A semistable Lefcshetz $(1, 1)$ theorem in equicharacteristic

joint with Ambrus Pál

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1. Introduction
2. Proof of the main result
3. p-adic cohomology in equicharacteristic
4. Global results

General problem: X/k smooth, projective variety. Want to describe the image of

$$\mathrm{cl} : \mathrm{CH}^n(X) \to H^{2n}(X)(n)$$

for H^* a well-behaved cohomology theory.

Example (Hodge conjecture)

$k = \mathbb{C}$, $H^* =$ Betti cohomology. Then

$$\mathrm{CH}^n(X)_{\mathbb{Q}} \twoheadrightarrow H^{2n}_B(X, \mathbb{Q}) \cap H^{n,n}.$$

When $n = 1$ this follows easily from the exponential sequence.

Example (Tate conjecture)

$k = \mathbb{F}_q$ (or more generally a finite generated field), $H^* =$ étale cohomology. Then

$$\mathrm{CH}^n(X)_{\mathbb{Q}_\ell} \twoheadrightarrow H^{2n}_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(n))^{G_k}$$

for any $\ell \neq \mathrm{char}(k)$. Wide open even for $n = 1$.

Variational version: $f : X \to S$ smooth projective morphism, $\alpha \in \Gamma(S, \mathcal{H}^{2n}(X/S)(n))$ a section of some 'relative cohomology sheaf'.

Conjecture (Grothendieck)

If $\alpha_s \in H^{2n}(X_s)(n)$ is algebraic for some s, then α_s is algebraic for all s.

Example (Variational Hodge conjecture)

If $S/\mathbb{C}$ then $\mathcal{H}^{2n}(X/S) = \mathbf{R}^{2n}f_*\mathbb{Q}_X$, with its natural VHS. In this case we take $\Gamma(S, \mathcal{H}^{2n}(X/S)) := \mathrm{Hom}_{\mathsf{VHS}_S}(\mathbb{Q}_S, \mathbf{R}^{2n}f_*\mathbb{Q}_X(n))$.

Example (Variational Tate conjecture)

If $S/\mathbb{F}_q$ (or over a finitely generated field) then we take $\mathcal{H}^{2n}(X/S) = \mathbf{R}^{2n}f_*\mathbb{Q}_{\ell,X}$. In this case we take $\Gamma(S, \mathcal{H}^{2n}(X/S)(n)) = \Gamma(S_{\text{ét}}, \mathbf{R}^{2n}f_*\mathbb{Q}_{\ell,X}(n))$.

As stated, these trivially follow from their absolute versions (e.g. the variational Hodge conjecture is known for $n = 1$) but we are interested in going the other way.

Today we'll look at:

- a variational Tate conjecture,
- for divisors;
- in p-adic cohomology,
- over a local base.

Notation

- k = perfect field, $\mathrm{char}(k) = p$
- $\mathcal{O}_K$ = complete DVR, $\mathcal{O}_K/\varpi = k$
- $K = \mathrm{Frac}(\mathcal{O}_K)$, $\mathrm{char}(K) = 0$
- R = complete DVR, $R/t = k$
- $F = \mathrm{Frac}(R)$, $\mathrm{char}(F) = p$
- $W = W(k)$, $K_0 = \mathrm{Frac}(W)$

Now take $\mathcal{X}/\mathcal{O}_K$ smooth and projective

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & X_0 \arrow[l] \arrow[d] \\ \mathrm{Spec}\,(K) \arrow[r] & \mathrm{Spec}\,(\mathcal{O}_K) & \mathrm{Spec}\,(k)\,. \arrow[l] \end{tikzcd}$$

For any $\mathcal{L} \in \mathrm{Pic}(X_0)_{\mathbb{Q}}$ we can consider

$$c_1(\mathcal{L}) \otimes 1 \in H^2_{\mathrm{cris}}(X_0/W) \otimes_W K \cong H^2_{\mathrm{dR}}(X/K).$$

Theorem (Berthelot–Ogus)

$\mathcal{L}$ lifts to $\mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ if and only if $c_1(\mathcal{L}) \otimes 1 \in F^1 H^2_{\mathrm{dR}}(X/K)$.

