homogeneous and harmonic on $\mathbb{R}^d$. (spherical harmonics of order k.)

$$\forall P \in \mathcal{H}_k, \quad \Delta_{\mathbb{S}^d} P = -k(k+d-1)P$$

$$L_k(x,y) = (1+\frac{k}{\nu})C_k^\nu(\langle x,y\rangle); \quad \nu = \frac{d-1}{2}$$

C_k^ν : Gegenbauer polynomials.

Exemple 3:The ball

Let $\quad M = B^d = \{\|x\| \leq 1\} \subset \mathbb{R}^d,\ d\mu(x) = W(x)dx,$

$W(x) = (1 - \|x\|^2)^{\mu - \frac{1}{2}};\ \mu > -\frac{1}{2}$

$$A(f) = \frac{1}{W} div((1 - \|x\|^2) W \nabla(f)) + \frac{1}{2} \sum_{i \neq j} D_{i,j}^2,$$

where $(D_{i,j} f(x) = (x_j \partial_i - x_i \partial_j) f(x))$

$$Af = \Delta f - \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \partial_i \partial_j f - (d + 2\mu)) x . \nabla_n f$$

Eigen spaces

$$-\int_M Af(x)f(x)W(x)dx =$$

$$\int_M (1-\|x\|^2)|\nabla f|^2 W dx + \frac{1}{2}\sum_{i\neq j}\int_M [D_{i,j}f]^2 W dx$$

So: $$\int_M Af(x)f(x)W(x)dx \geq 0.$$

One can easily verify that

$$A(\Pi_k(B^d)) \subset \Pi_k(B^d)$$

$$\forall P \in \mathcal{H}_k(B^d),\ AP = -k(k+2\mu+d-1)P$$

Approximation spaces for Jacobi, the sphere and the ball .

1. for $\quad f \in \mathbb{L}_p,\ 1 \le p \le \infty,$

$$\sigma_p(k, f) = \inf\{P \in \Pi_k, \quad \|f - P\|_p\}$$

2. For $1 \le q \le \infty$ (with the usual modification for $p = \infty$),

$$B^s_{p,q} : \quad \{f \quad \|f\|_{B^s_{pq}} = \|f\|_p + (\sum_{k\ge 1}(k^s\sigma_p(k, f))^q \frac{1}{k})^{1/q} < \infty\}$$

Actually $\|f\|_{B^s_{p,q}} \sim \|f\|_p + (\sum_{j\ge 0}(2^{js}\sigma_p(2^j, f))^q)^{1/q} < \infty.$

The polynomials (and their degrees) are the benchmark of regularity.

How to check $f \in B^s_{p,q}$.

Let P_j a sequence of operators verifying :

$$\exists C < \infty,\ \forall j \in \mathbb{N} \|P_j(f)\|_p \leq C\|f\|_p$$

$$\forall f \in \mathbb{L}_p,\ P_j(f) \in \Pi_{2^j}$$

$$\forall f \in \Pi_{2^{j-1}},\ P_j(f) = f.$$

Then $f \in B^s_{p,q}$ if and only if $2^{js}\|P_j(f) - f\|_p \in l_q$.

Remark:

Typically the family P_j : orthogonal projector on Π_{2^j},
NEVER verify the previous condition!

An important tool:Young Lemma.

Let $(X,\sigma),(Y,\mu)$ two measured spaces. Let $K(x,y)$ a mesurable function such that:

$$\int |K(x,y)|d\sigma(y) \leq C, \quad \int |K(x,y)|d\sigma(x) \leq C$$

$$\text{Let} \quad Kf(x) = \int K(x,y)f(y)d\mu(y),$$

then: $\forall 1 \leq p \leq \infty, \quad \|Kf\|_p \leq C\|f\|_p$

Minimax Estimation for the Radon Transform with respect to $B^s_{\pi,q}(B^d)$.

- Let $dY = R(f)d\mu(\theta,t) + \epsilon dW$. Let $1 \le p, \pi \le \infty$. Is it possible to find an estimator $\hat{f}$ such that

$$\{ \sup_{\|f\|_{B^s_{\pi,q}} \le M} \mathbb{E}\|\hat{f} - f\|_p^p\} \sim \inf_{\hat{h}} \{ \sup_{\|f\|_{B^s_{\pi,q}} \le M} \mathbb{E}\|\hat{h} - f\|_p^p\}$$

  when $\epsilon \to 0$?

- Is it possible to do it with NO knowledge of the regularity class $B^s_{r,q}$?

- A priori restriction for $B^s_{\pi,q} \subset \mathbb{L}^p$ (Sobolev injection) :

