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On the linear bounds on the genus of pointless curves

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Proof of the theorem, odd q

Proof of the theorem, even q

Conclusion

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$q+1-g\lfloor 2\sqrt{q} floor\leq q+1+g\lfloor 2\sqrt{q} floor.$

Let q, g be such that $q+1-g\lfloor 2\sqrt{q} \rfloor \leq 0$, i.e. $g \geq (q+1)/\lfloor 2\sqrt{q} \rfloor$. Question: Does there exist a curve over \mathbb{F}_q of genus g with no rational points (which is called *pointless*)? Main directions:

- 1. g is fixed (E. W. Howe, K. E. Lauter, J. Top, 2005),
- 2. *q* is fixed (R. Becker, D. Glass, 2013).



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$\boldsymbol{q} + \boldsymbol{1} - \boldsymbol{g} \lfloor 2\sqrt{\boldsymbol{q}} \rfloor \leq N_1(\mathcal{C}) \leq \boldsymbol{q} + 1 + \boldsymbol{g} \lfloor 2\sqrt{\boldsymbol{q}} \rfloor.$

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The case of fixed *q*

Given q, denote by g_q^{min} the number such that for all $g \ge g_q^{min}$ there is a smooth pointless genus g curve over \mathbb{F}_q .

Theorem (R. Becker and D. Glass, 2013) Let a be the least residue of g mod p. Suppose that

$$g \ge (p - a - 1)(q - 1)$$
, if $a ,$

or

$$g \ge (p-2a-2)(q-1)$$
, if $0 \le a \le (p-3)/2$.

Then there is a non-singular hyperelliptic pointless curve of genus g defined over \mathbb{F}_q .

Remarks. This result shows that $g_q^{min} \leq O(pq)$ when q is odd. However, under special assumptions they obtain that $g_q^{min} \leq O(q)$.



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The result of R. Becker and D. Glass possesses a generalization.

Theorem (I. Pogildiakov, 2017)

Let q be a prime power. Set

$$g_q = egin{cases} \max\{2,(q-3)/2\}, & q ext{ is odd}, \ \max\{2,q-1\}, & q ext{ is even}. \end{cases}$$

Suppose that $g \ge g_q$. Then there is a smooth genus g hyperelliptic curve over \mathbb{F}_q having no \mathbb{F}_q -points.

This implies a **linear bound** on the number g_a^{min} for all q.

Like R. Becker and D. Glass, we use explicit constructions, but

- 1. consider more cases (less assumptions),
- 2. involve more families of polynomials.

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- 1. consider more cases (less assumptions),
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 $2g+1 \leq \max\{2 \deg h(x), \deg f(x)\} \leq 2g+2.$

As a projective curve, C is the union of two affine patches:

$$y^{2} + h(x)y = f(x)$$
, and $y^{2} + x^{g+1}h(1/x)y = x^{2g+2}f(1/x)$.

The curve $\mathcal C$ is smooth if and only if

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Hyperelliptic curves over \mathbb{F}_q , odd q.

Let C be a hyperelliptic curve $y^2 = f(x)$ over \mathbb{F}_q , where q is odd. We have the following information:

1. ${\mathcal C}$ is the union of two affine patches

$$y^2 = f(x)$$
 and $y^2 = x^{2g+2}f(1/x)$.

- 2. C is smooth if and only if f(x) is square-free.
- 3. The number of rational points of ${\mathcal C}$ is

$$N_1(\mathcal{C}) = N_1(\mathcal{C})^{aff} + N_1(\mathcal{C})^{\infty},$$

where

- $N_1(\mathcal{C})^{aff}$ is the number of affine points in $\mathcal{C}(\mathbb{F}_q)$.
- $N_1(\mathcal{C})^{\infty}$ is the number of points in $\mathcal{C}(\mathbb{F}_q)$ that lie at ∞ .

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Counting rational points on C, odd q.

Let us define

$$\begin{split} N^0 &= \#\{\alpha \in \mathbb{F}_q \mid f(\alpha) = 0\},\\ N^r &= \#\{\alpha \in \mathbb{F}_q \mid f(\alpha) \text{ is a q.r.}\}. \end{split}$$

Then $N_1(\mathcal{C})^{aff} = N^0 + 2N^r$. Note, that $N_1(\mathcal{C})^{aff} \leq 2q$.

Points at infinity belong to the affine patch $y^2 = x^{2g+2}f(1/x)$ and correspond to the solutions with x = 0, i.e. $y^2 = LT(f(x))$.

- 1. If deg f(x) = 2g + 1, then $N_1(\mathcal{C})^{\infty} = 1$.
- 2. If $\deg f(x)=2g+2$ and LT(f) is a q.r., then $N_1(\mathcal{C})^\infty=2$.
- 3. If $\deg f(x)=2g+2$ and LT(f) is a q.n.r, then $N_1(\mathcal{C})^\infty=0$.