Consider the category of 'p-adic Hodge structures' on $\mathrm{Spec}\,(\mathcal{O}_K)$, that is:

- finite dimensional vector spaces V/K_0,
- plus a Frobenius $\varphi : V \to V$,
- plus a decreasing filtration $F^\bullet$ on $V \otimes_{K_0} K$.

crystalline cohomology comparison theorems $\Rightarrow \exists$ natural p-adic Hodge structure on $H^2_{\mathrm{cris}}(X_0/W)_{\mathbb{Q}}$, which plays the role of $\mathcal{H}^2(\mathcal{X}/\mathcal{O}_K)$. A 'global section' of this sheaf is then a morphism

$$K_0 \to H^2_{\mathrm{cris}}(X_0/W)_{\mathbb{Q}}(1)$$

of p-adic Hodge structures, in other words an element

$$\alpha \in H^2_{\mathrm{cris}}(X_0/W)_{\mathbb{Q}}^{\varphi=p} \cap F^1 H^2_{\mathrm{dR}}(X/K).$$

Such an element can be restricted to give cohomology classes $\alpha_0 \in H^2_{\mathrm{cris}}(X_0/W)_{\mathbb{Q}}$ on the special fibre and $\alpha_\eta \in H^2_{\mathrm{dR}}(X/K)$ on the generic fibre.

Corollary

If α_0 is algebraic, i.e. is in the image of $\mathrm{cl} : \mathrm{CH}^1(X_0)_{\mathbb{Q}} \to H^2_{\mathrm{cris})}(X_0/W)_{\mathbb{Q}}$ *then α_η is algebraic, i.e. is in the image of* $\mathrm{cl} : \mathrm{CH}^1(X)_{\mathbb{Q}} \to H^2_{\mathrm{dR}}(X/K)$.

There also exists a semistable version: if $\mathcal{X} \to \mathrm{Spec}\,(\mathcal{O}_K)$ is projective and semistable, then for any $\mathcal{L} \in \mathrm{Pic}(X_0)_{\mathbb{Q}}$ (resp. $\mathrm{Pic}(X_0^\times)_{\mathbb{Q}}$), we have

$$c_1(\mathcal{L}) \otimes 1 \in H^2_{\log\text{-cris}}(X_0^\times/W^\times) \otimes_W K \cong H^2_{\mathrm{dR}}(X/K).$$

Theorem (Yamashita)

$\mathcal{L}$ lifts to $\mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ *(resp.* $\mathrm{Pic}(\mathcal{X}^\times)_{\mathbb{Q}}$*) if and only if* $c_1(\mathcal{L}) \otimes 1 \in F^1 H^2_{\mathrm{dR}}(X/K)$.

Again, this can be phrased in terms of p-adic Hodge structures, stating that a ‘global section’ of the ‘relative cohomology’ is algebraic iff its special fibre is.

Now suppose we have $\mathcal{X}/R$ smooth and projective.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & X_0 \arrow[l] \arrow[d] \\ \operatorname{Spec}(F) \arrow[r] & \operatorname{Spec}(R) & \operatorname{Spec}(k) \arrow[l] \end{tikzcd}.$$

Then for $\mathcal{L} \in \operatorname{Pic}(X_0)_{\mathbb{Q}}$ we have $c_1(\mathcal{L}) \in H^2_{\text{cris}}(X_0/W)_{\mathbb{Q}}$.

Theorem (Morrow)

$\mathcal{L}$ lifts to $\operatorname{Pic}(\mathcal{X})_{\mathbb{Q}}$ if and only if $c_1(\mathcal{L})$ lifts to $H^2_{\text{cris}}(\mathcal{X}/W)_{\mathbb{Q}}$.

Today's goals:

1. Give a new proof of Morrow's result that easily generalises to the semistable case.
2. Explain how this result is the precise analogue of Berthelot–Ogus/Yamashita.
3. Deduce some global results.