$$s > (d+1)(\frac{1}{\pi} - \frac{1}{p})$$

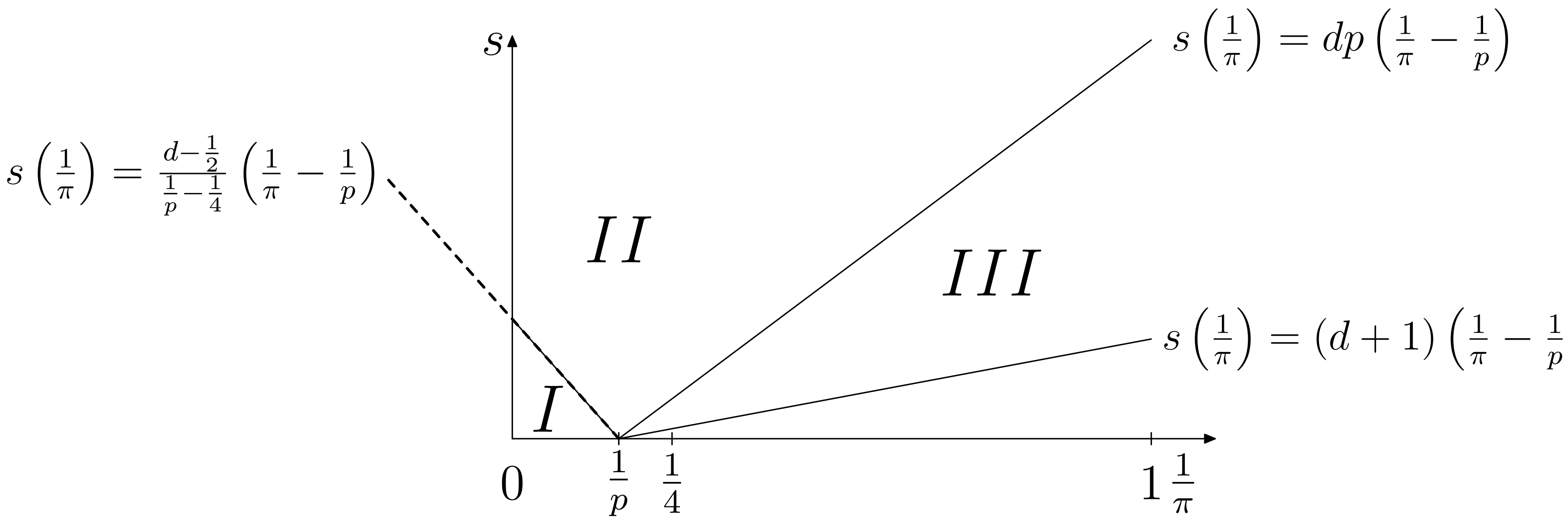
Lower Bound

- Let $1 \leq p \leq \infty$, and let $f \in B^s_{\pi,q}, \quad 1 \leq \pi \leq \infty, \quad 0 < s < \infty$. Let $s > (d+1)(\frac{1}{\pi} - \frac{1}{p})_+$ (such that $B^s_{\pi,q} \subset \mathbb{L}^p$.)

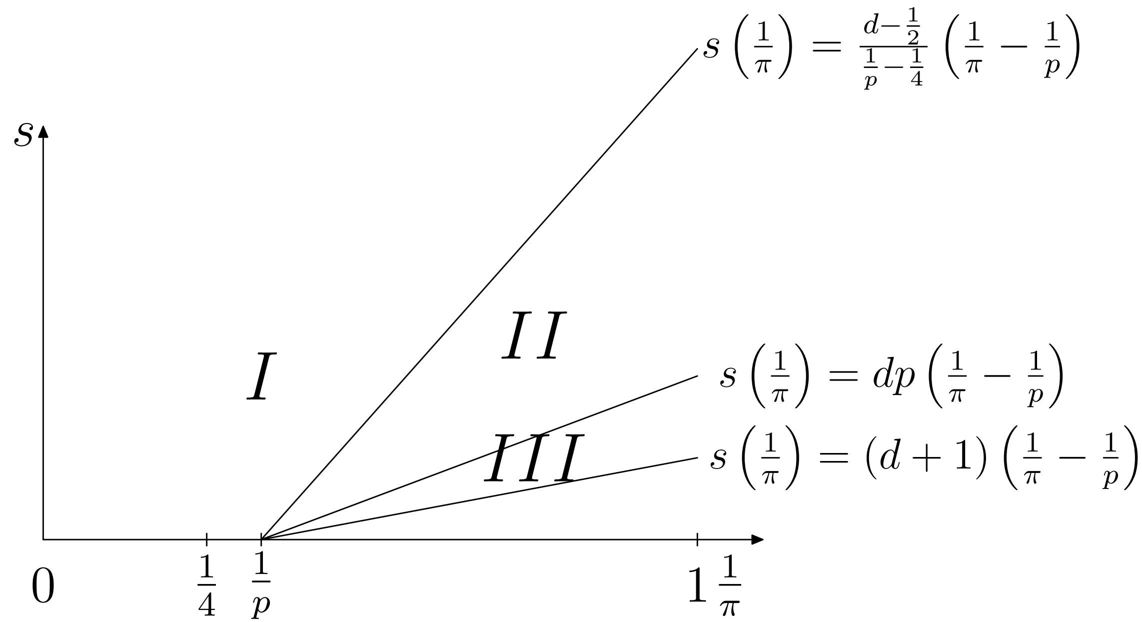
$$\inf_{\hat{h}} \{ \sup_{\|f\|_{B^s_{r,q}} \leq M} \mathbb{E}\|\hat{h} - f\|_p^p \} \geq C\epsilon^{\alpha p}$$

with

$$\alpha = \inf\{\frac{s}{s+d-\frac{1}{2}}, \frac{s-2(\frac{1}{\pi}-\frac{1}{p})}{s+d-\frac{2}{\pi}}, \frac{s-(d+1)(\frac{1}{\pi}-\frac{1}{p})}{s+d-\frac{d+1}{\pi}}\}$$



$$I : \frac{s}{s+d-\frac{1}{2}} \qquad II : \frac{s-2\left(\frac{1}{\pi}-\frac{1}{p}\right)}{s+d-\frac{2}{\pi}} \qquad III : \frac{s-(d+1)\left(\frac{1}{\pi}-\frac{1}{p}\right)}{s+d-\frac{d+1}{\pi}}$$



$$I : \frac{s}{s+d-\frac{1}{2}} \qquad II : \frac{s-2\left(\frac{1}{\pi}-\frac{1}{p}\right)}{s+d-\frac{2}{\pi}} \qquad III : \frac{s-(d+1)\left(\frac{1}{\pi}-\frac{1}{p}\right)}{s+d-\frac{d+1}{\pi}}$$

Agenda

- Construct a new 'basis' which is concentrated on the space AND on the spectral domain...
- How this can be used in statistical estimation? Project the process on it, and see...if some thresholding algorithm works ?

A General Paradigm

- In the case of : Jacobi,sphere, the ball (and the simplex), Narcowich, P. Petrushev, Ward , Y. Xu, .. have constructed a basis (in fact a frame) spectrally concentrated enough, but also well concentrated in the space domain. This is done busing heavily estimation on special functions, in 3 steps
  - smoothing the projection operator
  - splitting
  - discretization

Step I : Smoothing of the Projection Operator

- **Littlewood-Paley trick**

- Take $\phi \geq 0$, even, infinitely differentiable, supported in $[-1,1]$ and $|x| \leq \frac{1}{2} \Rightarrow \phi(x) = 1$ and $b(x) = \phi(\frac{x}{2}) - \phi(x)$. So

$$\forall x, \quad 1 = \phi(x) + \sum_j b(\frac{x}{2^j}) = \lim_{j \to \infty} \phi(\frac{x}{2^j})$$

$$B_j(x,y) = \sum_k b(\frac{k}{2^j}) L_k(x,y)$$

$$\Phi_j(x,y) = \sum_k \phi(\frac{k}{2^j}) L_k(x,y)$$

Good behavior of $\Phi_j(x,y)$ and $B_j(x,y)$.

