Overview On Some Recent Results about p-Adic Differential Equations over Berkovich curves

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- The equations of rigid cohomology usually have certain operators (**Frobenius**) plus some other **restrictions**.
- In comparison with ℓ-adic sheaves, p-adic differential equations are more “*explicit*”, and allow sometimes **direct computations.**

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- Results in this direction are essentially due (among other actors) to Dwork and Robba, then Christol and Mebkhout, and are (up to some exceptions) of **local nature** in the sense of Berkovich.

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Triviality over a disk

Over an open disk there are *non singular* differential equations with solutions that **do not converge on the whole disk.**

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Example

The equation $y' = y$ has solution

$$y = \exp(x) = \sum_{n\geq 0} \frac{x^n}{n!} . \quad (1)$$

Now, this series has a finite p-adic radius of convergence. However, the equation shows no singularities.

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1. Factorization/decomposition theorems by the radii ;
2. Finite dimensionality of the de Rham cohomology and index theorems.

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- Poineau-Bojkovic :

  2016 **Behavior of the radii by push-forward**+relation with **ramification**

Notation on Berkovich curves

Notation

$(K, |.|)$ is a complete valued field of **characteristic** 0.

To simplify, in this talk we assume that K is **algebraically closed.**

Berkovich curves

A K-analytic Berkovich curve is said **rig-smooth** or **quasi-smooth** if Ω^1_X is a **locally free** $\mathcal{O}_X$-module of rank one.

Berkovich curves

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- This definition allows boundary.

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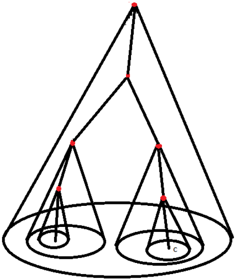

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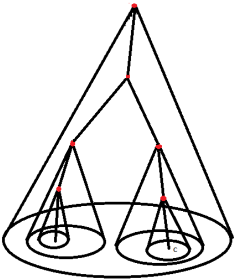
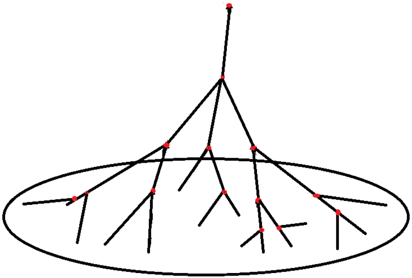
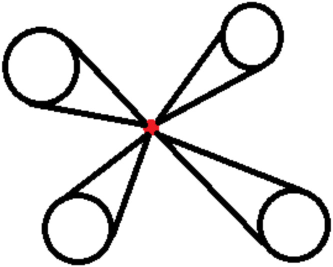




It is a arcwise connected space

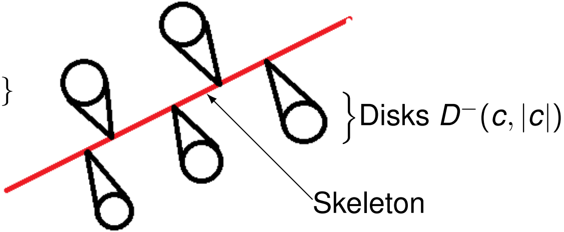
Closed disk



1. The union of all open sub-disks is an open, but **not a covering**
2. The space is **connected**
3. The red-point is the **boundary**

Open annuli

Annulus $\{r < |T| < 1\}$



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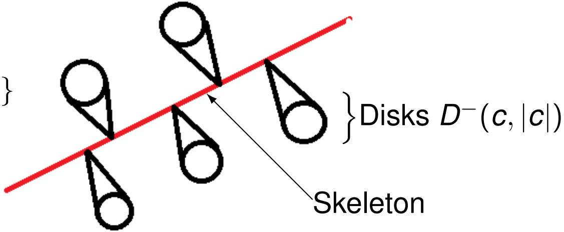
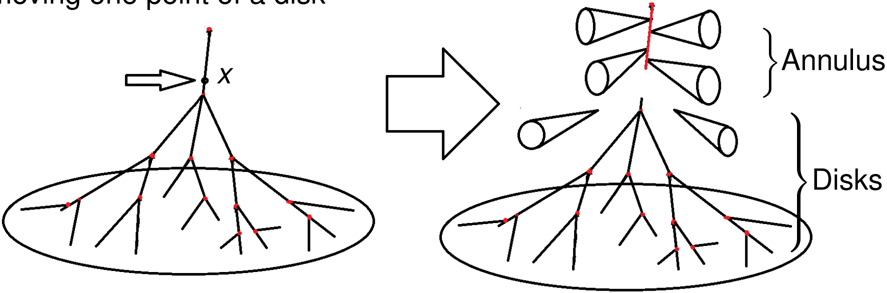


