Some contributions of the maxiset approach to the study of wavelet-based methods in nonparametric function estimation

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based on joint projects with
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Summary

1. Wavelet-based method and information pooling
2. Unidimensional second generation thresholding methods : maxiset approach
3. Multidimensional function estimation
4. Structure detection : asymptotic optimality

Wavelet nested multiscale structure

Definition (Compactly supported periodized wavelet bases)

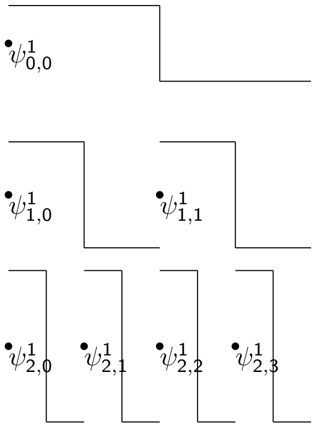
$$B_1 = \left\{ \phi := \psi^0, \psi^1_{j,k}(\cdot) = 2^{j/2}\psi^1\left(2^j \cdot - k\right); \right.$$
$$\left. j \in \mathbb{N}; 0 \leq k < 2^j \right\}.$$

Wlog, consider $f \in L_2[0,1]$,

$$f = \alpha\psi^0 + \sum_{j\geq 0}\sum_{k=0}^{2^j-1} \theta^1_{j,k}\psi^1_{j,k},$$

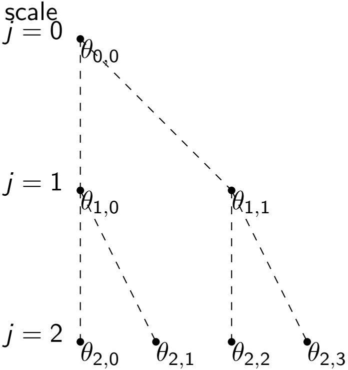
where,

$$\alpha = \left\langle f, \psi^0 \right\rangle$$
$$\theta_{j,k} = \left\langle f, \psi^1_{j,k} \right\rangle$$



Notion of heredity in the coefficient domain

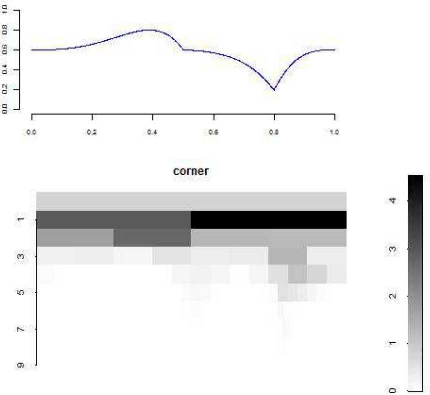
$\mathcal{T}_{j,k}$ the full tree rooted at (j, k)



$\theta_{0,0}$							
$\theta_{1,0}$				$\theta_{1,1}$			
$\theta_{2,0}$		$\theta_{2,1}$		$\theta_{2,2}$		$\theta_{2,3}$	
...	...	...	...	...	...	...	...

heredity : $\theta_{j,k}$ is the parents of its two children coefficients $\theta_{j+1,2k}$ and $\theta_{j+1,2k+1}$.

Clusters of large wavelet coefficients



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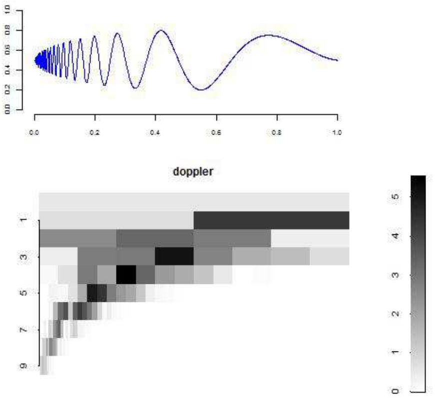

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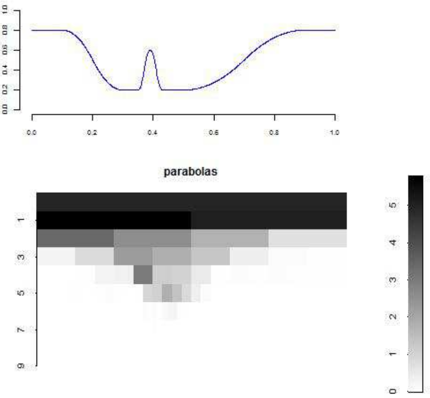

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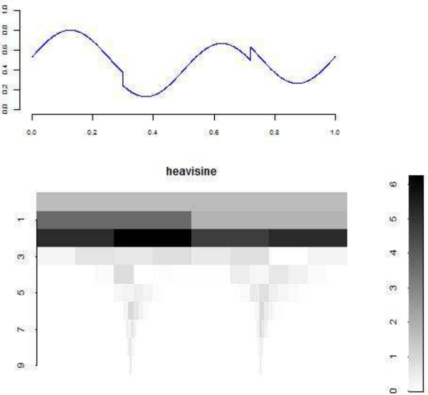

Exploiting the multiscale dependencies

- **Tree approximation** (Cohen et al 2001) is more restrictive than the usual $n-$term approximation but it comes with a little costs in terms of rates of approximation : nevertheless it enables efficient encoding strategies.
- **Multifractal analysis** In practice, multifractal analysis is almost always performed using the coefficients of a continuous or discrete wavelet transform. Jaffard et al introduced a multifractal formalism based on wavelet leaders $\sup_{I' \in 3\mathcal{T}_{j,k}} |\theta_{I'}|$ that allows for accurate estimation of the local hölder exponent,
- **Tree-structured wavelet estimation** : information pooling

- Cohen, A., Dahmen, W., Daubechies, I., Devore, R. (2001). Tree Approximation and Optimal Encoding, Applied and Computational Harmonic Analysis, 11, 192226.
- Jaffard, S., Lashermes, B., Abry, P. (2006). Wavelet leaders in multifractal analysis. In wavelet analysis and applications, Applied and numerical harmonic analysis, 201-246.

Nonparametric function estimation using wavelet thresholding under tree constraint

Consider the nonparametric regression model

$$Y_i = f\left(\frac{i}{n}\right) + \sigma\xi_i,\ \xi_i \sim \mathcal{N}(0,1),\ 1 \leq i \leq n,\ \sigma > 0.$$

- Hard thresholding does not take into account the relations among the wavelet coefficients (influence cones, hereditary structure. . .),

$$\hat{f}^H = \hat{\alpha}\psi^0 + \sum_j \sum_k \hat{\theta}_{j,k}\mathbf{1}_{\{|\hat{\theta}_{j,k}|>\lambda\}}\psi^1_{j,k}.$$

- ⇒ "second generation" wavelet thresholding : Information pooling within some geometric structures in the coefficient domain (horizontal/vertical blocks) to refine the choice of coefficients to Keep/kill,

Many theoretical results about horizontal block thresholding (Cai et al, 1997 ; Hall et al, 1999 ; Cai, 2009) ; not much on TSW.

- Cai, T. (1997), 'On Adaptivity of Blockshrink Wavelet Estimator over Besov Spaces', Technical Report 97-05, Purdue University.
- Hall, P.,Kerkycharian, G., and Picard, D. (1998a), 'Block Threshold Rules for Curve Estimation UsingKernel and Wavelet Methods', Annals of Statistics, 26(3), 922942.
-Hall, P., Kerkycharian, G., and Picard, D. (1998b), 'On the Minimax Optimality for Block Thresholded Wavelet Estimators', Statistica Sinica, 9, 3349.