First main result: $\exists C$ such that:

$$\forall j \in \mathbb{N}, \int |\Phi_j(x,y)| d\mu(x) \leq C$$

So

$$\bullet \, \forall 1 \leq p \leq \infty, \; \forall f \in \mathbb{L}^p, \Phi_j(f) \in \Pi_{2^j}$$

$$\bullet \, \forall f \in \Pi_{2^{j-1}}, \Phi_j(f) = f$$

$$\bullet \, \forall 1 \leq p \leq \infty, \; \forall f \in \mathbb{L}^p, \; \|\Phi_j(f)\|_p \leq C\|f\|_p$$

So :

$$f \in f \in B^s_{p,q} \Longleftrightarrow f \in \mathbb{L}^p, \quad \|f - \Phi_j(f)\|_p = \epsilon_j 2^{-js}, \; \epsilon_. \in l_q$$

Step II : The Spliting Procedure

- $$D_j(x,y) = \sum_{2^{j-1}<k<2^{j+1}} \sqrt{b(\frac{k}{2^j})} L_k(x,y)$$

- Due to $\int_M L_k(x,u)L_m(u,y)du = \delta_{k,m}L_k(x,y)$, we get

$$\int_M D_j(x,u)D_j(u,y)du = B_j(x,y)$$

$$= \sum_{2^{j-1}<k<2^{j+1}} b(\frac{k}{2^j}) L_k(x,y)$$

Step III : Discretization

- We have a quadrature formula . : for $\Pi_{2^{j+2}}$: $\exists \mathcal{X}_j \subset M$, and $\forall \xi \in \mathcal{X}_j,\ \lambda_{j,\xi} > 0$, such that

$$\forall f \in \quad \Pi_{2^{j+2}},\ \int_M f(u)du = \sum_{\xi \in \mathcal{X}_j} \lambda_{j,\xi} f(\xi)$$

So

$$B_j(x,y) = \int_M D_j(x,u)D_j(u,y)du = \sum_{\xi \in \mathcal{X}_j} \lambda_{j,\xi} D_j(x,\xi)D_j(\xi,y)$$

$$= \sum_{\xi \in \mathcal{X}_j} \sqrt{\lambda_{j,\xi}} D_j(x,\xi) \overline{D_j(y,\xi)} \sqrt{\lambda_{j,\xi}}$$

- So :

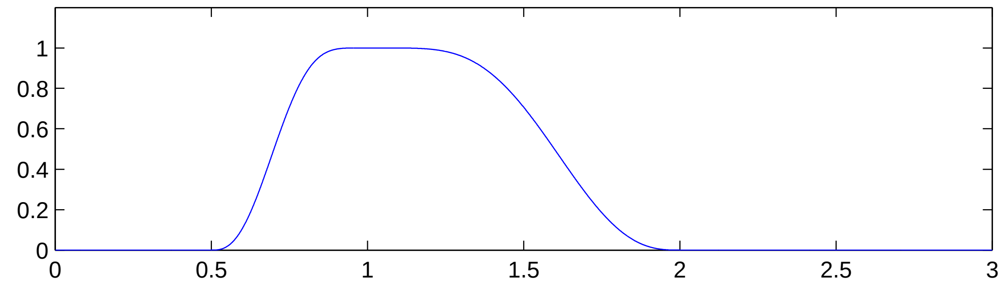
$$\int B_j(x,y)f(y)dy = \sum_{\xi\in\mathcal{X}_j} \lambda_{j,\xi} D_j(x,\xi) \int \overline{D_j(y,\xi)} f(y)dy$$

$$= \sum_{\xi\in\mathcal{X}_j} \sqrt{\lambda_{j,\xi}} D_j(x,\xi) \int \overline{\sqrt{\lambda_{j,\xi}} D_j(y,\xi)} f(y)dy$$

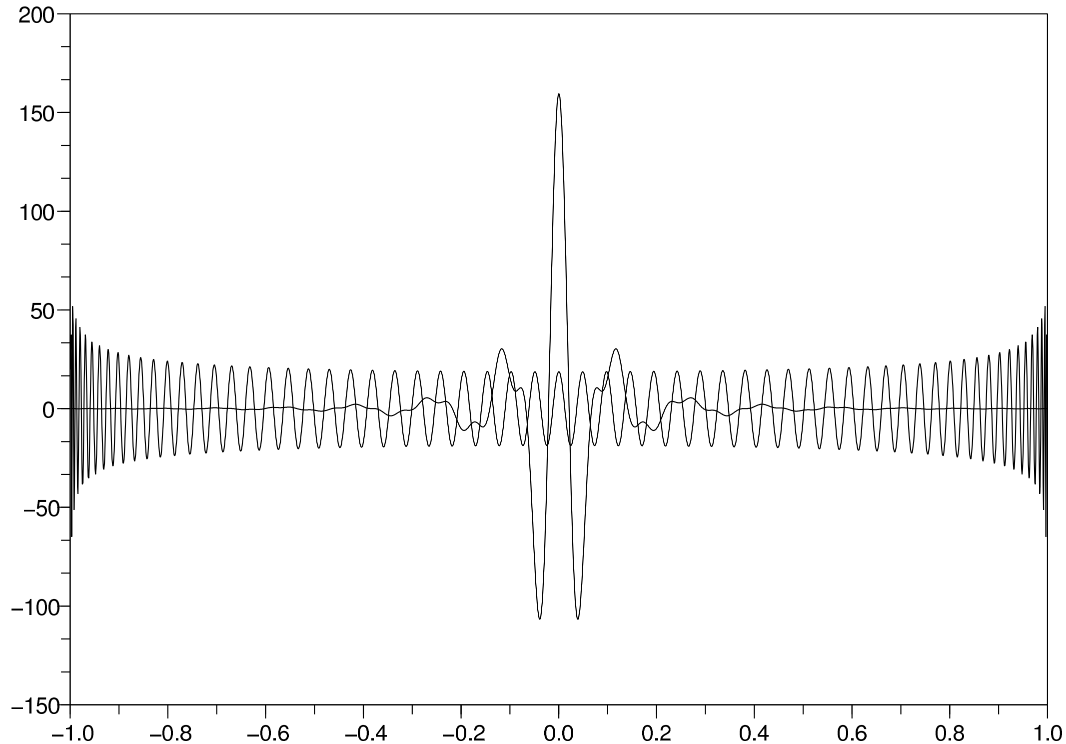
- Needlets frame.