Fix $R \cong k[\![t]\!]$, $\mathcal{X}/R$ smooth projective, X_0/k, X/F as before.
Set $R_n = R/(t^{n+1})$, X_n/R_n the base change, $\mathfrak{X} = \mathrm{colim}_n X_n$ the formal completion.

$$\begin{Bmatrix} \text{motivic cohomology} \\ \mathrm{Pic}(X_0)_{\mathbb{Q}},\ \mathrm{Pic}(\mathcal{X})_{\mathbb{Q}} \end{Bmatrix} \leftarrow \begin{Bmatrix} \text{de Rham–Witt} \\ \text{complex} \end{Bmatrix} \rightarrow \begin{Bmatrix} \text{crystalline cohomology} \\ H^2_{\mathrm{cris}}(X_0/W)_{\mathbb{Q}},\ H^2_{\mathrm{cris}}(\mathcal{X}/W)_{\mathbb{Q}} \end{Bmatrix}$$

Let $W_\bullet\Omega^*_{X_n}$ denote the de Rham–Witt complex of X_n, then $\forall r, n$ we have

$$d\log : \mathcal{O}^*_{X_n} \to W_r\Omega^1_{X_n}$$

with image $W_r\Omega^1_{X_n,\log}$.

Proposition

Fix $n \geq 0$. Then for $r \gg 0$ (depending on n) the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & 1 + t\mathcal{O}_{X_n} \arrow[r] \arrow[d, equal] & \mathcal{O}^*_{X_n} \arrow[r] \arrow[d, "d\log"'] & \mathcal{O}^*_{X_0} \arrow[r] \arrow[d, "d\log"'] & 1 \\
1 \arrow[r] & 1 + t\mathcal{O}_{X_n} \arrow[r, "d\log"] & W_r\Omega^1_{X_n,\log} \arrow[r] & W_r\Omega^1_{X_0,\log} \arrow[r] & 1
\end{tikzcd}$$

has exact rows.

Proof.

Exactness of the top row is well-known; since

$$d\log : \mathcal{O}_{X_0}^*/p^r \xrightarrow{\sim} W_r\Omega^1_{X_0,\log}$$

and the vertical maps are surjective by definition, the only thing that needs checking is injectivity of

$$d\log : 1 + t\mathcal{O}_{X_n} \to W_r\Omega^1_{X_n}.$$

By induction on n it suffices to prove that for $r \gg 0$ the map

$$d\log : 1 + t^n\mathcal{O}_{X_0} \to W_r\Omega^1_{X_n}$$

is injective. Vanishing of a section of $\mathcal{O}_{X_0}$ can be checked at closed points, so we may reduce to the case $\mathcal{X} = \mathrm{Spec}\,(R)$. Now a straightforward calculation shows that

$$d\log : 1 + t^n k \to W_r\Omega^1_{R_n}$$

is injective for $r \gg 0$. □

Corollary

- *$\mathcal{L} \in \mathrm{Pic}(X_0)$ lifts to $\mathrm{Pic}(\mathfrak{X})$ iff $c_1(\mathcal{L}) \in H^1_{\mathrm{cont}}(X_0, W_\bullet\Omega^1_{X_0,\log})$ lifts to $H^1_{\mathrm{cont}}(\mathfrak{X}, W_\bullet\Omega^1_{\mathfrak{X},\log})$.*
- *$\mathcal{L} \in \mathrm{Pic}(X_0)$ lifts to $\mathrm{Pic}(\mathcal{X})$ iff $c_1(\mathcal{L}) \in H^1_{\mathrm{cont}}(X_0, W_\bullet\Omega^1_{X_0,\log})$ lifts to $H^1_{\mathrm{cont}}(\mathcal{X}, W_\bullet\Omega^1_{\mathcal{X},\log})$.*