Removing one point of a disk



$D = \{|T| < 1\}$ $\qquad$ $D - \{x\}$

Type of a point

We can classify points in 4 types.
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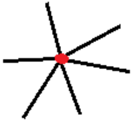

Type 2

Bifurcation

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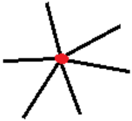

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Rational pts	Bifurcation	2 directions	

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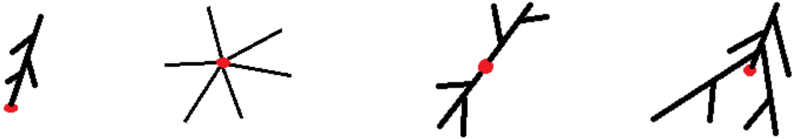

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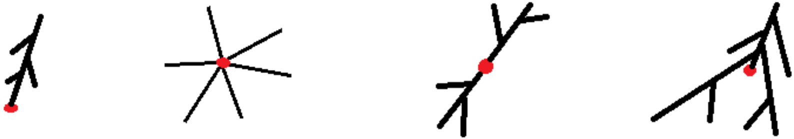

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Type 4 points are absent if the base field is spherically complete.

Semi-stable reduction

Theorem (V.Berkovich - A.Ducros)

*Let X be a quasi-smooth curve. There exists a **locally finite** subset $S \subseteq X$ formed by points of type 2 or 3 such that $X - S$ is a disjoint union of open **disks** and **annuli**.*

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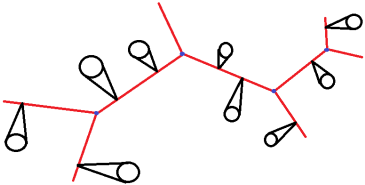
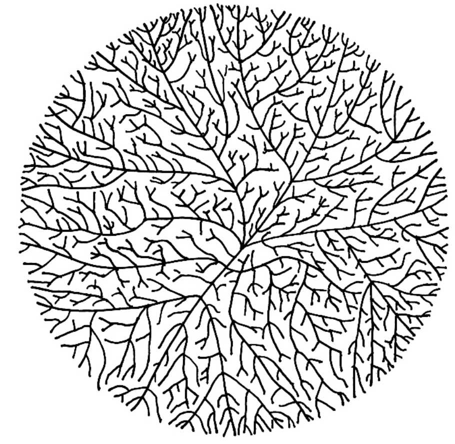


Skeleton

The union Γ_S of the skeletons of the annuli that are connected components of $X - S$ together with the points of S is a locally finite graph in X. Called the **skeleton** of S.

Projective line $\mathbb{P}^{1,\mathrm{an}}_K$



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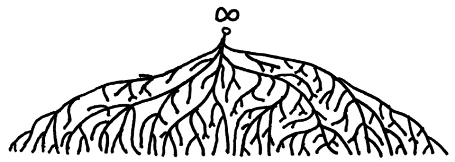

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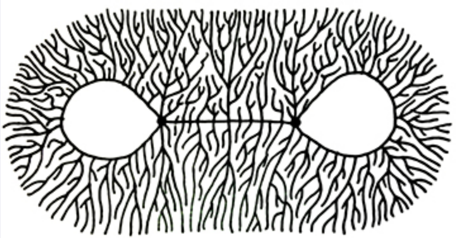

Open boundary

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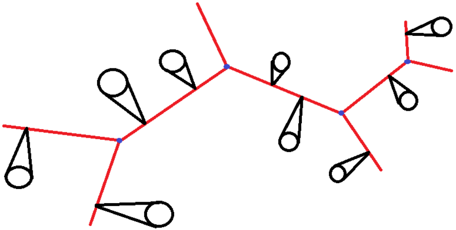

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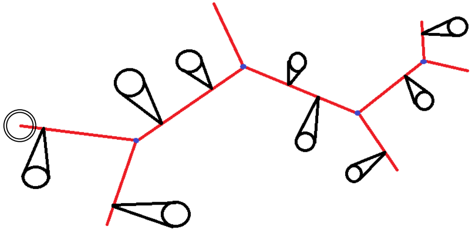

