Weierstrass Institute for
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Leibniz Association

Multiplier bootstrap for change point detection

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Algorithm

“ Materia – this is what is changing ”

B.-Gita

$$YYYYYYYY\overbrace{\underbrace{YYYYYYYY}_{L_1(\theta)}\underbrace{YYYYYYYY}_{L_2(\theta)}}^{L(\theta)}YYYYYYYY$$

Statistic foreach window position t:

$$T^2(t)/2 = L_1(t)(\widehat{\theta}_1) + L_2(t)(\widehat{\theta}_2) - L(t)(\widehat{\theta})$$

$$\widehat{\theta}_i = \operatorname*{argmax}_{\theta} L_i(\theta)$$

Quadratic likelihood

$$L(\theta) \approx L(\theta^*) + \xi^T(\theta - \theta^*) - \frac{1}{2}\left\|D(\theta - \theta^*)\right\|^2$$

Independent data $\{Y_i\}$

$$\xi = D^{-1}\nabla L(\theta^*) = D^{-1}\sum_i \nabla L(\theta^*, Y_i)$$

Useful properties: [1]

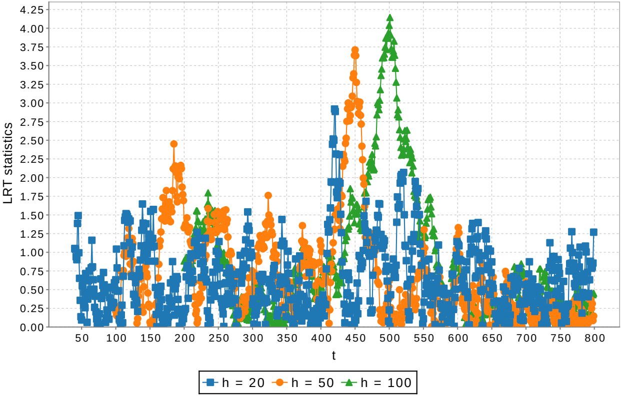
$$D(\widehat{\theta} - \theta^*) \approx \xi \quad \text{(Fisher Theorem)}$$

$$\sqrt{2L(\theta_2) - 2L(\theta_1)} \approx \|D(\theta_2 - \theta_1)\| \quad \text{(Sqrt Wilks Theorem)}$$

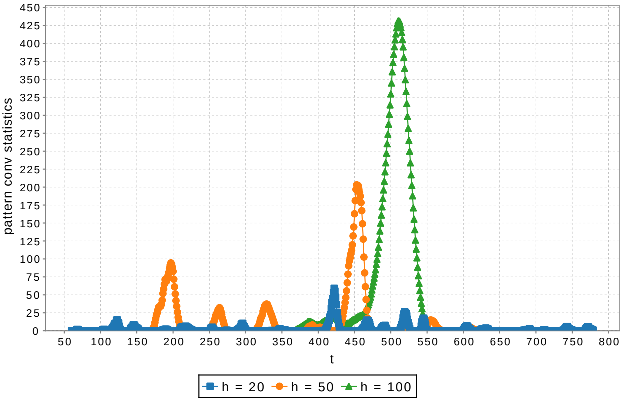
[1] V. Spokoiny. Penalized maximum likelihood estimation and effective dimension. 10 Aug 2015. arXiv:1205.0498

Different window sizes: $h = 20, 50, 100$

$$T(t) \approx \|\xi_{12}(t) + D_{12}(\theta_2^* - \theta_1^*)(t)\|$$



$$\mathbb{T}_h(\tau) = \sum_{t=\tau}^{\tau+h} (T(t) - \widehat{b}) * P_\tau(t), \quad P_\tau(t) = \widehat{a}(t - \tau)$$



$$T(t) \approx \|\xi_{12}(t) + D(\theta_2^* - \theta_1^*)(t)\|$$

Critical bound distribution ???

$$\max_\tau \mathbb{T}_h(\tau) = \max_\tau \sum_{t=\tau}^{\tau+h} P_\tau(t)\, \|\xi_{12}(t)\|$$

$$\xi_{12}(t) = \sum_{i=t}^{t+h} D_{ti}\xi_i = \sum_{i=t}^{t+h} D_{ti}\nabla L(\theta_i^*, Y_i)$$

Multiplier bootstrap for independent data $\{Y_i\}_{i=1}^n$

$$\xi_{12}^\flat(t) = \sum_{i=t}^{t+h} D_{ti}\xi_i\varepsilon_i^\flat = \sum_{i=t}^{t+h} D_{ti}\nabla L(\theta_i^*, Y_i)\varepsilon_i^\flat, \quad \varepsilon_i^\flat \sim \mathcal{N}(0,1)$$

Aim:

$$\left| I\!P\left(\max_\tau \mathbb{T}_h(\tau) < x\right) - I\!P_\flat\left(\max_\tau \mathbb{T}_h(\tau)^\flat < x\right)\right| \leq \ ???$$