$$\boxed{\psi_{j,\xi}(x) := \sqrt{\lambda_{j,\xi}} D_j(x,\xi) = \sqrt{\lambda_{j,\xi}} \sum_{2^{j-1}<k<2^{j+1}} \sqrt{b(\frac{k}{2^j})} L_k(x,\xi),}$$

- Here is $b(x)$:


1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3

-


200
150
100
50
0
–50
–100
–150
–1.0
–0.8
–0.6
–0.4
–0.2
0.0
0.2
0.4
0.6
0.8
1.0

Frame Properties of the Needlet

- $$f = \sum_{j \in \mathbb{N}} \sum_{\xi \in \mathcal{X}_j} \langle f, \psi_{j,\xi} \rangle \psi_{j,\xi}$$
$$\|f\|_2^2 = \sum_{j \in \mathbb{N}} \sum_{\xi \in \mathcal{X}_j} |\langle f, \psi_{j,\xi} \rangle|^2.$$
(In particular $\|\psi_{j,\xi}\|^2 \geq \|\psi_{j,\xi}\|^4$ so $\|\psi_{j,\xi}\|^2 \leq 1.$)

- But
$$\sum_{\xi \in \mathcal{X}_j} \sqrt{\lambda_{j,\xi}} \psi_{j,\xi} = \sum_{\xi \in \mathcal{X}_j} \lambda_{j,\xi} D_j(x, \xi) = \int_M D_j(x, u) du = 0$$

- So the family $(\psi_{j,\xi})_{j \in \mathbb{N}, \xi \in \mathcal{X}_j}$ is a tight frame, but is not linearly independent (redundancy).

Concentration Properties of the Needlet

- As :

$$\psi_{j,\xi} = \sqrt{\lambda_{j,\xi}} D_j(x,\xi) = \sqrt{\lambda_{j,\xi}} \sum_{2^{j-1}<k<2^{j+1}} \sqrt{b(\frac{k}{2^j})} L_k(x,\xi)$$

  the spectral localisation of $\psi_{j,\xi}$ is between 2^{j-1} and 2^{j+1}.

- $\exists 0 < c$ such that, $c \leq \|\psi_{j,\xi}\|_2^2 \leq 1$.

- SPATIAL LOCALISATION: $\exists C < \infty$ such that

$$\forall j \in \mathbb{N}, \sum_{\xi \in \mathcal{X}_j} \|\psi_{j,\xi}\|_1 |\psi_{j,\xi}(x)| \leq C$$

Concentration and sparsity.

As a consequence of Young lemma:
THEOREM : Let I a set of indexes . (X,μ) and (Y,ν) two measured spaces.

$$\text{Let } \phi_i(y),\ i \in I, y \in Y,\ h_i(x)\ i \in I, x \in X.$$

We suppose:

$$\exists 0 \leq C < \infty, \quad \sum_i \|\phi_i\|_1 |h_i(x)| \leq C; \quad \sup_{i \in I} \|\phi_i\|_\infty \|h_i\|_1 \leq C.$$

$\exists 0 < c$ such that, $\forall i \in I, c \leq \|\phi_i\|_2^2$. THEN:

•For $1 \leq p < \infty$, $\quad (\sum_i |\langle g, \overline{h_i}\rangle|^p \|\phi_i\|^p_{\mathbb{L}_p(Y)})^{\frac{1}{p}} \leq C\|g\|_{\mathbb{L}_p(X,\mu)}$

and for $p = \infty$, $\quad \sup_i |\langle g, \overline{h_i}\rangle| \|\phi_i\|_\infty \leq C\|g\|_{\mathbb{L}_\infty(X,\mu)}$

• Moreover, if $\forall a_i \in \mathbb{C}, i \in I$,

$$\forall 1 \leq p \leq \infty, \quad c\|\sum_i a_i h_i(x)\|_{\mathbb{L}^p(X,\mu)} \leq C(\sum_i |a_i|^p \|\phi_i\|^p_p)^{\frac{1}{p}};$$

(for $p = \infty$, $c\|\sum_i a_i h_i(x)\|_{\mathbb{L}_\infty(X,\mu)} \leq C \sup_i |a_i| \|\phi_i\|_\infty$

Consequence :characterization of $B^s_{p,q}$

- $$f = \sum_j \sum_{\xi \in \chi_j} \langle f, \psi_{j,\xi} \rangle \psi_{j,\xi} \in B^s_{p,q} \Longleftrightarrow$$

If $0 < q < \infty$:

$$(\sum_{\xi \in \chi_j} |\langle f, \psi_{j,\xi} \rangle|^p \|\psi_{j,\xi}\|_p^p)^{1/p} = u_j 2^{-js}, \quad u_. \in l_q$$

If $q = \infty$:

$$\sup_{\xi \in \chi_j} |\langle f, \psi_{j,\xi} \rangle| \|\psi_{j,\xi}\|_\infty^= u_j 2^{-js}, \quad u_. \in l_\infty$$

Thresholding strategy

Let us summarize: $\mathbb{L}^2(M) = \bigoplus \Pi_k$,

$e_k^i,\ i = 1, 2, .., dim(\Pi_k) = d_k$ orthonormal basis.

$$f = \sum_k \sum_{i=1}^{d_k} \alpha_k^i e_k^i;\ \alpha_k^i = \langle f, e_k^i \rangle$$

$$= \sum_{(j,\xi) j\in\mathbb{N}, \xi\in\chi_j} \beta_{j,\xi}\psi_{j,\xi}, \quad \beta_{j,\xi} = \langle f, \psi_{j,\xi} \rangle$$

$$\forall j \in \mathbb{N}, \xi \in \chi_j : \quad \psi_{j,\xi} = \sqrt{\lambda_{j,\xi}} \sum_k \sqrt{b(\frac{k}{2^j})} L_k(x,\xi) =$$

$$\sqrt{\lambda_{j,\xi}} \sum_k \sqrt{b(\frac{k}{2^j})} \sum_i e_k^i(x) \overline{e_k^i(\xi)}$$

$$\beta_{j,\xi} = \langle f, \psi_{j,\xi} \rangle = \sqrt{\lambda_{j,\xi}} \sum_k \sqrt{b(\frac{k}{2^j})} \sum_i \alpha_k^i e_k^i(\xi)$$