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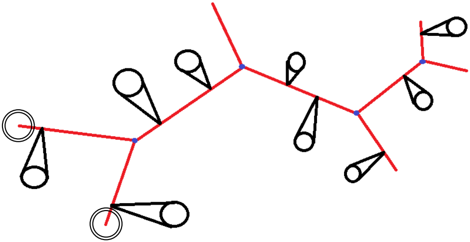

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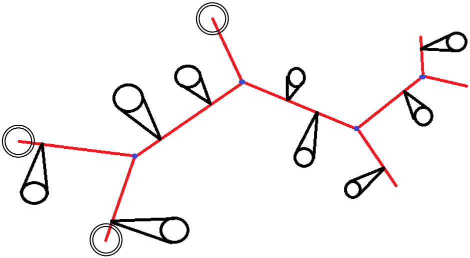

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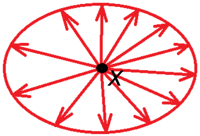
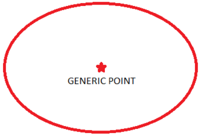
- If $g(x) > 0$, there is **no neighborhood** of x isomorphic to a domain of the line.
- Points of positive genus form a **locally finite set** in the curve.

There exists an **injective map**

$$\psi_x \; : \; \{\text{Directions out of } x\} \longrightarrow \{\widetilde{K}\text{-rational Pts of } \mathcal{C}_x\} \; . \qquad (2)$$





Genus and Euler characteristic of X

Genus

The **genus** $g(X)$ of the quasi-smooth curve X is by definition

$$g(X) \;=\; 1 - \chi_{top}(X) + \sum_{x \in X} g(x) \;\geq\; 0 \tag{3}$$

where $\chi_{top}(X) \leq 1$ is the Euler characteristic of the topological space underling X in the sense of **singular homology**.

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Characteristic of X

The **Euler characteristic of** X (following Q.Liu) is by definition

$$\chi_c(X) \;=\; 2-2g(X)-N(X) \tag{4}$$

where $N(X)$ is the number of germ of segments at the **open boundary** of X.

If $X = C^{an}$ is the analytification of a smooth algebraic curve C then

$$g(X) \ = \ \text{algebraic genus of } C\ . \tag{5}$$

Differential equations over quasi-smooth Berkovich curves

Differential equations

Let X be a quasi-smooth curve

Definition (Differential equation)

A **differential** equation over X is a locally free $\mathcal{O}_X$-module of finite rank $\mathscr{F}$, endowed with a **connection**

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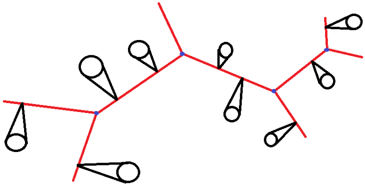
Contrary to the complex case, a differential equation of this type is not always analytically trivial over an open disk. The reason is that the radii of convergence of the Taylor solutions are not always **maximal**.

Radii of convergence and convergence Newton polygon

Radii of convergence



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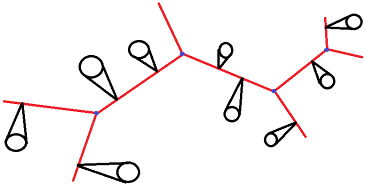

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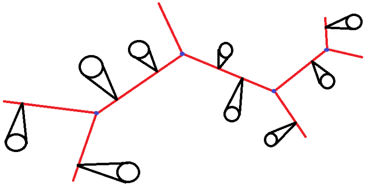

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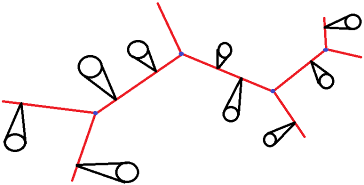

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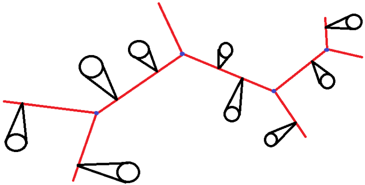

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Definition

Let $r := \mathrm{rang}_x(\mathscr{F})$. Denote by $D_{S,i}(x, \mathscr{F}) \subseteq D(x, S)$ the **largest open sub-disk** on which $\mathscr{F}$ has at least $r - i + 1$ linearly independent solutions :

$$\{x\} \neq D_{S,1}(x, \mathscr{F}) \subseteq D_{S,2}(x, \mathscr{F}) \subseteq \cdots \subseteq D_{S,r}(x, \mathscr{F}) \subseteq D(x, S) \,. \qquad (7)$$

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Fix a coordinate on $D(x, S)$.