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- $\Rightarrow$ "second generation" wavelet thresholding : Information pooling within some geometric structures in the coefficient domain (horizontal/vertical blocks) to refine the choice of coefficients to Keep/kill,

the tree / hereditary constraint imposes that if a coefficient is used to compute the estimator, all its ancestors have to be included. Tree-Structured Wavelets (TSW).
How to impose tree structure :

- Greedy Tree Approximation (GTA)
- Condensing Sort and Select Algorithm (CSSA)
- Complexity Penalized Sum if Squares (CPSS)

Model and family of estimators

Definition (Sequential Gaussian White noise)

$$\hat{\alpha} = \alpha + \varepsilon\xi \qquad \hat{\theta}_{j,k} = \theta_{jk} + \varepsilon\xi_{j,k}, j \in \mathbb{N}; 0 \leq k < 2^j.$$

where $\xi, \xi_{j,k}$ are i.i.d. $\mathcal{N}(0,1)$ and $\varepsilon \in]0,1[$ (noise level).

Definition ((λ, q)-VBT estimators)

Use a coarse to fine dynamic programming algorithm on the tree. Attributes ℓ_q scores to overlapping vertical blocs of coefficients.
Let $q \in [1, +\infty]$. The estimator $\hat{f}_q$ is defined by :

$$\hat{f}_q = \hat{\alpha}\psi^0 + \sum_{j<j_{\lambda_\varepsilon}} \sum_{k=0}^{2^j-1} \hat{\theta}_{j,k} \, \mathbf{1}\left\{(j,k) \in \tau_q(\hat{\theta}, \lambda_\varepsilon)\right\} \psi^1_{j,k}.$$

Model and family of estimators

Remark :

- $\hat{f}_2$: Oracle approach (Donoho, 1997).
- $\hat{f}_\infty$: hard thresholding + hereditary constraint (Autin, 2008).

Definition (HARD TREE estimator ($\hat{f}_\infty = \hat{f}^{HT}$))

$$\hat{f}^{HT}(\cdot) = \hat{\alpha}\psi^0(\cdot) + \sum_{j<j_{\lambda_\varepsilon}} \sum_{k=0}^{2^j-1} \hat{\theta}_{j,k} \mathbf{1}\left\{ \max_{(j',k')\in\mathcal{T}_{j,k}} \left|\hat{\theta}_{j',k'}\right| > \lambda_\varepsilon \right\} \psi^1_{j,k}(\cdot).$$

- Tomassi, D., Milone, D., Nelson, J.D.B. (2015) Wavelet shrinkage using adaptive structured sparsity constraint. Signal Processing, 106, 73-87

Use the 'recent' Dual Tree Complex Wavelet Transform (Kingsbury, 2001) : nearly shift invariant and has low redundancy.

Tree-structured estimation using a lasso-type algorithm with a mixed norm regularizer that induces structured sparsity of an overcomplete representation (overlapping group lasso) with adaptive weights.

It has been shown that there is more information to retrieve intra-inter scale within DTCWT coefficients domain than DWT.

- Kingsbury, N. (2001). Complex Wavelets for Shift Invariant Analysis and Filtering of Signals. ACHA, 10, 234253

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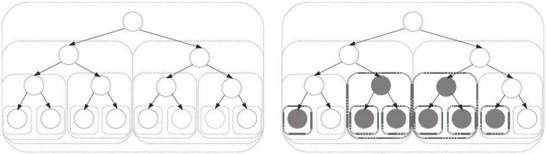

Fig. 2. (a) Grouping scheme for the proposed mixed-norm regularisation and (b) example of estimate induced by the adopted grouping scheme; its support is the complement of the union of the sets pushed to zero during optimisation which in turn sets to zero the coefficients represented by shaded circles.

FIGURE: forest like hierarchical tree structure

MAXISET THEORY

<u>Remark :</u> Minimax approach has some drawbacks :

- choice of $\mathcal{F}$,
- pessimistic approach (at least w.r.t maxiset approach),
- no comparison of *optimal* estimators.

Definition (Maxiset approach (Cohen, Kerkyacharian, Picard (2001)))

The maxiset of an estimator $\hat{f}_\varepsilon$ is the largest functional space where its risk attains a given convergence rate v_ε :

$$MS_2\left(\hat{f}_\varepsilon, v_\varepsilon\right) = \left\{ f : \sup_{0<\varepsilon<1} {v_\varepsilon}^{-1}\mathbb{E}\|\hat{f}_\varepsilon - f\|_2^2 < +\infty \right\}.$$

Remarks :

- Estimator with good *performance* $\Longleftrightarrow$ *large* Maxiset.
- Rate $\nearrow$ Maxiset $\searrow$.
- Usually $v_\varepsilon = v_{\mathcal{F},\varepsilon}$ or $v_{\mathcal{F},\varepsilon}|\ln\varepsilon|^\delta$
- estimator $\leftrightarrow$ maxiset

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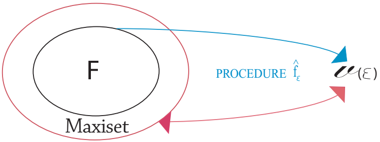

VERTICAL BLOC THRESHOLDING

Theorem (Autin, F. et von Sachs (2011))

Let $s > 0$, $q \in \ [1, +\infty]$, $m \geq 4\sqrt{3}$ *and* $v_\varepsilon = \left(\varepsilon\sqrt{|\ln \varepsilon|}\right)^{\frac{4s}{1+2s}}$.

$$MS_2\left(\hat{f}_q, v_\varepsilon\right) = \mathcal{B}_{2,\infty}^{\frac{s}{1+2s}} \cap \mathcal{W}^V_{\frac{2}{1+2s},q}.$$

$$\mathcal{B}_{2,\infty}^s = \left\{ f; \sup_{J\in\mathbb{N}} 2^{2Js} \sum_{j\geq J} \sum_{k=0}^{2^j-1} |\theta_{j,k}|^2 < \infty \right\},$$

$$\mathcal{W}^V_{r,q} = \left\{ f : \sup_{0<\lambda<1} \lambda^{r-2} \sum_j \sum_k \theta_{jk}^2 \, \mathbf{1}\left\{(j,k) \notin \tau_q(\theta,\lambda)\right\} < +\infty \right\},$$

$$\bigcap$$

$$\mathcal{W}^V_{r,q'} = \left\{ f : \sup_{0<\lambda<1} \lambda^{r-2} \sum_j \sum_k \theta_{jk}^2 \, \mathbf{1}\left\{(j,k) \notin \tau_{q'}(\theta,\lambda)\right\} < +\infty \right\}, q' > q.$$

Autin, F., F, J-M., von Sachs, R. (2011). Ideal denoising within a family of tree-structured wavelet estimators. *Electronic Journal of Statistics*, 5, 829–855.

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$$MS_2\left(\hat{f}_q, v_\varepsilon\right) = \mathcal{B}^{\frac{s}{1+2s}}_{2,\infty} \cap \mathcal{W}^{V}_{\frac{2}{1+2s},q}.$$

Theorem (Autin, F. and von Sachs (2011))

From the maxiset point of view :

1. *the best estimator is :* $\hat{f}_\infty$ *(Hard Tree).*
2. *moreover* $\hat{f}_\infty$ *is better estimator than* $\hat{f}^H$.

Autin, F., F, J-M., von Sachs, R. (2011). Ideal denoising within a family of tree-structured wavelet estimators. *Electronic Journal of Statistics*, 5, 829–855.