Use $\max_{\|\gamma\|=1} \gamma^T \xi \approx \|\xi\|$ yields

$$\max_{\tau} \mathbb{T}_h(\tau) = \max_{\tau} \sum_{t=\tau}^{\tau+h} P_\tau(t) \, \|\xi_{12}(t)\|$$

$$= \max_{\tau,\gamma} \left(\sum_{t=\tau}^{\tau+h} P_\tau(t) \gamma_t^T \sum_{i=1}^{h} D_{ti} \xi_{t+i} \right)$$

$$= \max_{\tau,\gamma} \left(\sum_{k=1}^{2h} X_{\tau\gamma k} \right)$$

Reduced aim:

$$\left| I\!P \left(\max_{\tau,\gamma} \left(\sum_{k=1}^{2h} X_{\tau\gamma k} \right) < x \right) - I\!P_\flat \left(\max_{\tau,\gamma} \left(\sum_{k=1}^{2h} X_{\tau\gamma k}^\flat \right) < x \right) \right| \leq \; ???$$

$$X_{\tau\gamma} = \sum_{k=1}^{2h} X_{\tau\gamma k}$$

compare to

$$\widetilde{X} \in \mathcal{N}(0, \mathrm{Var}\, X)$$

compare to

$$X^{\flat} \in \mathcal{N}(0, \mathrm{Var}\, X^{\flat})$$

Intermediate aim:

$$\left| I\!P\left(\max_{\tau,\gamma}\left(\sum_{k=1}^{2h} X_{\tau\gamma k}\right) < x\right) - I\!P\left(\max_{\tau,\gamma}\left(\sum_{k=1}^{2h} \widetilde{X}_{\tau\gamma k}\right) < x\right)\right| \le \;???$$

$$\mathbb{P}\left(\max_{1\le m\le M} (X_m) < x \pm \triangle \right) \approx \mathbb{E} g_{\triangle}\left(h_\beta(X)\right), \quad m = (\tau, \gamma)$$

$$g_\triangle(X) \to \mathbb{1}[X < x], \quad \triangle \to 0$$

$$h_\beta(X) = \beta^{-1} \log \left(\sum_{m=1}^{M} e^{\beta X_m} \right) \quad \text{(smooth max)}$$

$$\beta = \frac{1}{\triangle} \log(M)$$

Use Lindeberg's **telescoping sums** for $X_m = \sum_k X_{km}$:

$$\left| \mathbb{E} g_\triangle\left(h_\beta(X)\right) - \mathbb{E} g_\triangle\left(h_\beta(\widetilde{X})\right) \right| \le \left\| \nabla^3 \mathbb{E}(g_\triangle \circ h_\beta) \right\|_1 \mu_X^3$$

$$\mu_X^3 = \sum_{k=1}^{2h} \mathbb{E}\left\| X_k \right\|_\infty^3 + \sum_{k=1}^{2h} \mathbb{E}\left\| \widetilde{X}_k \right\|_\infty^3 \sim \frac{\log(M)}{\sqrt{h}}$$

$$\left\|\nabla^3 \mathbb{E}(g_\triangle \circ h_\beta)\right\|_1 \leq ???$$

Simple estimation

$$\left\|\nabla^k h_\beta\right\|_1 \leq \beta^k, \quad |g_\triangle^{(k)}| \leq \frac{1}{\triangle^k}$$

Leads to

$$\left\|\nabla^3 (g_\triangle \circ h_\beta)\right\|_1 \lesssim \frac{\log(M)}{\triangle^3}$$

and

$$\left|\mathbb{E} g_\triangle\left(h_\beta(X)\right) - \mathbb{E} g_\triangle\left(h_\beta(\widetilde{X})\right)\right| \lesssim \frac{\log(M)}{\triangle^3}\mu_X^3$$

$$\left|\mathbb{P}\left(\max_{\tau,\gamma}\left(\sum_{k=1}^{2h} X_{\tau\gamma k}\right) < x\right) - \mathbb{P}\left(\max_{\tau,\gamma}\left(\sum_{k=1}^{2h} \widetilde{X}_{\tau\gamma k}\right) < x\right)\right| \lesssim \frac{\log(M)}{h^{1/8}}$$

V. Chernozhukov, D. Chetverikov, K. Kato. Gaussian approximations and multiplier bootstrap for maxima of sums of high-dimensional random vectors. 2013.

Improvement

$$\left\| \nabla^3 \mathbb{E}(g_\triangle \circ h_\beta)(X) \mathbb{1}[x < \max(X) < x + \triangle] \right\|_1 \lesssim \frac{\log(M)}{\triangle^2}$$

Leads to

$$\left| \mathbb{P}\left(\max_{\tau,\gamma} \left(\sum_{k=1}^{2h} X_{\tau\gamma k} \right) < x \right) - \mathbb{P}\left(\max_{\tau,\gamma} \left(\sum_{k=1}^{2h} \widetilde{X}_{\tau\gamma k} \right) < x \right) \right| \lesssim \frac{\log(M)}{h^{1/6}}$$

V. Bentkus. On the dependence of the Berry–Esseen bound on dimension. 2001.

$$\sup_{\mathcal{A}} \left| \mathbb{P}\left(\sum_k X_k \in \mathcal{A} \right) - \mathbb{P}\left(\sum_k \widetilde{X}_k \in \mathcal{A} \right) \right| \lesssim \frac{M^{1/4}}{h^{1/2}}$$

Thank you for attention!