Now we use the exact sequences

$$0 \to W_\bullet\Omega^1_{X_0,\log} \to W_\bullet\Omega^1_{X_0} \stackrel{1-F}{\to} W_\bullet\Omega^1_{X_0} \to 0$$
$$0 \to W_\bullet\Omega^1_{\mathcal{X},\log} \to W_\bullet\Omega^1_{\mathcal{X}} \stackrel{1-F}{\to} W_\bullet\Omega^1_{\mathcal{X}} \to 0$$

to deduce that

$$H^1_{\mathrm{cont}}(X_0, W_\bullet\Omega^1_{X_0,\log})_{\mathbb{Q}} \cong H^2_{\mathrm{cris}}(X_0/W)^{\varphi=p}_{\mathbb{Q}}$$
$$H^1_{\mathrm{cont}}(\mathcal{X}, W_\bullet\Omega^1_{\mathcal{X},\log})_{\mathbb{Q}} \twoheadrightarrow H^2_{\mathrm{cris}}(\mathcal{X}/W)^{\varphi=p}_{\mathbb{Q}}.$$

Corollary (Morrow)

$\mathcal{L}$ lifts to $\mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ if and only if $c_1(\mathcal{L})$ lifts to $H^2_{\mathrm{cris}}(\mathcal{X}/W)_{\mathbb{Q}}$.

For $\mathcal{X}/R$ projective, semistable: replace $W_\bullet\Omega^*$ everywhere by its logarithmic analogue $W_\bullet\omega^*$ (in this generality introduced by Matsuue).

Notation

- $\mathcal{X}^\times = (\mathcal{X}, M)$ log structure from special fibre
- $X_n^\times = (X_n, M_n)$ base change to R_n
- $\mathrm{Pic}(\mathcal{X}^\times) = H^1(\mathcal{X}_{\text{ét}}, M^{\mathrm{gp}})$
- $\mathrm{Pic}(X_n^\times) = H^1(X_{n,\text{ét}}, M_n^{\mathrm{gp}})$
- $R_n^\times$ = log structure from $\{t = 0\}$
- $k^\times$ = punctured point

Then for all n, r we have

$$d\log : M_n^{\mathrm{gp}} \to W_r\omega^1_{X_n^\times}$$

with image $W_r\omega^1_{X_n^\times,\log}$.

Proposition

Fix $n \geq 0$. Then for $r \gg 0$ (depending on n) the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & 1 + t\mathcal{O}_{X_n} \arrow[r] \arrow[d, equal] & \mathcal{O}^*_{X_n} \arrow[r] \arrow[d] & \mathcal{O}^*_{X_0} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & 1 + t\mathcal{O}_{X_n} \arrow[r] \arrow[d, equal] & M_n^{\mathrm{gp}} \arrow[r] \arrow[d, "d\log"'] & M_0^{\mathrm{gp}} \arrow[r] \arrow[d, "d\log"'] & 1 \\
1 \arrow[r] & 1 + t\mathcal{O}_{X_n} \arrow[r] & W_r\omega^1_{X_n^\times,\log} \arrow[r] & W_r\Omega^1_{X_0^\times,\log} \arrow[r] & 1
\end{tikzcd}$$

has exact rows.

Main difficulty is proving 'logarithmic' analogues of standard properties of the de Rham–Witt complex.

Corollary

If we let $\mathcal{K}_{n,r}$ denote the kernel of the surjective map

$$W_r\omega^1_{X_n^\times,\log} \to W_r\Omega^1_{X_0^\times/k^\times,\log}$$

then there is a split exact sequence

$$1 \to 1 + t\mathcal{O}_{X_n} \to \{\mathcal{K}_{n,r}\}_r \to \{\mathbb{Z}/p^r\mathbb{Z}\}_r \to 0$$

of pro-sheaves on $X_{n,\text{ét}}$.

Corollary (L.–Pál)

$\mathcal{L} \in \mathrm{Pic}(X_0)_\mathbb{Q}$ *(resp.*$\mathrm{Pic}(X_0^\times)_\mathbb{Q}$*) lifts to* $\mathrm{Pic}(\mathcal{X})_\mathbb{Q}$ *(resp.*$\mathrm{Pic}(\mathcal{X}^\times)_\mathbb{Q}$*)* $\Leftrightarrow$ $c_1(\mathcal{L}) \in H^2_{\log\text{-cris}}(X_0^\times/W^\times)_\mathbb{Q}$ *lifts to* $H^2_{\log\text{-cris}}(\mathcal{X}^\times/W)_\mathbb{Q}$

Again, to obtain this we need to prove 'logarithmic' analogues of well–known results concerning $W_\bullet\Omega^*$.