NUMERICAL EXPERIMENTS

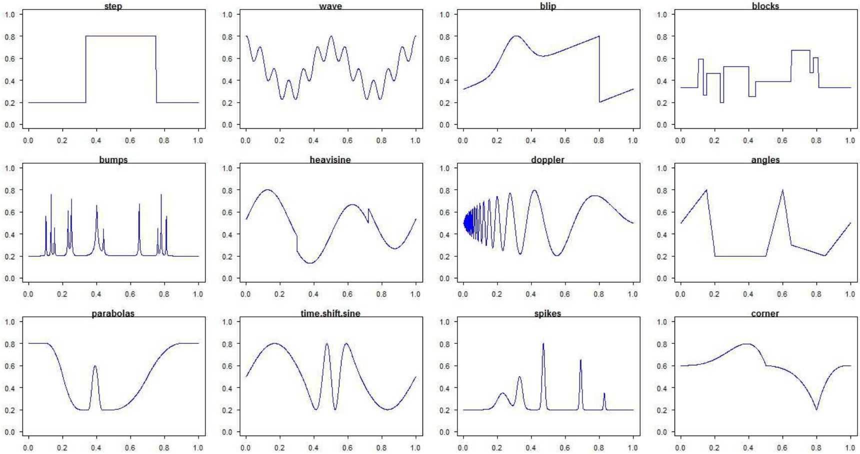


FIGURE: Test functions

NUMERICAL EXPERIMENTS

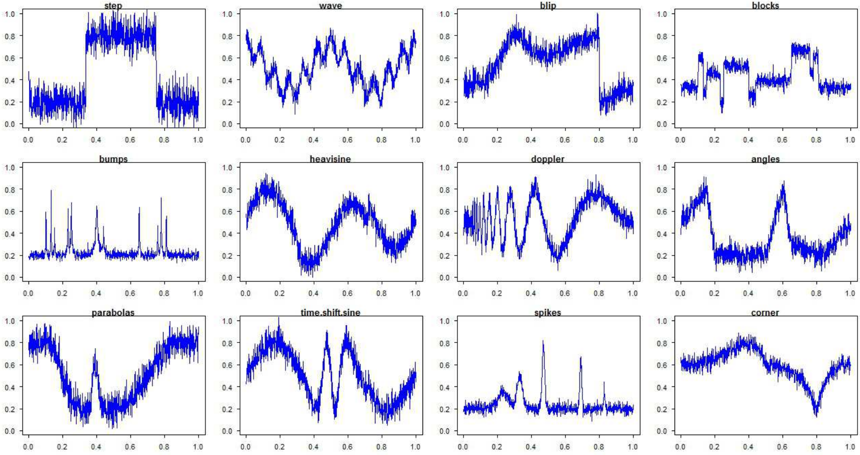


FIGURE: Noisy functions

NUMERICAL EXPERIMENTS

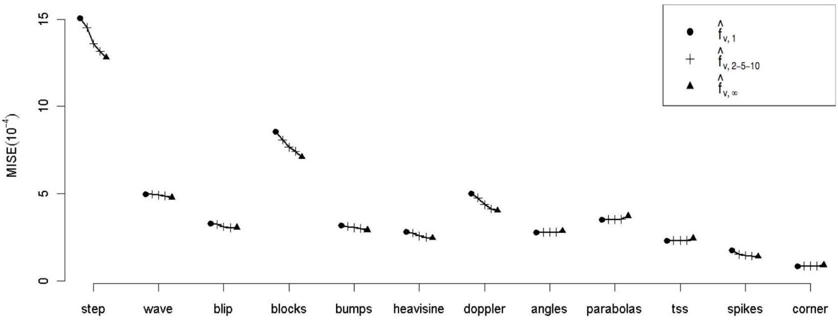


FIGURE: *MISE* estimators $\hat{f}_q$ (Db8, SNR = 10db, N=1024)

Multidimensional function estimation

Multivariate wavelet bases

Motivation : multivariate wavelet bases are well adapted to described functions in $\mathbb{L}_2([0,1]^d)$ in a parsimonious way. Atoms are localized both in time and frequency.

Two popular multidimensional wavelet bases :

1. Standard wavelet basis.
2. Hyperbolic wavelet basis.

Actually, a large regain of interest in :

- Statistics (Neumann (2000), Benhaddou et al. (2013)).
- Approximation theory (Kerkyacharian et al. (2006)).
- Compressive sensing (Duarte, Baraniuk (2012).
- Multifractal analysis (Abry et al. (2014))

- Benhaddou, R., Pensky, M., Picard, D. (2013) Anisotropic denoising in functional deconvolution model with dimension-free convergence rates, EJS, 7, 1686-1715.
- Duarte, M.F., Baraniuk, R. (2012) Kronecker Compressive Sensing. IEEE Transactions on Image Processing, 21(2), 494-504.
- P. Abry, M. Clausel, S. Jaffard, S.G. Roux, B. Vedel, Hyperbolic wavelet transform : an efficient tool for multifractal analysis of anisotropic textures.Revista Matematica Iberoamericana, vol 31(1), pp 313-348 (2015)
- Kerkyacharian, G., Picard, D., Temlyakov, V. (2006) Some inequalities for the tensor product of greedy bases and weight-greedy bases. East journal on approximations, 12(1), 103-118.

Standard and hyperbolic wavelet bases

Definition

From a periodized univariate wavelet basis

$$\mathcal{B}_1 = \left\{ \psi^0, \psi^1_{j,k} : j \in \mathbb{N}, 0 \le k < 2^j \right\},$$

- *the standard wavelet basis is defined as :*
$$\mathcal{I}_d = \left\{ \psi^{\underline{0}}_{\underline{0},\underline{0}}, \psi^{\underline{i}}_{\underline{j},\underline{k}} : \underline{i} \in \{0,1\}^d \setminus \underline{0},\ \underline{j} \in \mathbb{J},\ \underline{k} \in \mathbb{K}_{\underline{j}} \right\},$$
- *the hyperbolic wavelet basis is defined as :*
$$\mathcal{H}_d = \left\{ \psi^{\underline{0}}_{\underline{0},\underline{0}}, \psi^{\underline{i}}_{\underline{j},\underline{k}} : \underline{i} \in \{0,1\}^d \setminus \underline{0},\ \underline{j} \in \mathbb{J}^{\underline{i}},\ \underline{k} \in \mathbb{K}_{\underline{j}} \right\}.$$

$$\begin{aligned}
\psi^{\underline{0}}_{\underline{0},\underline{0}} &= \psi^0 \times \cdots \times \psi^0 \quad \text{and} \quad \psi^{\underline{i}}_{\underline{j},\underline{k}} = \psi^{i_1}_{j_1,k_1} \times \cdots \times \psi^{i_d}_{j_d,k_d}, \\
\mathbb{J} &= \left\{ \underline{j} = (j, \cdots, j) : j \in \mathbb{N} \right\}, \\
\mathbb{J}^{\underline{i}} &= \left\{ \underline{j} = (j_1 i_1, \cdots, j_d i_d) : \forall u, j_u \in \mathbb{N} \right\}, \\
\mathbb{K}_{\underline{j}} &= \left\{ \underline{k} = (k_1, \cdots, k_d) : \forall u, 0 \le k_u < 2^{j_u} \right\}.
\end{aligned}$$

2-D hyperbolic wavelet basis

Let ψ^1 be a 1-D wavelet and ψ^0 its associated scaling function.