Let us suppose we are in the simplest case :Π_k is an eigenspace of R^*R:

$$R(e_k^i) = \mu_k h_k^i; \quad R^*R(e_k^i) = \mu_k^2 e_k^i \quad \mu_k \sim k^{-r}, \ r = \text{illposedness.}$$

$$Y = Rf + \epsilon W,$$

$$\langle Y, h_k^i \rangle = \mu_k \alpha_k^i + \epsilon \langle W, h_k^i \rangle$$

$$\hat{\alpha}_k^i = \frac{1}{\mu_k} \langle Y, h_k^i \rangle = \alpha_k^i + \frac{\epsilon}{\mu_k} Z_{k,i}; \quad Z_{k,i} \text{iid } N(0,1).$$

Estimator of $\beta_{j,\xi}$.

$$\hat{\beta}_{j,\xi} = \sqrt{\lambda_{j,\xi}} \sum_k \sqrt{b(\frac{k}{2^j})} \sum_i \hat{\alpha}_k^i e_k^i(\xi)$$

$$= \sqrt{\lambda_{j,\xi}} \sum_k \sqrt{b(\frac{k}{2^j})} \sum_i \frac{1}{\mu_k} \langle Y, h_k^i \rangle e_k^i(\xi)$$

$$= \beta_{j,\xi} + \sqrt{\lambda_{j,\xi}} \sum_k \sqrt{b(\frac{k}{2^j})} \frac{\epsilon}{\mu_k} \sum_i Z_{k,i} e_k^i(\xi)$$

$$\mathbb{E}(\hat{\beta}_{j,\xi} - \beta_{j,\xi})^2 = \lambda_{j,\xi} \epsilon^2 \sum_k \frac{1}{\mu_k^2} \sum_i b(\frac{k}{2^j}) |e_k^i(\xi)|^2$$

$$= \lambda_{j,\xi} \epsilon^2 \sum_k \frac{1}{\mu_k^2} b(\frac{k}{2^j}) L_k(\xi, \xi)$$

Thresholding strategy

$$\mathbb{E}(\hat{\beta}_{j,\xi} - \beta_{j,\xi})^2 \leq \epsilon^2 [\sup_{2^{j-1} \leq k \leq 2^{j+1}} \frac{1}{\mu_k^2}] \lambda_{j,\xi} \sum_k \frac{1}{\mu_k^2} b(\frac{k}{2^j}) L_k(\xi, \xi)$$

$$= \epsilon^2 [\sup_{2^{j-1} \leq k \leq 2^{j+1}} \frac{1}{\mu_k^2}] \|\psi_{j,\xi}\|_2^2$$

$$\lesssim \epsilon^2 2^{jr}$$

So we have the threholding strategy: We keep $\hat{\beta}_{j,\xi}$ up to some threshold $t_{\epsilon,j} = \kappa \sqrt{2^{jr} \epsilon^2 \log \frac{1}{\epsilon}}$:

$$\hat{\beta}_{j,\xi} \mapsto \hat{\beta}_{j,\xi} 1_{|\hat{\beta}_{j,\xi}| > t_{\epsilon,j}}$$

Stopping rule strategy and thresholding estimator.

We define now the thresholding estimator:

$$\hat{f}_\epsilon = \sum_{j \leq J_\epsilon} \sum_{\xi \in \chi_j} \hat{\beta}_{j,\xi} 1_{|\hat{\beta}_{j,\xi}| > t_{\epsilon,j}} \psi_{j,\xi}$$

with the following stopping rule for J :

$$\epsilon^2 2^{J_\epsilon r} \, \#(\chi_{J_\epsilon}) \log(\frac{1}{\epsilon}) \sim 1$$

Part II Wavelet frame on $\mathbb{R}$.

Wavelet, atomic decompositions, sparse representation of functions spaces, appears in the eighties . Let us cite

• M. Frazier, B. Jawerth, Decomposition of Besov spaces, Indiana Univ. Math. J. 34 (1985),

• M. Frazier, B. Jawerth, A discrete transform and decomposition of distribution spaces, J. Funct. Anal. 93 (1990),

• M. Frazier, B. Jawerth, and G. Weiss, Littlewood-Paley theory and the study of function spaces, CBMS No 79 (1991), AMS.

• and then of course all the works of Yves Meyer , Stephane Mallat, Ingrid Daubechies, Ronald Coifman, Victor Wickerhauser, ...

• In the geometric framework, 2000-2010, Petrushev, Ward, Xu, Narcowich, Pesenson.

For application, wavelet is a tool which give a discrete representation of mathematical object through a denumerable family of coefficients($=$ scalar products), and the sparsity of the representation is directly linked to the regularity.

Let us observe that a scalar product is actually an experiment in physics, and the aim is to obtain the more economical family of experiment.

On the other side an object has
two representation:
The physical representation and the spectral world :
•The "real" world : functions, distributions...
•The "spectral " world : an object is the superposition of waves : the Fourier world. The two world are equivalent but some informations could be obtain more easily in one world or in the other.

Moreover the Fourier transform is stable for $\mathbb{L}^2$ but not for other $\mathbb{L}^p$.

Actually: irregularity is the consequence of too much high frequencies.

Regularity on Euclidian spaces.

Simplest concept of regularity on metric space : Lipschitz spaces.

Let (X, d) a metric space, one can define:

$$\forall 0 < \alpha \leq 1, \; Lip_\alpha : \{f : \|f\|_{Lip_\alpha} =$$

$$\|f\|_\infty + \sup_{x,y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} < \infty\}$$

Low frequency approximation.

Let X a Banach space : (here $X = \mathbb{L}^p(\mathbb{R}),\ 1 \leq p \leq \infty$.)
and let $\Sigma : \{\Sigma_t, t \in \mathbb{R}_+\}$ a non decreasing family of "regular" subspace.

$$\text{here } \Sigma^p : \{\Sigma_t^p = \{f \in \mathbb{L}^p,\ supp(\mathcal{F}(f)) \subset \{\xi, |\xi| \leq t\}\}$$

the space of "low frequencies" functions.)
Let us define :

$$\sigma_X(f, t, \Sigma) = \inf_{\phi \in \Sigma_t} \|f - \phi\|_X \quad \text{(The "best" } \Sigma_t \text{ approximation)}$$

Approximation spaces :Besov spaces.