Question

Does the result hold for line bundles with $\mathbb{Q}_p$-coefficients?

Unfortunately, the answer is no.

The reason is that if the answer were yes, then for any elliptic curves $E_1, E_2/F$ with semistable reduction, the map

$$\mathrm{Hom}(E_1, E_2)_{\mathbb{Q}_p} \to \mathrm{Hom}_{\mathbf{BT}_F}(E_1[p^\infty], E_2[p^\infty])_{\mathbb{Q}}$$

would be an isomorphism. This is well-known to be false.

1. Introduction
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$\mathcal{X}/R$ semistable, X_0/k, X/F as before. Want to understand the p-adic cohomology of X, and how it relates to $H^n_{\log\text{-cris}}(X_0^\times/W^\times)_\mathbb{Q}$.
Let

$$\Gamma = W[\![t]\!]\langle t^{-1}\rangle = \left\{ \sum_i a_i t^i \,\middle|\, a_i \in W,\;\; a_i \to 0 \text{ as } i \to -\infty \right\},$$

this is a *Cohen ring* for F. Since X/F is smooth, projective we get finite dimensional cohomology groups

$$H^n_{\text{cris}}(X/\mathcal{E}) := H^n_{\text{cris}}(X/\Gamma)_\mathbb{Q}$$

over the p-adic field $\mathcal{E} := \Gamma_\mathbb{Q}$.

Now choose a Frobenius lift $\sigma : \mathcal{E} \to \mathcal{E}$ and let $\Omega^1_\mathcal{E}$ denote the module of p-adically continuous differentials.

Definition

A (φ, ∇)-module over $\mathcal{E}$ is a finite dimensional vector space M together with a connection

$$\nabla : M \to M \otimes \Omega^1_\mathcal{E}$$

and a *horizontal* Frobenius $\sigma^* M \xrightarrow{\sim} M$. The category of these objects will be denoted $\underline{\mathbf{M}\Phi}^\nabla_\mathcal{E}$.

Standard constructions in crystalline cohomology:

$$H^n_{\mathrm{cris}}(X/\mathcal{E}) \in \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}}.$$

Now let

$$\mathcal{E}^\dagger = \left\{ \sum_i a_i t^i \in \mathcal{E} \,\middle|\, \exists \lambda < 1 \text{ s.t. } |a_i|\,\lambda^i \to 0 \text{ as } i \to -\infty \right\}$$
$$\mathcal{R} = \mathrm{colim}_{\lambda<1}\mathcal{O}(\lambda \leq |t| < 1)$$

Therefore we have diagrams

$$\begin{tikzcd} \mathcal{R} & & \mathcal{E} \\ & \mathcal{E}^\dagger \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{R}} & & \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}} \\ & \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}^\dagger} \arrow[ul] \arrow[ur] & \end{tikzcd}.$$

Theorem (Kedlaya)

1. *The functor* $\underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}^\dagger} \to \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}}$ *is fully faithful.*

Base change in log-crystalline cohomology $\Rightarrow H^n_{\mathrm{cris}}(X/\mathcal{E})$ descends to $H^n_{\mathrm{cris}}(X/\mathcal{E}^\dagger) \in \underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{E}^\dagger}$.