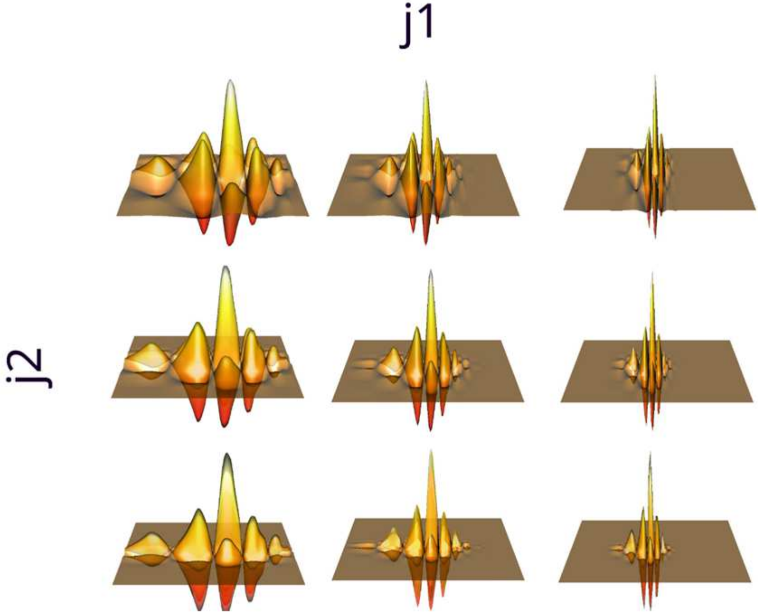
$$\begin{array}{rcll} \psi^{(0,0)}(x,y) & = & \psi^0(x)\,\psi^0(y), & |\underline{i}| = 0 \\ \psi^{(1,0)}_{j_1,k_1}(x,y) & = & \psi^1_{j_1 k_1}(x)\,\psi^0(y), & |\underline{i}| = 1 \\ \psi^{(0,1)}_{j_2,k_2}(x,y) & = & \psi^0(x)\,\psi^1_{j_2 k_2}(y), & |\underline{i}| = 1 \\ \psi^{(1,1)}_{j_1,j_2,k_1,k_2}(x,y) & = & \psi^1_{j_1 k_1}(x)\,\psi^1_{j_2 k_2}(y), & |\underline{i}| = 2 \end{array}$$

The support is a rectangle of size $2^{-j_1} \times 2^{-j_2}$: optimal to adapt to anisotropic smoothness.

Consider $f \in L_2([0,1]^2)$.

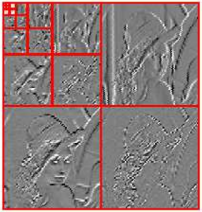
$$\begin{aligned} f &= \underbrace{\text{avg approx}} + \underbrace{\text{horizontal details}} + \underbrace{\text{vertical details}} + \underbrace{\text{diagonal details}} \\ &= \alpha\phi_{00} + \sum_{j_1 \geq 0}\sum_{k_1} \theta^{(1,0)}_{j_1 k_1}\psi^{(1,0)}_{j_1 k_1} + \sum_{j_2 \geq 0}\sum_{k_2} \theta^{(0,1)}_{j_2 k_2}\psi^{(0,1)}_{j_2 k_2} + \sum_{j_1,j_2 \geq 0}\sum_{k_1,k_2} \theta^{(1,1)}_{j_1 j_2 k_1 k_2}\psi^{(1,1)}_{j_1,j_2,k_1,k_2} \\ &= \alpha\phi_{00} + \sum_{\underline{i}=\{h,v,d\}}\sum_{\underline{j}\in\mathbb{J}^{\underline{i}}}\sum_{\underline{k}\in\mathbb{K}_{\underline{j}}} \theta^{\underline{i}}_{\underline{j},\underline{k}}\psi^{\underline{i}}_{\underline{j},\underline{k}} \end{aligned}$$

Hyperbolic wavelet functions

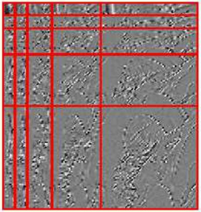


Standard vs Hyperbolic wavelet transforms

lena



Isotropic



Hyperbolic

Wavelet decomposition of a multivariate function

Consider $f \in \mathbb{L}_2([0,1]^d)$:

$$f = \text{avg approx} + \sum_{\underline{i} \in \{0,1\}^d \setminus \underline{0}} \text{details in orientation } \underline{i}$$

$$= \theta_{\underline{0},\underline{0}}^{\underline{0}} \psi_{\underline{0},\underline{0}}^{\underline{0}} + \sum_{\underline{i} \in \{0,1\}^d \setminus \underline{0}} \left(\sum_{\underline{j} \in \mathcal{J}} \sum_{\underline{k} \in \mathbb{K}_{\underline{j}}} \theta_{\underline{j},\underline{k}}^{\underline{i}} \psi_{\underline{j},\underline{k}}^{\underline{i}} \right),$$

where

- $\mathcal{J} = \mathbb{J}$ for the standard wavelet decomposition,
- $\mathcal{J} = \mathbb{J}^{\underline{i}}$ for the hyperbolic wavelet decomposition.

Remark : in the multivariate wavelet setting

$$\|f\|_2^2 = (\theta_{\underline{0},\underline{0}}^{\underline{0}})^2 + \sum_{\underline{i} \in \{0,1\}^d \setminus \underline{0}} \sum_{\underline{j} \in \mathcal{J}} \sum_{\underline{k} \in \mathbb{K}_{\underline{j}}} \left(\theta_{\underline{j},\underline{k}}^{\underline{i}} \right)^2 .$$

Besov spaces of smooth functions

Definition (Characterisation of Besov space (Abry et al. (2014)))

The Besov space $\mathcal{B}^{\underline{s}}$ is defined as follows :

$$\mathcal{B}^{\underline{s}} = \left\{ f \in \mathbb{L}_2([0,1]^d) : \sup_{\underline{i} \neq \underline{0}} \sup_{\underline{j} \in \mathbb{J}^{\underline{i}}} \left(\max_{1 \le u \le d} 2^{2 j_u s_u} \right) \sum_{\underline{k} \in \mathbb{K}_{\underline{j}}} \left(\theta^{\underline{i}}_{\underline{j},\underline{k}} \right)^2 < +\infty \right\}.$$

- Isotropic Besov spaces : $\underline{s} = (s_1, \cdots, s_d) : \forall (u,v),\ s_u = s_v.$
- Anisotropic Besov spaces : $\underline{s} = (s_1, \cdots, s_d) : \exists (u,v),\ s_u \neq s_v.$

Theorem (Neumann (2000))

Minimax rate over Besov spaces

$$\mathcal{R}^*_\varepsilon(\mathcal{B}^{\underline{s}}) \asymp \varepsilon^{\frac{4\gamma}{1+2\gamma}}.$$

$$\text{where } \gamma = \left(s_1^{-1} + \cdots + s_d^{-1} \right)^{-1} > \frac{1}{2}.$$

Hierarchical structure

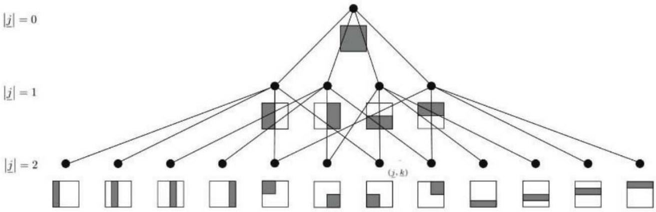


FIGURE: Hierarchical structure $\mathcal{T}^{\underline{i}}$ coming from the MRA ($d = 2, \underline{i} = (1, 1)$)

Proposition (Autin, Claeskens and F.(2014))

Consider $\underline{i} \in \{0, 1\}^d \setminus \underline{0}$. *Each node* $(\underline{j}, \underline{k})$ *of* $\tau_{\infty}^{\underline{i}}$,

1. *has* $2|\underline{i}|$ *children at scales* $|\underline{j}| + 1$ *;*
2. *has at most* $|\underline{i}|$ *ancestors at scales* $|\underline{j}| - 1$ *and a total number of ancestors equals to* $\prod_{u=1}^{d}(j_u + 1)^{i_u}$.