And let us define: for $0 < s < \infty$, $0 < q < \infty$. $B^s_q(X, \Sigma)$:

$$\{f \in X, \|f\|_{B^s_q(X)} = \|f\|_X + [\int_1^\infty (t^s \sigma_X(f, t, \Sigma))^q \frac{dt}{t}]^{\frac{1}{q}} < \infty\}$$

and $B^s_\infty(X, \Sigma)$:

$$\{f \in X, \|f\|_{B^s_q(X)} = \|f\|_X + \sup_{1 \le t < \infty} t^s \sigma_X(f, t, \Sigma) < \infty\}$$

Discrete characterization of $B^s_{p,q}$.

By discretization of $[\int_1^\infty (t^s\sigma_X(f,t,\Sigma))^q \frac{dt}{t}]^{\frac{1}{q}}$ we get easily the following characterization:

$$\text{for } \; 0 < q < \infty, \; f \in B^s_q(X,\Sigma) \Longleftrightarrow$$

$$\|f\|_X + [\sum_{j\geq 0}(2^{js}\sigma_X(f,2^j,\Sigma))^q]^{\frac{1}{q}} < \infty$$

$$\text{for } q = \infty, \; f \in B^s_\infty(X,\Sigma) \Longleftrightarrow$$

$$\|f\|_X + \sup_{j\geq 0}(2^{js}\sigma_X(f,2^j,\Sigma) < \infty$$

Operator characterization of $B^s_q(X,\Sigma)$.

Again let us recall: If:

$$P_j \in \mathcal{L}(X) \quad \text{and } \forall j \geq 0,\ \|P_j\|_{\mathcal{L}(X)} \leq C$$

$$P_j(X) \subset \Sigma_{2^j}$$

$$P_j|\Sigma_{2^{j-1}} = I_d$$

Then

$$f \in B^s_q(X,\Sigma) \Longleftrightarrow f \in X,\ \|P_j(f) - f\|_X = \epsilon_j 2^{-js}\ \epsilon_{.} \in l_q$$

Littlewood-Paley decomposition

Let us recall the classical Littlewood-Paley functions :

$$\hat{\Phi} \geq 0, \text{ even}, \ \hat{\Phi} \in \mathcal{D}(\mathbb{R}), \quad \text{for } |u| \leq \frac{\pi}{4}, \ \hat{\Phi}(u) = 1,$$

$$supp(\hat{\Phi}) \subset \{|u| \leq \frac{\pi}{2}\}, .$$

$$\hat{\Psi}(u) = \hat{\Phi}(\frac{u}{2}) - \hat{\Phi}(u); \quad \hat{\Psi} \geq 0, \ \hat{\Psi} \in \mathcal{D}(\mathbb{R}),$$

$$supp(\hat{\Psi} \subset \{\frac{\pi}{4} \leq |u| \leq \pi\}.$$

$$\text{Then :} \quad 1 = \hat{\Phi}(\xi) + \sum_j \hat{\Psi}(\frac{\xi}{2^j})$$

Littlewood-Paley characterization of $B^s_{p,q}$

So, for all $T \in \mathcal{S}'(\mathbb{R}), \Phi, \Psi \in \mathcal{S}(\mathbb{R})$

$$T = T \star \Phi + \sum_j T \star \Psi_j, \quad \Psi_j(x) = 2^j \Psi(2^j x)$$

As :by YOUNG lemma $\quad \|f \star \Psi_j\|_p \leq \|\Psi_j\|_1 \|f\|_p = \|\Psi\|_1 \|f\|_p$

$$f \in B^s_{p,q} \longleftrightarrow \|f\|_p < \infty,\ \|f - f \star \Phi_j\|_p = \epsilon_j 2^{-js},\ \epsilon_. \in l_q$$

$$f \in B^s_{p,q} \longleftrightarrow \|f\|_p < \infty,\ 2^{js}\|f - f \star \Phi_j\|_p \in l_q.$$

$$f \in B^s_{p,q} \longleftrightarrow \|f\|_p < \infty,\ 2^{js}\|f \star \Psi_j\|_p \in l_q.$$

Interpolation spaces. (Lions-Peetre).

Let $Y \subseteq X$ two Banach space. ($\forall f \in Y,\ \|f\|_X \leq C\|f\|_Y$).
Let us define:

$$\forall f \in X,\ K(t, f, X, Y) = \inf_{\phi \in Y} \|f - \phi\|_X + t\|\phi\|_T)$$

One define the interpolation space between X and Y : $[X, Y]_{\theta,q}$ depending on $0 < \theta < 1,\ 0 < q \leq \infty$:

$$[X, Y]_{\theta,q} :$$

$$\{f \in X,\ \|f\|_{[X,Y]_{\theta,q}} = \|f\|_X + [\int_1^\infty (\frac{K(t, f, X, Y)}{t^\theta})^q \frac{dt}{t}]^{\frac{1}{q}} < \infty\}$$

The interpolations spaces have reiteration properties and continuity properties.

Link between interpolation and best approximation.

Let $\Sigma_t \subset Y \subseteq X,\ t \in \mathbb{R}_+, \Sigma_t$ non decreasing family of subspa

Let $0 < N < \infty$.

$$A^s_q(X,\Sigma) = [X,Y]_{\theta,q}, \quad s = \Theta N$$

under the following properties :

Jackson: $\exists C < \infty,\ \forall f \in Y,\ \sigma_X(f,t,\Sigma) \leq Ct^{-N}\|f\|_Y$

Bernstein: $\exists D < \infty, \forall t > 0, \forall f \in \Sigma_t, \quad \|f\|_Y \leq Dt^N\|f\|_X$

Let us observe that $A^s_q(X,\Sigma)$ does not depend on Y.

Typically if $N \in \mathbb{N}$ the two previous properties are verified for $W_p^N(\mathbb{R}^n)$ and $H_p^N(\mathbb{R}^N)$. So

$$\forall 0 < q \leq \infty, 1 \leq p \leq \infty, s = \theta N,$$

$$A_q^s(\mathbb{L}^p(\mathbb{R}^n), \Sigma_p) = [\mathbb{L}^p, W_p^N(\mathbb{R}^n)]_{\theta,q} = [\mathbb{L}^p, H_p^N(\mathbb{R}^n)]_{\theta,q}$$

The semi-group point of view.