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Then, set

$$\mathcal{R}_{S,i}(x,\mathscr{F}) \;:=\; \frac{R_i}{R} \;\leq\; 1 \,. \qquad (9)$$

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We then define the radii after base change to X_Ω.

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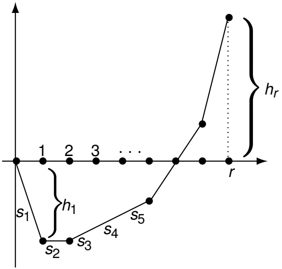

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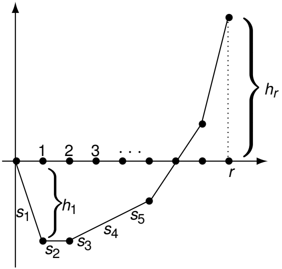

(10)

Define the **partial heights** as $h_0 = 0$ and

$$h_i := s_1 + s_2 + \cdots + s_i \ .$$

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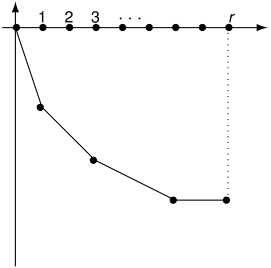

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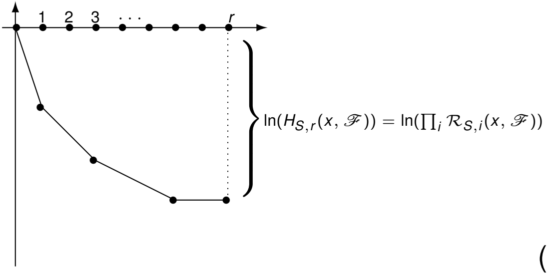

(12)

Continuity and finiteness of the radii

Continuity of the first radius

Remember that $\mathcal{R}_{S,1}(x, \mathscr{F})$ is the radius of the largest disk "at x" where **all the solutions converge.**
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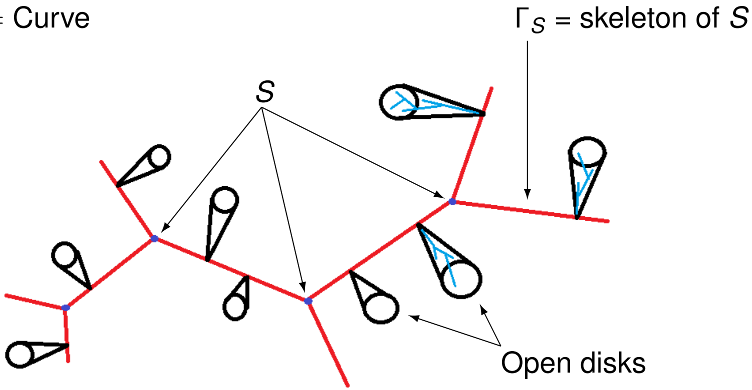
In 2013 Kedlaya

- re-proved the same result (similar methods)
- showed that $\Gamma_{S,i}(\mathscr{F})$ has **no points of type** 4 (Tannakian methods).

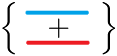
The locally finite graph $\Gamma_{S,i}(\mathscr{F})$

$X :=$ Curve



S = weak triangulation

$\Gamma_S \subseteq \Gamma_S(\mathscr{F})$



= Controlling graph

Global decomposition by the radii

The informal idea is that the filtration by the radii of the space of the solutions implies a **decomposition** of the differential equation itself by sub-differential equations.

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2013 Poineau-P. : **global decomposition** over curves.

Global decomposition/factorization by the radii

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For differential equations over $\mathbb{C}((T))$ this "***is***" the classical decomposition of B.Malgrange.

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It result important to **measure** the size of these graphs.

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- Over a **projective curve**, the bound **only depends on the rank** of the differential equation.
- Under appropriate conditions on the exponents, the bound is related to the de Rham **index.**

Global irregularity and index theorem

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3. $\chi(x,S) := 2 - 2g(x) - N_S(x)$, where $N_S(x)$ is the number of directions of Γ_S out of x. It is a certain **characteristic** related to the residual curve of x.

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Then,

$$\dim \mathrm{H}^{\bullet}_{\mathrm{dR}}(X, \mathscr{F}) \;<\; +\infty$$

and we have the following index formula

$$\chi_{\mathrm{dR}}(X, \mathscr{F}) \;=\; \chi_c(X) \cdot \mathrm{rank}(\mathscr{F}) - \mathrm{Irr}_X(\mathscr{F}) \,. \tag{18}$$

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- We also treat the case of **meromorphic singularities.**