HEREDITARY STRUCTURE

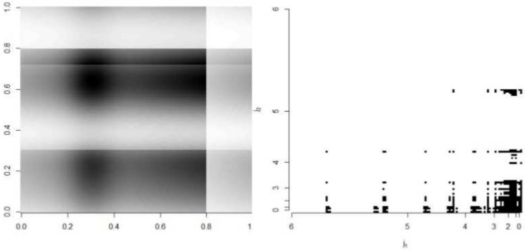


FIGURE: Hereditary structure among θ_{j_1,j_2,k_1,k_2}

Estimation in Hyperbolic wavelet basis

Hyperbolic basis :

$$H_d = \left\{ \psi^{\underline{0}}_{\underline{0},\underline{0}},\ \psi^{\underline{i}}_{\underline{j},\,\underline{k}} : \quad \underline{i} \in \{0,1\}^d \setminus \underline{0};\quad \underline{j} \in \mathbb{J}^{\underline{i}};\ \underline{k} \in \mathbb{K}_{\underline{j}} \right\}.$$

Definition

Hard Thresholding estimator ($c \geq 1/2$) :

$$\hat{f}^H_c \;=\; \hat{\alpha}\phi_{\underline{0},\underline{0}} + \sum_{\underline{i}\neq\underline{0}} \sum_{\underline{j}\in\mathbb{J}^{\underline{i}},\ |\underline{j}|<j_{\lambda_{\varepsilon,c}}} \sum_{\underline{k}} \hat{\theta}^{\underline{i}}_{\underline{j},\underline{k}} \mathbf{1}\left\{ |\hat{\theta}^{\underline{i}}_{\underline{j},\underline{k}}| > \lambda_{\varepsilon,c} \right\} \psi^{\underline{i}}_{\underline{j},\underline{k}}$$

Hard Tree estimator ($c \geq 1/2$) :

$$\hat{f}^{HT}_c = \hat{\alpha}\phi_{\underline{0},\underline{0}} + \sum_{\underline{i}\neq\underline{0}} \sum_{\underline{j}\in\mathbb{J}^{\underline{i}},\ |\underline{j}|<j_{\lambda_{\varepsilon,c}}} \sum_{\underline{k}} \hat{\theta}^{\underline{i}}_{\underline{j},\underline{k}} \mathbf{1}\left\{ \max_{\underline{j}',\underline{k}'\in\mathcal{T}^{\underline{i}}_{\underline{j},\underline{k}}(\lambda_{\varepsilon,c})} |\hat{\theta}^{\underline{i}}_{\underline{j}',\underline{k}'}| > \lambda_{\varepsilon,c} \right\} \psi^{\underline{i}}_{\underline{j},\underline{k}}$$

$$\text{with} \quad \lambda_{\varepsilon,c} \;=\; \varepsilon|\ln\varepsilon|^c \ \text{and}\ 2^{-j_{\lambda_{\varepsilon,c}}} = (\lambda_{\varepsilon,c})^2.$$

RESULTATS MAXISETS

Theorem (Autin, Claeskens et Freyermuth (2014))

Let $\gamma > 0$ $p \geq 2$ *et* $c \geq 1/2$. *Si* $m \geq 4\sqrt{p+1}$, *then :*

$$MS_p\left(\hat{f}^H, (\varepsilon|\ln\varepsilon|^c)^{\frac{2\gamma p}{1+2\gamma}}\right) = \mathcal{A}_{p,\infty}^{\frac{\gamma}{1+2\gamma}} \cap \mathcal{W}^H_{\frac{p}{1+2\gamma}},$$

$$MS_p\left(\hat{f}^{HT}, (\varepsilon|\ln\varepsilon|^c)^{\frac{2\gamma p}{1+2\gamma}}\right) = \mathcal{A}_{p,\infty}^{\frac{\gamma}{1+2\gamma}} \cap \mathcal{W}^{HT}_{\frac{p}{1+2\gamma}} \cap \mathcal{W}^{HT,\star}_{\frac{p}{1+2s},c}.$$

$$\mathcal{A}^r_{p,\infty} = \left\{ f : \sup_{J\in\mathbb{N}} \sum_{\underline{i}\neq 0} \sum_{\underline{j}\in\mathbb{J}^{\underline{i}};\ |\underline{j}|\geq J} 2^{Jrp+|\underline{j}|(p/2-1)} \sum_{\underline{k}\in\mathbb{K}_{\underline{j}}} \left|\theta^{\underline{i}}_{\underline{j},\underline{k}}\right|^p < +\infty \right\},$$

$$\mathcal{W}^H_r = \left\{ f : \sup_{0<\lambda<1} \lambda^{r-p} \sum_{\underline{i}\neq 0} \sum_{\underline{j}\in\mathbb{J}^{\underline{i}}} 2^{|\underline{j}|(p/2-1)} \sum_{\underline{k}\in\mathbb{K}_{\underline{j}}} \left|\theta^{\underline{i}}_{\underline{j},\underline{k}}\right|^p \mathbf{1}\left\{\left|\theta^{\underline{i}}_{\underline{j},\underline{k}}\right| \leq \lambda\right\} < +\infty \right\},$$

$$\cap$$

$$W^{HT}_r = \left\{ f : \sup_{0<\lambda<1} \lambda^{r-p} \sum_{\underline{i}\neq 0} \sum_{\underline{j}\in\mathbb{J}^{\underline{i}}} 2^{|\underline{j}|(p/2-1)} \sum_{\underline{k}\in\mathbb{K}_{\underline{j}}} \left|\theta^{\underline{i}}_{\underline{j},\underline{k}}\right|^p \mathbf{1}\left\{\max_{\underline{j}',\underline{k}'\in\mathcal{T}^{\underline{i}}_{\underline{j},\underline{k}}(\lambda)} |\theta^{\underline{i}}_{\underline{j}',\underline{k}'}| \leq \lambda\right\} < +\infty \right\},$$

$$W^{HT,\star}_{r,c} = \left\{ f : \sup_{0<\lambda<1} \lambda^{r} \log(\lambda^{-1})^{-pc} \sum_{\underline{i}\neq 0} \sum_{\underline{j}\in\mathbb{J}^{\underline{i}}} 2^{|\underline{j}|(p/2-1)} \sum_{\underline{k}\in\mathbb{K}_{\underline{j}}} \mathbf{1}\left\{\max_{\underline{j}',\underline{k}'\in\mathcal{T}^{\underline{i}}_{\underline{j},\underline{k}}(\lambda)} |\theta^{\underline{i}}_{\underline{j}',\underline{k}'}| > \lambda\right\} < +\infty \right\}.$$

Autin, F., Claeskens, G., Freyermuth, J-M. (2014). Hyperbolic wavelet thresholding rules : the curse of dimensionality through the maxiset approach. *Applied and Computational Harmonic Analysis*, 36, 239-255.

Curse of dimensionality

Constat : Dimension $d \nearrow$ sequence space $\mathcal{W}^{\star}_{r,c} \searrow$.
Is *information pooling* still interesting for $d > 1$?

Theorem (Autin, Claeskens et F. (2014))

Let $p \geq 2$ and $c \geq 1/2$. Then

$$MS_p\left(\hat{f}^H, (\varepsilon|\ln\varepsilon|^c)^{\frac{2\gamma p}{1+2\gamma}}\right) \subset MS_p\left(\hat{f}^{HT}, (\varepsilon|\ln\varepsilon|^c)^{\frac{2\gamma p}{1+2\gamma}}\right),$$

once one of the following properties is satisfied :

1. $d \leq pc$ *(small dimension),*
2. $d > pc$ *(large dimension) and structural constraint* $[H_p(c)]$.