If X is a Banach space and $T_t \in \mathcal{L}(X), t > 0$, a contraction semi group with generator A :

$$\forall f \in X, t > 0,\ \|T_t(f)\|_X \leq \|f\|_X,\ \forall 0 < t, 0 < s,\ T_t \circ T_s = T_{t+s}$$

$$D(A) = \{f \in X, A(f) = \lim_{t \mapsto 0} \frac{T_t(f) - f}{t} \text{ exists}\}$$

$$\forall m \in \mathbb{N},\ \|f\|_{D(A^m)} = \|f\|_X + \|A^m(f)\|_X$$

Moreover we suppose T_t is an holomorphic semi group:

$$\forall t > 0, T_t(X) \subset D(A),\ \exists C,\ t\|A(T_t(f)\|_X \leq C$$

Then if $m \in \mathbb{N}, 0 < \theta < 1, 0 < q \leq \infty$. $[X, D(A^m)]_{\theta,q} =$

$$\{f \in X, \|f\|_X + [\int_0^1 (t^{-\theta m}\|(tA)^m T_t(f)\|_X)^q \frac{dt}{t}]^{\frac{1}{q}} < \infty\}$$

The Laplacian on the real line.

$-\Delta$ is a positive operator on $\mathbb{L}^2(\mathbb{R})$.

$$-\int_{\mathbb{R}} \Delta(f)(x)\overline{f(x)}dx = \int_{\mathbb{R}} |\nabla(f)(x)|^2 dx$$

The associated semi-group is given by :

$$e^{t\Delta} f(x) = g_t \star f(x), \quad g_t(u) = \frac{e^{-u^2/4t}}{2\sqrt{\pi t}}$$

$\sqrt{-\Delta}$ is the generator of the subordinate semi-group :

$$e^{-t\sqrt{-\Delta}} f(x) = h_t \star f(x), \quad h_t(u) = \frac{t}{\pi(t^2+u^2)}$$

The spectral resolution (dim 1)

Let $P_\lambda(f) = \frac{\sin \lambda .}{\pi .} \star f$ be the orthogonal projector operator on Λ_λ

$$\Sigma_\lambda = \{f \in \mathbb{L}^2,\ supp(\widehat{f}) \subset [-\lambda, \lambda]\}$$

$$\langle \sqrt{-\Delta}(f), f \rangle = \int_0^\infty \lambda d \langle E_\lambda f, f \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} |\xi| |\widehat{f}(\xi)|^2 d\xi =$$

$$\int_0^\infty \lambda d (\frac{1}{2\pi} \int_{-\lambda}^{\lambda} |\widehat{f}(\xi)|^2 d\xi) = \int_0^\infty \lambda d \langle P_\lambda(f), f \rangle$$

The spectral decomposition of Id associated to $\sqrt{-\Delta}$ is given by the P_λ.
The spectral decomposition of Id associated to $-\Delta$ is given by the $P_{\sqrt{\lambda}}$.

Semi- group characterization of $B^s_{p,q}$

Let $m \in \mathbb{N}$. One can prove that Bernstein and Jackson are verified with $N = 2m$ for Δ^m and Σ_t So for $0 < s < 2m,\ 2m\theta = s$ we have : taking $D_p(\Delta^m)$ the domain in $\mathbb{L}^p$ of Δ^m,

$$B^s_{p,q} = [\mathbb{L}^p, D_p(\Delta^m)]_{\theta,q}$$

$$= \{f \in \mathbb{L}^p,\ \|f\|_p + [\int_0^1 (t^{-\frac{s}{2}} \|(t\Delta)^m(f)\|)^q \frac{dt}{t}]^{\frac{1}{q}}\}$$

Finite difference characterization.

$$\Delta_y f(x) = f(x+y) - f(x)$$

$$\Delta_y^N f(x) = \Delta_y(\Delta_y^{N-1}) f(x) = \sum_{l=0}^{N} C_N^l (-1)^{N+l} f(x+ly)$$

$$\forall t > 0, \quad \omega_p^N(f,t) = \sup_{|h|\le t} \|\Delta_h^N f\|_p,$$

$f \in B^s_{p,q} \Longleftrightarrow f \in \mathbb{L}^p$ and $\quad \exists N \in \mathbb{N}, N > s,$

and if $0 < q < \infty, \quad [\int_0^1 (\frac{\omega_p^N(t,f)}{t^s})^q \frac{dt}{t}]^{1/q} < +\infty,$

and if $\quad q = +\infty, \quad \sup_{0<t<1} \frac{\omega_p^N(t,f)}{t^s} < +\infty$

$$\Longleftrightarrow \quad \forall j \in \mathbb{N},\ 2^{js}\omega_p^N(2^{-j},f) \in l_q(\mathbb{N})$$

Second tool: Cubature formula.

$$\bullet \forall f \in \Sigma^1_{2\Omega}(\mathbb{R}), \quad \int_{\mathbb{R}} f(u)du = \frac{\pi}{\Omega} \sum_{k\in\mathbb{Z}} f(k\frac{\pi}{\Omega})$$

$$\bullet : \forall f, g \in \Sigma^2_{\Omega}(\mathbb{R}), \quad \int_{\mathbb{R}} f(u)g(u)du = \frac{\pi}{\Omega} \sum_{k\in\mathbb{Z}} f(k\frac{\pi}{\Omega})g(k\frac{\pi}{\Omega})$$

Shannon Wavelet

$$\mathbb{L}^2 = \Sigma_\pi \oplus \{\overline{\oplus_{j\in\mathbb{N}} G_{2^j\pi}}\}, G_{2^j\pi} = \{f, supp(\mathcal{F}(f)) \subset \{2^j\pi \leq |\xi| \leq$$

$$\text{Let} \quad \phi(t) = \frac{\sin \pi t}{\pi t},\ \mathcal{F}(\phi)(\xi) = 1_{[-\pi,\pi]}(\xi)$$

$$\psi_0(t) = 2\phi(2t) - \phi(t) = \frac{\sin \pi t(2\cos \pi t - 1)}{\pi t},\ \ \mathcal{F}(\psi_0)(\xi) = 1_{\pi\leq|\xi|}$$

$$\psi_j(t) = 2^j\psi_0(2^j t),\ \ \mathcal{F}(\psi_j)(\xi) = 1_{2^j\pi\leq|\xi|\leq 2^{j+1}\pi}(\xi)$$

Then, by Shannon sampling theorem :

$$(\phi_k(t) = \phi(t-k))_{k\in\mathbb{Z}} \quad \text{is an orthonormal basis of } \Lambda_\pi,$$

$$\{2^{j/2}\psi(2^j(t-\frac{k}{2^j}))\}_{k\in\mathbb{Z}} \quad \text{is an orthonormal basis of } G_{2^j\pi}.$$

We obtain the Shannon "wavelet basis" which is perfectly localized spectrally, but VERY BADLY localized in space and unfortunatly CANNOT catch the $\mathbb{L}^p$ regularities .