Now can base change to get $H^n_{\text{cris}}(X/\mathcal{R}) := H^n_{\text{cris}}(X/\mathcal{E}^\dagger) \otimes \mathcal{R}$. Can construct a connection on

$$H^n_{\log\text{-cris}}(X_0^\times/W^\times)_{\mathbb{Q}} \otimes \mathcal{R}$$

using the monodromy operator N. Concretely

$$\nabla(v \otimes r) = v \otimes dr + N(v) \otimes r d\log t.$$

Theorem

There exists an isomorphism

$$H^n_{\text{cris}}(X/\mathcal{R}) \cong H^n_{\log\text{-cris}}(X_0^\times/W^\times)_{\mathbb{Q}} \otimes \mathcal{R}$$

in $\underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{R}}$. *In particular*

$$H^n_{\text{cris}}(X/\mathcal{R})^{\nabla=0} \cong H^n_{\log\text{-cris}}(X_0^\times/W^\times)^{N=0}_{\mathbb{Q}}.$$

Now take $\mathcal{L} \in \mathrm{Pic}(X_0)_{\mathbb{Q}}$ (resp. $\mathrm{Pic}(X_0^{\times})_{\mathbb{Q}}$). Since $c_1(\mathcal{L})$ is killed by N, we can therefore view

$$c_1(\mathcal{L}) \otimes 1 \in H^2_{\mathrm{cris}}(X/\mathcal{R})^{\nabla=0} \subset H^2_{\mathrm{cris}}(X/\mathcal{R}).$$

Theorem

$\mathcal{L}$ lifts to $\mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ (resp. $\mathrm{Pic}(\mathcal{X}^{\times})_{\mathbb{Q}}$) if and only if

$$c_1(\mathcal{L}) \in H^2_{\mathrm{cris}}(X/\mathcal{E}^{\dagger}) \subset H^2_{\mathrm{cris}}(X/\mathcal{R}).$$

Proof.

Hard Lefschetz $\Rightarrow$ the Leray spectral sequence

$$E_2^{p,q} = H^q_{\log\text{-cris}}(\mathrm{Spec}\,(R)\,/W, \mathbf{R}^p f_{\log\text{-cris}*}\mathcal{O}_{\mathcal{X}^{\times}/W})_{\mathbb{Q}} \Rightarrow H^{p+q}_{\log\text{-cris}}(\mathcal{X}^{\times}/W)_{\mathbb{Q}}$$

degenerates and we have a surjective edge map

$$H^2_{\log\text{-cris}}(\mathcal{X}^{\times}/W)_{\mathbb{Q}} \twoheadrightarrow H^0_{\log\text{-cris}}(\mathrm{Spec}\,(R)\,/W, \mathbf{R}^2 f_{\log\text{-cris}*}\mathcal{O}_{\mathcal{X}^{\times}/W})_{\mathbb{Q}} \cong H^2_{\mathrm{cris}}(X/\mathcal{E}^{\dagger})^{\nabla=0}.$$

$\square$

Mixed characteristic - deformations of lines bundles controlled by the Hodge filtration $F^\bullet$ on $H^2_{\log\text{-cris}}(X_0^\times/W^\times)_{\mathbb{Q}} \otimes K$

Equicharacteritic - deformations of line bundles controlled by the $\mathcal{E}^\dagger$-lattice $H^2_{\text{cris}}(X/\mathcal{E}^\dagger) \subset H^2_{\log\text{-cris}}(X_0^\times/W^\times)_{\mathbb{Q}} \otimes \mathcal{R}$

This is reminiscent of Serre–Tate theory:

Mixed characteristic - deformations of abelian varieties controlled by the Hodge filtration $F^\bullet$ on $H^1_{\text{cris}}(A_0/W)_{\mathbb{Q}} \otimes K$

Equicharacteritic - deformations of abelian varieties controlled by $\mathcal{E}^\dagger$-lattices in $H^1_{\text{cris}}(A_0/W) \otimes \mathcal{R}$

$$\left\{ \begin{matrix} \text{Hodge filtrations } F^\bullet \\ \text{on } (\varphi, G_K, N)\text{-modules} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \mathcal{E}^\dagger\text{-lattices in} \\ (\varphi, \nabla)\text{-modules over } \mathcal{R} \end{matrix} \right\}$$

In equicharacteristic, relative cohomology $\mathcal{H}^2(X/S)(1)$ is given by

$$H^2_{\mathrm{cris}}(X/\mathcal{E}^\dagger)(1) \in \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^\dagger}.$$