Definition

A function $f \in L_p([0,1])$ satisfies the **structural constraint** $[H_p(c)]$ iff, for all $|\underline{i}| \geq pc$ and all $0 < \lambda < 1$,

$$|\langle f, \psi^{\underline{i}}_{\underline{j},\underline{k}}\rangle| > \lambda \implies \underline{j} \text{ is such that } \max\{j_u;\ 1 \leq u \leq d\} \leq |\ln\lambda|^{\frac{pc}{|\underline{i}|}}.$$

CURSE OF DIMENSIONALITY

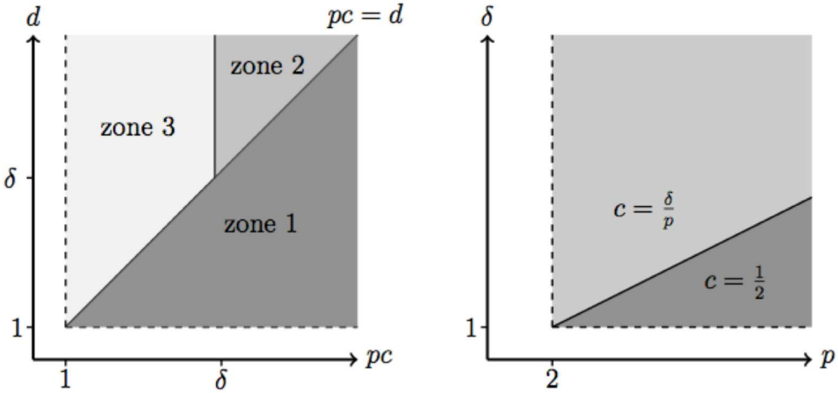


FIGURE: *information pooling*? (Y/N)

Zone 1 : $d \leq pc$. Zone 2 : $d > pc$ and $\delta \leq pc$. Zone 3 : $\delta > pc$.

Conclusion

- Information pooling in signal and image processing has promising future in applications
- still lot of challenges to precisely characterized the performances methods using information pooling
- the maxiset approach is an effective way to asses the performance of methods offering a different perspective to problem

Time-varying spectrum

Following R.Dahlhaus the evolutionary spectrum of locally stationary process can be written as :

$$f(u,\omega) \;=\; \lim_{T\to\infty} f_W([uT],\omega)\,, \qquad u=\frac{t}{T} \;\in\; [0,1]$$

with the *Wigner–Ville* spectrum

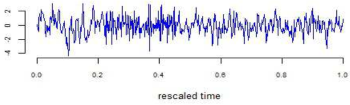
$$f_W(t,\omega) = \sum_s \mathbb{C}\mathrm{ov}\left(X_{t-s/2}\,,X_{t+s/2}\right)\;\exp(-i\omega s)$$

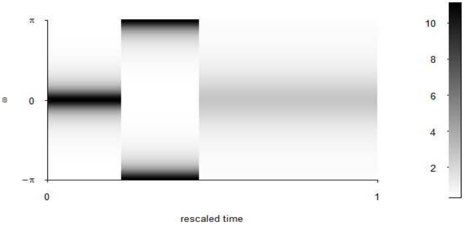
Estimation :
First approaches : Classical periodograms on segments of length N

$$I_N(u,\omega) \;=\; \frac{1}{2\pi N}\left|\sum_{s=0}^{N-1} X_{[uT-N/2+s+1]}\,\exp(-is\omega)\right|^2$$

Drawback : Choice of the length of the blocks N ?

Example : a Piecewise stationary AR process





Time-varying spectrum

Example : a Piecewise stationary AR process

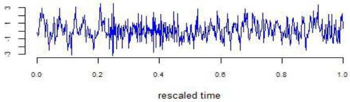
Definition (Preperiodogram/empirical Wigner Ville distribution)

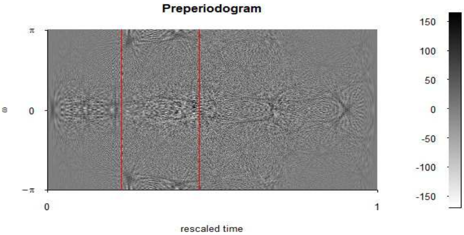
Fourier transform of empirical local autocovariance

Rescaled preperiodogram of a process X

$$I_X(u,\omega) := (2\pi)^{-1} \sum_{s:1\leq uT-s/2,\, uT+s/2\leq T} X_{[uT-s/2],T}\, X_{[uT+s/2],T}\, \exp(-is\omega)$$

$\mathbb{E}\ I_X(u,\omega) \rightarrow f(u,\omega)$ as $T \rightarrow \infty$.





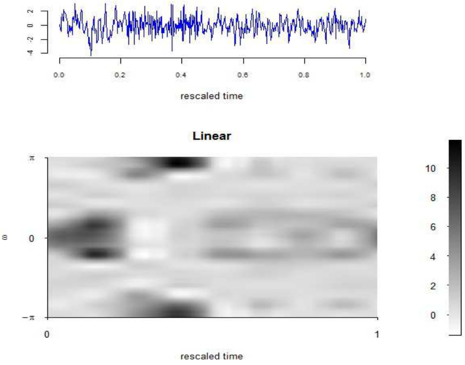
Time-varying spectrum

Smoothing of the preperiodogram

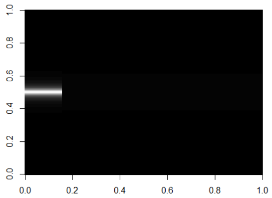
$$\hat{f}(u,\omega) \;:=\; I\,(u,\omega) ** K(u,\omega)$$

where $**$ is the two dimensional convolution operator with a kernel K.

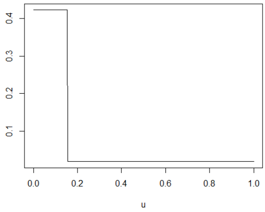
Example : a Piecewise stationary AR process



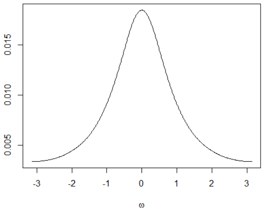
Anisotropy



TimeVaryingspectrum



cut along time



cut along frequency

Structure detection

Sobol decomposition of $f \in L_2\left[0,1\right]^d$

$$f(x_1,\ldots,x_d) = f_0 + \sum_{u=1}^{d} f_u\left(x_u\right) + \sum_{u<v} f_{uv}\left(x_u,x_v\right) + \ldots + f_{1,\ldots,d}\left(x_1,\ldots,x_d\right) \qquad (1)$$

Marginal components (main effect) $f_u : \ [0,1) \longrightarrow \mathbb{R} \ \ (1 \leq u \leq d)$

Dalalyan, A., Ingster, Y., Tsybakov, A. (2014). Statistical inference in compound functional models. *Probability Theory and Related Fields*, 158(3), pp. 512–532.

The atomic dimension δ of f reflects the maximal degree of interaction between the d variables within f.

- Additive structure : $f(\underline{x}) = \sum_{u=1}^{d} f_u(x_u) \ \rightarrow \delta = 1.$
- Sructure with maximal degree of interaction $m < d$:

$$f(\underline{x}) = \sum_{u_1<\cdots<u_m} f_{u_1,\cdots,u_m}(x_{u_1},\cdots,x_{u_m}) \ \rightarrow \delta = m.$$

Structure detection

Sobol decomposition of $f \in L_2[0,1]^d$

$$f(x_1,\ldots,x_d) = f_0 + \sum_{u=1}^{d} f_u(x_u) + \sum_{u<v} f_{uv}(x_u,x_v) + \ldots + f_{1,\ldots,d}(x_1,\ldots,x_d) \quad (1)$$

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The atomic dimension δ of f reflects the maximal degree of interaction between the d variables within f.

Back to Time-Varying spectrum

Question : is the process stationary ?

Multivariate/Multiple nonparameric regression model :

$$Y_{ij} = f(u_i,\omega_j) + \text{error}_{ij}, \ f : [0,1]^2 \to \mathbb{R}$$

i.e., is the variable t_i relevant

$$f(u,\omega) = f_2(\omega) \quad (2)$$

Structure detection

Sobol decomposition of $f \in L_2 [0,1]^d$

$$f(x_1, \ldots, x_d) = f_0 + \sum_{u=1}^{d} f_u(x_u) + \sum_{u<v} f_{uv}(x_u, x_v) + \ldots + f_{1,\ldots,d}(x_1, \ldots, x_d) \quad (1)$$

Marginal components (main effect) $f_u : \ [0,1) \longrightarrow \mathbb{R} \ \ (1 \leq u \leq d)$

Dalalyan, A., Ingster, Y., Tsybakov, A. (2014). Statistical inference in compound functional models. *Probability Theory and Related Fields*, 158(3), pp. 512–532.