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1. If deg f(x) = 2g + 1, then $N_1(\mathcal{C})^{\infty} = 1$. 2. If deg f(x) = 2g + 2 and LT(f) is a q.r., then $N_1(\mathcal{C})^{\infty} = 2$. 3. If deg f(x) = 2g + 2 and LT(f) is a q.n.r, then $N_1(\mathcal{C})^{\infty} = 0$.

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\mathbb{F}_{q} -maximal hyperelliptic curves

Let us call f(x) to be a *good polynomial*, if it satisfies the following conditions:

- 1. $\deg f(x) = 2g + 2$ is even,
- 2. f(x) is squarefree,
- 3. LT(f(x)) is a quadratic residue,
- 4. $f(\alpha)$ is a q.r. for all $\alpha \in \mathbb{F}_q \ (\Rightarrow N^0_{f(x)} = 0, \ N^r_{f(x)} = 2q).$

Let C be a curve $y^2 = f(x)$ over \mathbb{F}_q , where f(x) is good. Then C is smooth of genus g having

$$N_1(\mathcal{C}) = N_1(\mathcal{C})^{aff} + N_1(\mathcal{C})^{\infty} = N_{f(x)}^0 + 2N_{f(x)}^r + N_1(\mathcal{C})^{\infty} = 2q + 2.$$

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Let $\mathcal C$ be a curve $y^2 = f(x)$ over $\mathbb F_q$, where f(x) is good. Then $\mathcal C$ is smooth of genus g having

 $N_1(\mathcal{C}) = N_1(\mathcal{C})^{aff} + N_1(\mathcal{C})^{\infty} = N_{f(x)}^0 + 2N_{f(x)}^r + N_1(\mathcal{C})^{\infty} = 2q + 2.$

We will call such a curve \mathbb{F}_q -maximal, since $N_1 \leq 2q+2$ for every hyperelliptic curve over \mathbb{F}_q .

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- 1. $\deg f(x) = 2g + 2$ is even,
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- 3. LT(f(x)) is a quadratic residue,
- 4. $f(\alpha)$ is a q.r. for all $\alpha \in \mathbb{F}_q \ (\Rightarrow N^0_{f(x)} = 0, \ N^r_{f(x)} = 2q).$

Let C be a curve $y^2 = f(x)$ over \mathbb{F}_q , where f(x) is good. Then C is smooth of genus g having

$$N_1(\mathcal{C}) = N_1(\mathcal{C})^{aff} + N_1(\mathcal{C})^{\infty} = N_{f(x)}^0 + 2N_{f(x)}^r + N_1(\mathcal{C})^{\infty} = 2q + 2.$$

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Construction of a pointless curve over \mathbb{F}_{q} , odd q

Let C be a smooth \mathbb{F}_q -maximal hyperelliptic curve over \mathbb{F}_q . It can be defined by $y^2 = f(x)$, where f(x) is good of degree 2g + 2, where g is the genus of C.

Let C' be its quadratic twist: $\alpha y^2 = f(x)$, α is a q.n.r.

The Weil theorem implies that

$$\begin{split} & \operatorname{Tr}(\mathit{Fr}_{\mathcal{C}}) = q + 1 - \mathit{N}_1(\mathcal{C}) = -q - 1, \\ & \mathit{N}_1(\mathcal{C}') = q + 1 - \operatorname{Tr}(\mathit{Fr}_{\mathcal{C}'}) = q + 1 + \operatorname{Tr}(\mathit{Fr}_{\mathcal{C}}) = 0, \end{split}$$

where *Fr* stands for Frobenius endomorphism.

Thus, C' is smooth pointless hyperelliptic genus g curve over \mathbb{F}_q . The idea of the search of pointless curves:

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Construction of a pointless curve over \mathbb{F}_q , odd q

Let C be a smooth \mathbb{F}_q -maximal hyperelliptic curve over \mathbb{F}_q . It can be defined by $y^2 = f(x)$, where f(x) is good of degree 2g + 2, where g is the genus of C.

Let \mathcal{C}' be its quadratic twist: $lpha y^2 = f(x)$, lpha is a q.n.r.

The Weil theorem implies that

$$\begin{split} & \operatorname{Tr}(\mathit{Fr}_{\mathcal{C}}) = q + 1 - \mathit{N}_1(\mathcal{C}) = -q - 1, \\ & \mathit{N}_1(\mathcal{C}') = q + 1 - \operatorname{Tr}(\mathit{Fr}_{\mathcal{C}'}) = q + 1 + \operatorname{Tr}(\mathit{Fr}_{\mathcal{C}}) = 0, \end{split}$$

where *Fr* stands for Frobenius endomorphism.