Again: Littlewood-Paley decomposition

Let us recall the classical Littlewood-Paley functions :

$$\hat{\Phi} \geq 0,\ \text{even},\ \hat{\Phi} \in \mathcal{D}(\mathbb{R}), \quad \text{for } |u| \leq \frac{\pi}{4},\ \hat{\Phi}(u) = 1,$$

$$supp(\hat{\Phi}) \subset \{|u| \leq \frac{\pi}{2}\},.$$

$$\hat{\Psi}(u) = \hat{\Phi}(\frac{u}{2}) - \hat{\Phi}(u); \quad \hat{\Psi} \geq 0,\ \hat{\Psi} \in \mathcal{D}(\mathbb{R}),$$

$$supp(\hat{\Psi} \subset \{\frac{\pi}{4} \leq |u| \leq \pi\}.$$

$$\hat{\Phi} = \mathcal{F}(\phi)^2 = \mathcal{F}(\phi \star \phi);\ \hat{\Psi} = (\mathcal{F}(\psi))^2 = \mathcal{F}(\psi \star \psi),$$

$$\phi, \psi \in \mathcal{S}(\mathbb{R});.$$

Then : $\quad 1 = \hat{\Phi}(\xi) + \sum_j \hat{\Psi}(2^{-j}\xi) = \hat{\phi}^2(\xi) + \sum_j \hat{\psi}^2(2^{-j}\xi)$

$$\hat{f}(\xi) = \hat{f}(\xi)\hat{\Phi}(\xi) + \sum_j \hat{f}(\xi)\hat{\Psi}(\frac{\xi}{2^j})$$

$$= \hat{f}(\xi)\hat{\phi}^2(\xi) + \sum_j \hat{f}(\xi)\hat{\psi}^2(\frac{\xi}{2^j}).$$

i.e $\quad f = f \star \Phi + \sum_j \Psi_j \star f = \phi \star \phi \star f + \sum_j \psi_j \star \psi_j \star f$

$$\psi_j(x) = 2^j \psi(2^j x)$$

Wavelet "a la" Frazier, Jawerth, Weiss.

Let us recall the Shannon sampling theorem:

$$\forall f, g \in \Lambda_T,\ \langle f, g \rangle = \sum_k \frac{\pi}{T} f(k\frac{\pi}{T})\overline{g(k\frac{\pi}{T})} \quad \text{(here } T = 2^j \pi)$$

$$\psi_j \star \psi_j (x-y) = \int \psi_j(x-u)\psi_j(y-u)du = \frac{1}{2^j} \sum \psi_j(x-\frac{k}{2^j})\psi_j(y-$$

$$\Psi_j \star f = \psi_j \star \psi_j \star f = \sum_k \langle f, \psi_{j,k} \rangle \psi_{j,k}(x); \quad \psi_{j,k}(x) = 2^{j/2}\psi(2^j(x-$$

$\psi \in \mathcal{S}(\mathbb{R})$, the $\psi_{j,k}(x)$ are LOCALIZED around $\frac{k}{2^j}$ and SPECTRALLY LOCALIZED:

$$supp(\mathcal{F}(\psi_{j,k}) \subset \{\xi,\ 2^{j-2}\pi \leq |\xi| \leq 2^j \pi\}$$

Space concentration.

Moreover

$$\frac{1}{2^j}\sum_k \psi_j(x-\frac{k}{2^j})\psi_j(y-\frac{k}{2^j}) = \sum_k 2^{\frac{j}{2}}\psi(2^jx-k)2^{\frac{j}{2}}\psi(2^jy-k)$$

$$\text{But} \quad \sum_k \|\psi_{j,k}\|_1|\psi_{j,k}(x)| =$$

$$\sum_k \int_{\mathbb{R}} |2^{\frac{j}{2}}\psi(2^jx-k)|dx|2^{\frac{j}{2}}\psi(2^jy-k)| = \|\psi\|_1 \sum_k |\psi(2^jy-k)|$$

$$\leq \|\psi\|_1 \sup_{y\in\mathbb{R}} \sum_k |\psi(2^jy-k)| \leq C$$

So the family $\psi_{j,k}$ (which is a tight frame but not an orthogonal basis) could handle, by the discretization tool : all the $\mathbb{L}_p$ regularity:

For example using as previously the localization :

$$f \in B^s_{p,q} \Longleftrightarrow$$

If $1 \leq p < \infty$,

$$f \in \mathbb{L}^p;\ (\sum_k |\langle f, \psi_{j,k}\rangle|^p \|\psi_{j,k}\|_p^p)^{1/p} = \epsilon_j 2^{-js},\ \epsilon_. \in l_q$$

If $p = \infty$,

$$f \in \mathbb{L}^\infty;\ (\sup_k |\langle f, \psi_{j,k}\rangle| \|\psi_{j,k}\|_\infty = \epsilon_j 2^{-js},\ \epsilon_. \in l_q$$

$$\|\psi_{j,k}\|_p = \|\psi\|_p 2^{j(\frac{1}{2}-\frac{1}{p})}$$

Summary

Main points :

• Regularized spectral decomposition : Littlewood-Paley decomposition. $\Longrightarrow$ Concentration.

• Shannon formula $\Longrightarrow$ Discretization, and frame construction.

This provides a frame

$$\psi_{j,k} = 2^{j/2}\psi(2^j(x - \frac{k}{2^j}))$$

which analyzes and represents regularity spaces in a sparse way.

Our aim is to mimick this ,

BUT WITHOUT FOURIER TRANSFORM.