The atomic dimension δ of f reflects the maximal degree of interaction between the d variables within f.

Back to Time-Varying spectrum

Question : the process is non stationary but is the behavior similar across the different frequency bands ?

Multivariate/Multiple nonparameric regression model :

$$Y_{ij} = f(u_i, \omega_j) + \text{error}_{ij}, \ \ f : [0,1]^2 \to \mathbb{R}$$

i.e., is there interaction between time and frequency ?

$$f(u, \omega) = f_1(u) + f_2(\omega) \quad (3)$$

Approach

2 characteristic ingredients of our multivariate data :

- general structure (atomic dimension, variable selection)
- anisotropy

Objective : Propose procedures to test for structure of multivariate data with special emphasis of being 'optimal' (or near optimal) under anisotropic smoothness properties.

Method : Find appropriate representation in a wavelet basis → design procedures for detecting some structural characteristics of the estimand :

- maximal degree of interaction of variables (atomic dimension),
- variable selection...

Construct (near) minimax optimal tests

- Abramovich, F. De Feis, I., Sapatinas, T. (2009). Optimal testing for additivity in multiple nonparametric regression. *Ann. Inst. Stat. Math.*, 61, pp. 691–714.
- Comminges, L. Dalalyan, A. (2013). Minimax testing of a composite null hypothesis defined via a quadratic functional in the model of regression *EJS*, 7, pp. 146–190.
- Lepski, O. Pouet, C. (2008). Hypothesis testing under composite function alternative. In *Topics in Stochastic Analysis and nonparametric estimation*, Springer.

The approach

physical domain

d-variate function f = sum of functional components $f_{\underline{i}}$ (related to the Sobol decomposition)

$\Updownarrow$ Hyperbolic wavelet transform

wavelet coefficient domain

coefficients within orientations $\underline{i}$

A : form test statistics within each orientation to detect $f_{\underline{i}}$ (theoretical properties of these elementary test statistics)

B : combine them in appropriate way $\Rightarrow$ test of more general structure (atomic dimension) | Anisotropy

Functional ANOVA components and orientations in the coefficient domain

$$f = \alpha\phi_{00} + \sum_{\underline{i}=\{\underline{i}\neq 0\}} \sum_{\underline{j}\in\mathbb{J}^{\underline{i}}} \sum_{\underline{k}\in\mathbb{K}_{\underline{j}}} \theta^{\underline{i}}_{\underline{j},\underline{k}} \psi^{\underline{i}}_{\underline{j},\underline{k}}$$

Example variable selection for u :

$$Y = f(u,\omega) + \text{error}, \tag{4}$$

$$\text{with } f(u,\omega) = f_1(u) + f_2(\omega) + f_{12}(u,\omega) \tag{5}$$

In a wavelet bases, relation between functional components and wavelet coefficients of different orientations :

$$\underline{i} = (1,1); \quad \left\langle f; \psi^1_{j_1,k_1}\psi^1_{j_2,k_2} \right\rangle = 0, \forall j_1, j_2, k_1, k_2 \tag{6}$$

$$\underline{i} = (0,1); \quad \left\langle f; \psi^0 \psi^1_{j_2,k_2} \right\rangle \neq 0, \forall j_2, k_2 \tag{7}$$

$$\underline{i} = (1,0); \quad \left\langle f; \psi^1_{j_1,k_1}\psi^0 \right\rangle = 0, \forall j_1, k_1 \tag{8}$$

Functional ANOVA components and orientations in the coefficient domain

$$f = \alpha\phi_{00} + \sum_{\underline{i}=\{\underline{i}\neq 0\}} \sum_{\underline{j}\in\mathbb{J}^{\underline{i}}} \sum_{\underline{k}\in\mathbb{K}_{\underline{j}}} \theta^{\underline{i}}_{\underline{j},\underline{k}} \psi^{\underline{i}}_{\underline{j},\underline{k}}$$

Example first order interaction, ie. atomic dimension $= 1$:

$$Y = f(u,\omega) + \text{error}, \quad (9)$$

$$\text{with } f(u,\omega) = f_1(u) + f_2(\omega) + f_{12}(u,\omega) \quad (10)$$

In a wavelet bases, relation between functional components and wavelet coefficients of different orientations :

$$\underline{i} = (1,1); \quad \left\langle f; \psi^1_{j_1,k_1}\psi^1_{j_2,k_2} \right\rangle = 0, \forall j_1, j_2, k_1, k_2 \quad (11)$$

$$\underline{i} = (0,1); \quad \left\langle f; \psi^0 \psi^1_{j_2,k_2} \right\rangle \neq 0, \forall j_2, k_2 \quad (12)$$

$$\underline{i} = (1,0); \quad \left\langle f; \psi^1_{j_1,k_1} \psi^0 \right\rangle \neq 0, \forall j_1, k_1 \quad (13)$$

Test coefficient in orientation such that $\{|\underline{i}| > \delta\}$

Wavelet decomposition of a multivariate function

Consider $f \in \mathbb{L}_2([0,1]^d)$:

$$\begin{aligned} f &= \text{avg approx} + \sum_{\underline{i} \in \{0,1\}^d \setminus \underline{0}} \text{details in orientation } \underline{i} \\ &= \theta^{\underline{0}}_{\underline{0},\underline{0}} \psi^{\underline{0}}_{\underline{0},\underline{0}} + \sum_{\underline{i} \in \{0,1\}^d \setminus \underline{0}} \left(\sum_{\underline{j} \in \mathcal{J}} \sum_{\underline{k} \in \mathbb{K}_{\underline{j}}} \theta^{\underline{i}}_{\underline{j},\underline{k}} \psi^{\underline{i}}_{\underline{j},\underline{k}} \right), \end{aligned}$$

where

- $\mathcal{J} = \mathbb{J}$, $\mathbb{J} = \left\{ \underline{j} = (j, \cdots, j) : j \in \mathbb{N} \right\}$, for the standard wavelet decomposition,
- $\mathcal{J} = \mathbb{J}^{\underline{i}}$ for the hyperbolic wavelet decomposition.

Summary

1. Wavelet-based method and information pooling
2. Unidimensional second generation thresholding methods : maxiset approach
3. Multidimensional function estimation
4. Structure detection : asymptotic optimality

The Model

Model : We observe n^d independent random variables under the multivariate nonparametric regression problem over an equidistant design grid :

$$Y = f\left(\underline{x}\right) + \sigma\xi,\ \xi \sim \mathcal{N}(0,1)$$

where $\underline{x} \in [0,1]^d$. The model is equivalent with the multivariate Gaussian white noise model.

$$dY_\varepsilon(\underline{x}) = f(\underline{x})d\underline{x} + \varepsilon dW(\underline{x}),$$

where $\underline{x} = (x_1, \ldots, x_d) \in [0,1]^d$, $f \in \mathbb{L}_2([0,1]^d)$, $W(\underline{x})$ is the Brownian sheet and ε is the known noise level with the calibration $\varepsilon = \sigma n^{-\frac{d}{2}}$.

Definition (Sequential white noise model)

$$\hat{\theta}^{\underline{i}}_{\underline{j},\underline{k}} = \theta^{\underline{i}}_{\underline{j},\underline{k}} + \varepsilon\xi^{\underline{i}}_{\underline{j},\underline{k}},$$

where $\xi^{\underline{i}}_{\underline{j},\underline{k}}$ are i.i.d. $\mathcal{N}\left(0,1\right)$ and $(\underline{i},\underline{j},\underline{k}) \in \{0,1\}^d \times \mathbb{N}^d \times \mathbb{Z}^d$.