Thus, C' is smooth pointless hyperelliptic genus g curve over \mathbb{F}_q . The idea of the search of pointless curves:

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Linear bound on the genus for odd *q*

Theorem

Given odd q, for all $g \ge (q-3)/2$ (or $g \ge 2$) there is a pointless smooth hyperelliptic curve over \mathbb{F}_q of genus g.

Idea: for each step find a monic polynomial of degree 2g + 2 having good values and then check whether it is square-free or not.

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The proof is divided into three parts, depending on a family of good polynomials.

Note:

- 1. We omit technical details of proofs of smoothness.
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Let $g \geq (q-1)/2$.

Special condition on *q* and *g* :

- 1. $\mathcal{D}(q,g)>$ 2, or $\mathcal{D}(q,g)=$ 2 and $\mathcal{L}(q,g)>$ 1, or
- 2. $\mathcal{D}(q,g)=$ 2 and $\mathcal{L}(q,g)=$ 1, but one of the following holds:

$$q=p^{2n}$$
 or $q=p^{2n+1}$, $p\equiv 1 \pmod{8}$ or $2g+2\not\equiv -1/2 \pmod{p}$,

where $\mathcal{D}(q,g)=\gcd(2g+2,q-1),$ $\mathcal{L}(q,g)=\lfloor(2g+2)/(q-1)\rfloor.$ The polynomial

 $f(x) = x^{2g+2} - x^{2g+2-l(q-1)} + a^2, \quad a \in \mathbb{F}_q^*, \quad 1 \le l \le \mathcal{L}(q,g).$

1. is monic of even degree,

- 2. has good values over \mathbb{F}_q , since $f(x)=\left(x^{\prime(q-1)}-1
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Sketch of proof: part II

Let $g \ge (q-1)/2$ and the special condition does not hold. Let n = 2g + 2 - (q-1), and $b, \xi \in \mathbb{F}_p^*$, ξ is a q.n.r. The polynomial

$$f(x) = x^{q-1+n} + b^2 x^{2n} - (2b^2\xi + 1)x^n + b^2\xi^2$$

= $(x^{q-1} - 1)x^n + b^2(x^n - \xi)^2$

1. is monic of even degree,

- 2. has good values (since n is even and ξ is not a q.n.r),
- 3. is square-free, if one chooses b, given ξ , in the way that

$$s^2 = b^4 \xi^2 + 4b^2 \xi + 1$$

and $s \neq 0$, $b \neq 0$.

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Remark. We have treated all cases for $g \ge (q-1)/2$, so that it suffices to prove the theorem for g = (q-3)/2.

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Sketch of proof: part III

Let g = (q-3)/2. We can assume that q > 5.

The curve $x^2 + y^2 = \delta^2$ over \mathbb{F}_q has a rational point (x, y) such that $xy \neq 0$.

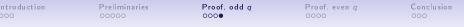
Let α , β , $\gamma \in \mathbb{F}_q$ be residues such that $\alpha + \beta = \gamma$.

The polynomial

$$f(x) = \frac{\alpha}{\gamma} \left(x^{(q-1)/2} + 1 \right)^2 + \frac{\beta}{\gamma} \left(x^{(q-1)/2} - 1 \right)^2$$
$$= x^{q-1} + 2\frac{\alpha - \beta}{\gamma} x^{(q-1)/2} + 1.$$

is monic, of even degree and has good values (we have either $f(a) = 4\alpha/\gamma$, or $f(a) = 4\beta/\gamma$, or f(a) = 1 for all $a \in \mathbb{F}_q$). The condition $\alpha\beta \neq 0$ implies that f(x) is square-free.

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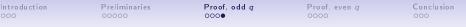
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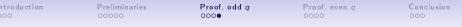
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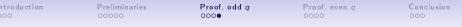


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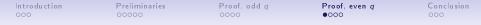


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Hyperelliptic curves over \mathbb{F}_q , q is even

This is a completely different case: for every hyperelliptic curve over \mathbb{F}_{2^n}

$$y^2 + h(x)y = f(x)$$

we can not let h(x) = 0 and it is not trivial to count rational points.

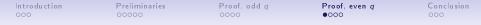
The existence of pointless curves over \mathbb{F}_2 is known.

Theorem (H. Stichtenoth, 2011)

For every $g \ge 2$ there is a non-singular pointless curve over \mathbb{F}_2 .

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Goal: study the case q > 2.



Hyperelliptic curves over \mathbb{F}_q , q is even

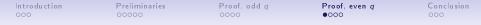
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we can not let h(x) = 0 and it is not trivial to count rational points. The existence of pointless curves over \mathbb{F}_2 is known. Theorem (H. Stichtenoth, 2011)

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Hyperelliptic curves over \mathbb{F}_q , q is even

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Linear bound on the genus for even *q*

Theorem

Let $q = 2^n$, q > 2. For every $g \ge q - 1$ there is a pointless smooth curve over \mathbb{F}_q of the form $y^2 + a f(x)y = b h(x)$, where

$$f(x) = x^{g+1} + x^{g+1-(q-1)} + c, \quad h(x) = x^{2g+2} + x^{2g+2-2(q-1)} + d,$$

for some a, b, c, $d \in \mathbb{F}_q^*$.