A global section $\Gamma(S, \mathcal{H}^2(X/S)(1))$ is a homomorphism

$$\mathcal{E}^\dagger \to H^2_{\mathrm{cris}}(X/\mathcal{E}^\dagger)(1).$$

Concretely, this is some $\alpha \in H^2_{\mathrm{cris}}(X/\mathcal{E}^\dagger)^{\nabla=0,\varphi=p}$. Can 'specialise' such an element via

$$H^2_{\mathrm{cris}}(X/\mathcal{E}^\dagger)^{\nabla=0} \subset H^2_{\mathrm{cris}}(X/\mathcal{R})^{\nabla=0} \cong H^2_{\log\text{-cris}}(X_0^\times/W^\times)_{\mathbb{Q}}^{N=0}$$

to obtain $\alpha_0 \in H^2_{\log\text{-cris}}(X_0^\times/W^\times)_{\mathbb{Q}}$.

Corollary

α is algebraic iff α_0 is algebraic.

Notation

k, W as before, $K = W[1/p]$
$\mathcal{C}/k$ smooth, projective, geom. conn. curve, $F = k(\mathcal{C})$
$v \in |\mathcal{C}|$, F_v = completion, k_v = residue field
$W_v = W(k_v)$, $K_v = W_v[1/p]$, $\mathcal{E}_v^\dagger$, $\mathcal{R}_v$ (bounded) Robba ring 'at v'

X/F smooth projective, have

$$\mathcal{H}^i_{\mathrm{rig}}(X/K) \in \text{2-colim}_{U \subset \mathcal{C}} F\text{-Isoc}^\dagger(U/K)$$

Since $\mathrm{Isoc}^\dagger(U/K) \to \mathrm{Isoc}^\dagger(V/K)$ is fully faithful, we get

$$H^i_{\mathrm{rig}}(X/K) := \mathcal{H}^i_{\mathrm{rig}}(X/K)^{\nabla=0} \in F\text{-Isoc}(K)$$

well-defined. There exists a Chern class map

$$c_1 : \mathrm{Pic}(X)_{\mathbb{Q}} \to H^2_{\mathrm{rig}}(X/K).$$

Now suppose that X has semistable reduction $X_v^\times/k_v$ at v.
Then we have

$$\begin{tikzcd}
H^2_{\mathrm{rig}}(X/K) \arrow[r, "r_v"] \arrow[drr, "\mathrm{sp}_v"] & H^2_{\mathrm{rig}}(X_{F_v}/\mathcal{E}_v^\dagger)^{\nabla=0} \subset H^2_{\mathrm{rig}}(X_{F_v}/\mathcal{R}_v)^{\nabla=0} \arrow[dr] & \cong H^2_{\log\text{-cris}}(X_v^\times/W_v^\times)_{\mathbb{Q}}^{N=0} \\
& & H^2_{\log\text{-cris}}(X_v^\times/W_v^\times)_{\mathbb{Q}}.
\end{tikzcd}$$

Theorem (L.–Pál)

Let $\alpha \in H^2_{\mathrm{rig}}(X/K)$. The following are equivalent:

1. $\alpha \in c_1(\mathrm{Pic}(X)_{\mathbb{Q}})$;
2. $\mathrm{sp}_v(\alpha) \in c_1(\mathrm{Pic}(X_v)_{\mathbb{Q}})$;
3. $\mathrm{sp}_v(\alpha) \in c_1(\mathrm{Pic}(X_v^\times)_{\mathbb{Q}})$.

Proof.

(1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) is straightforward. (3) $\Rightarrow \exists\, \mathcal{L} \in \mathrm{Pic}(X_{F_v})_{\mathbb{Q}}$ such that $c_1(\mathcal{L}) = r_v(\alpha)$.
Standard approximations arguments $\Rightarrow$ descend $\mathcal{L}$ to X. $\square$

Question

Does this theorem hold with $\mathbb{Q}_p$-coefficients?

Thank-you!