Testing hypotheses

We are interested in testing whether a component in a certain orientation $\underline{i}$ is zero.

Definition (Hypotheses of the testing problem)

$$\mathcal{H}_{\underline{i},0}: \quad f \in \mathcal{N}_{\underline{i}}(R) = \left\{f : \|f\|_2 \leq R,\ f_{\underline{i}} = 0\right\}$$

$$\mathcal{H}_{\underline{i},a}: \quad f \in \mathcal{A}_{\underline{i}}(R, C, \underline{s}, r_\varepsilon) = \left\{f : f \in \mathcal{B}^{\underline{s}}(R),\ \|f_{\underline{i}}\|_2 \geq C r_\varepsilon\right\},$$

where $C > 0$, $(r_\varepsilon)_\varepsilon$ is a decreasing and continuous sequence of real numbers tending to 0 when ε goes to 0 and $\mathcal{B}^{\underline{s}}(R)$ denotes the ball of radius R of the Besov space with $\underline{s}$ as smoothness parameter.

We consider two cases :

- Non adaptive case : $\underline{s}$ is such that s_u are known for all $i_u = 1$.
- Semi adaptive case : $\gamma_{\underline{i}} = \left(i_1 s_1^{-1} + \cdots + i_d s_d^{-1}\right)^{-1}$ is known. Example : Anisotropic Holder ($s_1 = 0.5, s_2 = 0.8$), $\gamma_{(1,0)} = 0.5, \gamma_{(0,1)} = 0.8, \gamma_{(1,1)} = 0.31$

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- Adaptive case : no information available

Testing procedure and related errors

Aim at : providing testing procedures which ensure that the sum Error-I + Error-II does not exceed a chosen $\alpha > 0$ and which are optimal or near optimal in the minimax sense (see Ingster (1986)).

Definition

A testing procedure is a random variable Δ *with value in* $\{0, 1\}$ *for which :*

- $\Delta = 1$ *"*$\mathcal{H}_{\underline{i},0}$ *is rejected"*
- $\Delta = 0$ *else.*

Definition

For the testing hypotheses considered and a fixed Δ *:*

- *Error-I :* $$\sup_{f \in \mathcal{N}_{\underline{i}}(R)} \mathbb{P}_f(\Delta = 1).$$
- *Error-II :* $$\sup_{f \in \mathcal{A}_{\underline{i}}(R, C, \underline{s}, r_\varepsilon)} \mathbb{P}_f(\Delta = 0).$$

Minimax results for hypothesis testing : case NA

Theorem (Autin, Claeskens, Freyermuth and Pouet (2015))

1. *there exists a constant* $C' > 0$ *such that for any* $C < C'$ *the testing problem has* *no solution* *for the rate* $r_\varepsilon = \left(\varepsilon^4\right)^{\frac{4\gamma_{\underline{i}}}{1+4\gamma_{\underline{i}}}}$
2. *there exists a constant* $C'' > 0$ *such that for any* $C > C''$ *the testing problem as a* *solution* *for the rate* $r_\varepsilon = \left(\varepsilon^4\right)^{\frac{4\gamma_{\underline{i}}}{1+4\gamma_{\underline{i}}}}$. *Then, the solution of the testing problem is given through the testing procedure :*

$$\Delta_{\underline{i},t} = \left\{ T_{\underline{i}}^{na} > t_{\alpha}^{na} \right\}$$

where the test statistic is the nonnegative random variable

$$T_{\underline{i}}^{na} \;=\; \sum_{\underline{j} \in \mathbb{J}_{na}^{\underline{i}}} \sum_{\underline{k} \in \mathbb{K}_{\underline{j}}} \left(\hat{\theta}_{\underline{j},\underline{k}}^{\underline{i}}\right)^2 .$$

Minimax results for hypothesis testing : case NA

Definition (calibration of the test $\Delta_{\underline{i}}^{na}$)

- $\mathbb{J}_{na}^{\underline{i}} = \left\{ \underline{j} \in \mathbb{J}^{\underline{i}} : 2^{j_u s_u} < \left(\varepsilon^4\right)^{-\frac{i_u \gamma_{\underline{i}}}{1+4\gamma_{\underline{i}}}}, \forall u \right\}$,
- t_{α}^{na} *is the quantile of order* $1 - \frac{\alpha}{2}$ *of the Chi-squared distribution* $\# \left\{ (\underline{j}, \underline{k}) : \underline{j} \in \mathbb{J}_{na}^{\underline{i}}, \underline{k} \in \mathbb{K}_{\underline{j}} \right\}$ *with degrees of freedom.*

Remark :

- $\Delta_{\underline{i}}^{na}$ is defined through the knowledge of $\underline{s}$.
- Minimax rates of separation depends on the orientation.

Minimax results for hypothesis testing : case SA

Theorem (A., Claeskens, Freyermuth and Pouet (2015))

1. *there exists a constant* $C' > 0$ *such that for any* $C < C'$ *the testing problem has* *no solution* *for the rate* $r_\varepsilon = \left(\varepsilon^4 \log\log \varepsilon^{-1}\right)^{\frac{4\gamma_{\underline{i}}}{1+4\gamma_{\underline{i}}}}$
2. *there exists a constant* $C'' > 0$ *such that for any* $C > C''$ *the testing problem as a* *solution* *for the rate* $r_\varepsilon = \left(\varepsilon^4 \log\log \varepsilon^{-1}\right)^{\frac{4\gamma_{\underline{i}}}{1+4\gamma_{\underline{i}}}}$. *Then, the solution of the testing problem is given through the testing procedure :*

$$\Delta^a_{\underline{i}} = \left\{ T^a_{\underline{i}} > t^a_\alpha \right\}$$

where the test statistic is the nonnegative random variable

$$T^a_{\underline{i}} \;=\; \sum_{\underline{j} \in \mathbb{J}^{\underline{i}}_a} \sum_{\underline{k} \in \mathbb{K}_{\underline{j}}} \left(\hat{\theta}^{\underline{i}}_{\underline{j},\underline{k}}\right)^2 .$$

Minimax results for hypothesis testing : case SA

Definition (calibration of the test $\Delta^a_{\underline{i}}$)

- $\mathbb{J}^{\underline{i}}_a = \left\{ \underline{j} \in \mathbb{J}^{\underline{i}} : 2^{j_1+\cdots+j_d} < \left(\varepsilon^4 \log\log \varepsilon^{-1}\right)^{-\frac{1}{1+4\gamma_{\underline{i}}}} \right\}$,
- t^a_α *is the quantile of order* $1-\frac{\alpha}{2}$ *of the Chi-squared distribution* $\#\left\{(\underline{j},\underline{k}) : \underline{j} \in \mathbb{J}^{\underline{i}}_a, \underline{k} \in \mathbb{K}_{\underline{j}}\right\}$ *with degrees of freedom.*

remark

- $\Delta^a_{\underline{i}}$ is only defined through the knowledge of $\gamma_{\underline{i}}$.
- There is a loss of rate through the $\log\log$ term.
- At this near optimal rate, $\Delta^a_{\underline{i}}$ is able to detect more functions than the than the previous test for the chosen precision α.

About the questions on structure (selection of variables or atomic dimension) :

1. "Is x_m a true variable for f ?"
   The answer depends on the observation of the testing procedure :

$$\Delta_{sel}(m) = \max\left(\Delta^*_{\underline{i}} : \underline{i} \text{ s.t. } i_m = 1\right).$$

2. "Is the atomic dimension of f smaller than δ ?"
   The answer depends on the observation of the testing procedure :

$$\Delta_{atom}(\delta) = \max\left(\Delta^*_{\underline{i}} : |\underline{i}| \geq \delta\right).$$

Notation : $* \in \{na, a\}$.

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