The goal: find auxiliary a, b, c, and d in \mathbb{F}_q^* . Proof steps:

- 1. The smoothness implies a condition on *c* and *d*.
- Choose b, d by a, c (and by two good elements in F^{*}_q) in order to guaranty the pointlessness.

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The condition q > 2 is essential for the last step.

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Sketch of proof: the smoothness

Let C be a hyperellptic curve over \mathbb{F}_q , $q = 2^n$, of the form $y^2 + af(x)y = bh(x)$, where

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The curve \mathcal{C} is smooth $\Leftrightarrow R(x)$ and Q(x) are comprime, where

$$R(x) = a^2 b f'(x)h(x) + b^2 h'(x)^2, \quad Q(x) = a f(x).$$

One can show that

 $gcd(R(x), Q(x)) = gcd(h(x), f(x)) = gcd(f(x)^2 + c^2 + d, f(x)).$

This implies a condition for the smoothness: $c^2 \neq d$.

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$\ensuremath{\mathcal{C}}$ is the union of

$$y^{2} + a f(x)y = b g(x), \quad y^{2} + a x^{g+1} f(x)y = b x^{2g+2} g(x),$$
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Any rational point corresponds to a solution of

either
$$y^2 + y + bd(ac)^{-2} = 0$$
, or $y^2 + y + ba^{-2} = 0$.

By Hilbert'90, ${\cal C}$ has no rational points iff

$$\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_2}\left(rac{b}{a^2}\cdotrac{d}{c^2}
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		The gap		

- 1. The Weil-Serre bound implies that if $g < (q+1)/\lfloor 2\sqrt{q} \rfloor$, then the answer is NO.
- 2. The result implies that if

$$g \geq (q-3)/2$$
, q is odd, or $g \geq q-1$, q is even,

then the answer is YES.

It remains to resolve the question in the case when, given odd or even q, the number g belongs to the gap

$$\left(\frac{q+1}{\lfloor 2\sqrt{q} \rfloor}, \frac{q-3}{2}\right)$$
 or $\left(\frac{q+1}{\lfloor 2\sqrt{q} \rfloor}, q-1\right)$, resp.

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More good polynomials

Let q be odd. Let g(x), h(x) be two monic polynomials of degree n and m, an let $b \in \mathbb{F}_q$. Define

$$F_{g,h}(x) = \frac{g(x)^2 - b^2 h(x)^2}{2} x^{\frac{q-1}{2}} + \frac{g(x)^2 + b^2 h(x)^2}{2}.$$

$$F_{g,h}(\alpha) = \begin{cases} g(\alpha)^2, & \alpha \text{ is a q.r.,} \\ b^2 h(\alpha)^2, & \alpha \text{ is a q.n.r.,} \\ \frac{g(0)^2 + b^2 h(0)^2}{2}, & \alpha = 0. \end{cases}$$

- We can choose *m* compare to *n*, depending on *q* (mod 4).
- We can choose b using the intersection of two quadrics in 𝔽³_q (to make F_{g,h}(0) and LT(F_{g,h}(x)) to be q.rr.), that is an elliptic curve over 𝔽_q!
- Is F_{g,h}(x) square-free? In the greate majority of cases YES (experiments).

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- We can choose *m* compare to *n*, depending on *q* (mod 4).
- We can choose *b* using the intersection of two quadrics in \mathbb{F}_q^3 (to make $F_{g,h}(0)$ and $LT(F_{g,h}(x))$ to be q.rr.), that is an elliptic curve over $\mathbb{F}_q!$
- Is F_{g,h}(x) square-free? In the greate majority of cases YES (experiments).

Conclusion

Work in progress

Let *q* be odd.

- 1. There is a binary linear code over \mathbb{F}_q defined by parity-check matrix such that:
 - This matrix depends only on the arithmetic of \mathbb{F}_q possessing several properties.
 - Each codeword of weight w corresponds to a pointless smooth curve over \mathbb{F}_q of genus w 1.
 - **Problem**: find the minimum distance, the weight enumerator and so on.
- 2. The generation function $\sum a_n z^n$ of the class of polynomials with good values.
 - a_n , $n \ge q$, are computed.
 - $a_n > 0$, $(q-1)/2 \le n \le q-1$ (more explicit constructions!).
 - a_n, 0 ≤ n ≤ q − 1 depends only on the generation function ∑ b_nzⁿ of the class of square-free polynomials with good values.
 - **Problem**: find the minimal n_0 such that for all $n \ge n_0$ we have $b_n > 0$.