M-functions associated with modular forms

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- This was later generalized to $L(f, s)$ and $\zeta_K(s)$ by Matsumoto.
- Ihara and the Euler–Kronecker constant : starting from the study of $L'(1, \chi)/L(1, \chi)$ he obtained a whole range of beautiful results on the value distribution of L'/L and $\log L$ (many of them with Matsumoto).

Flavour of Ihara's results (2008)

Given a global field K, i.e. a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_q(t)$, and a family of characters χ of K Ihara considered the distribution of $L'(s,\chi)/L(s,\chi)$ in the following cases:

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He obtains equidistribution results of the type

$$\operatorname{Avg}'_{\chi} \Phi\left(\frac{L'(s,\chi)}{L(s,\chi)}\right) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw|,$$

for $\sigma = \operatorname{Re} s > 1$ for number fields, and for $\sigma > 3/4$ for function fields, under significant restrictions on the test function Φ.

Improvements with Matsumoto (∼2010)

- Case (A) : still valid for both families $\mathfrak{L}(s,\chi) = L'(s,\chi)/L(s,\chi)$ and $\log L(s,\chi)$, for Φ of at most polynomial growth and $\sigma > 1/2$, assuming GRH for number fields:

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- Mourtada and Murty (2015) : some equidistribution result conditional on GRH for averages over quadratic characters.
- Matsumoto and Umegaki (2016): similar results for differences of logarithms of two symmetric power L-functions under the GRH.

The density function M_σ

- The above results give rise to the density functions $M_\sigma(z)$ and related functions $\tilde{M}_s(z_1, z_2)$ (which is the inverse Fourier transform of M_σ, when $z_2 = \bar{z}_1$, $s = \sigma \in \mathbb{R}$).
  Under optimal circumstances we have

$$M_\sigma(z) = \mathrm{Avg}_\chi \, \delta_z \left(\mathfrak{L}(s, \chi)\right), \quad \tilde{M}_\sigma(z_1, z_2) = \mathrm{Avg}_\chi \, \psi_{\mathbf{z_1}, \mathbf{z_2}} \left(\mathfrak{L}(s, \chi)\right),$$

  where $\mathfrak{L}(s, \chi)$ is either $L'(s, \chi)/L(s, \chi)$ or $\log L(s, \chi)$, δ_z is the Dirac delta function, and $\psi_{\mathbf{z_1}, \mathbf{z_2}}(w) = \exp\left(\frac{i}{2}(z_1 \bar{w} + z_2 w)\right)$.

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- Properties of $\tilde{M}$
  - it has an Euler product expansion,
  - it admits an analytic continuation to the left of $\operatorname{Re} s > 1/2$,
  - its zeroes and the "Plancherel volume" $\int_{\mathbb{C}} |\tilde{M}_\sigma(z, \bar{z})|^2 |dz|$ are interesting objects to investigate.

Why is it interesting for us here?

- It complements the asymptotic theory of global fields of Ihara, Tsfasman and Vladuts, giving information on $\{\zeta_{K_i}\}$, where K_i runs through abelian families of global fields (it explains the behaviour of Euler–Kronecker constants in cyclotomic fields for instance)

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- It is challenging to understand what the notion of M-function should be.
- It may lead us to a better (higher dimensional) asymptotic theory.

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  where
  $$\begin{cases} |\alpha_f(p)| = 1,\ \beta_f(p) = \alpha_f(p)^{-1} & \text{if } (p,N) = 1, \\ \alpha_f(p) = \pm p^{-\frac{1}{2}}, \beta_f(p) = 0 & \text{if } p \parallel N, \\ \alpha_f(p) = \beta_f(p) = 0 & \text{if } p^2 \mid N. \end{cases}$$

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- define $\mathfrak{L}(f \otimes \chi, s)$ to be either $(L'/L)(f \otimes \chi, s)$ or $\log L(f \otimes \chi, s)$,
- $\mathfrak{g}(f \otimes \chi, s, z) = \exp\left(\frac{iz}{2} \mathfrak{L}(f \otimes \chi, s)\right)$,
- Write $\mathfrak{g}(f,s,z) = \sum_{n \geq 1} \mathfrak{l}_z(n) n^{-s}$ with $\mathfrak{l}_z(n) = \sum_{x \geq 1} c_{z,x}^N(n) \eta_f(x)$, where $c_{z,x}^N(n)$ depend only on the level N. Put $c_{z,x}(n) = c_{z,x}^1(n)$.

Our results (f is fixed, χ varies) at the $\tilde{M}$-side

Theorem 1

Assume that m is a prime number and let Γ_m denote the group of Dirichlet characters modulo m. Let $0 < \epsilon < \frac{1}{2}$ and $T, R > 0$. Let $s = \sigma + it$ belong to the domain $\sigma \geq \epsilon + \frac{1}{2}$, $|t| \leq T$, let z and z' be inside the disk $\mathcal{D}_R = \{z \mid |z| \leq R\}$. Then, assuming the Generalized Riemann Hypothesis (GRH) for $L(f \otimes \chi, s)$, we have

$$\lim_{m \to \infty} \frac{1}{|\Gamma_m|} \sum_{\chi \in \Gamma_m} \overline{\mathfrak{g}(f \otimes \chi, s, z)} \mathfrak{g}(f \otimes \chi, s, z') = \sum_{n \geq 1} \overline{\mathfrak{l}_z(n)} \mathfrak{l}_{z'}(n) n^{-2\sigma}$$
$$=: \tilde{M}_\sigma(-\bar{z}, z').$$

Our results (f is fixed, χ varies) at the M-side

Theorem 2

Let $\mathrm{Re}\, s = \sigma > \frac{1}{2}$ and let m run over prime numbers. Let Φ be either a continuous function on $\mathbb{C}$ with at most exponential growth, or the characteristic function of a bounded subset of $\mathbb{C}$ or of a complement of a bounded subset of $\mathbb{C}$. Define M_σ as the inverse Fourier transform of $\tilde{M}_\sigma(z, \bar{z})$. Then under GRH for $L(f \otimes \chi, s)$ we have

$$\lim_{m \to \infty} \frac{1}{|\Gamma_m|} \sum_{\chi \in \Gamma_m} \Phi(\mathfrak{L}(f \otimes \chi, s)) = \int_{\mathbb{C}} M_\sigma(w) \Phi(w) |dw|.$$

Our results (f varies) at the $\tilde{M}$-side

Theorem 3

Assume that N is a prime number and that k is fixed. Let $0 < \epsilon < \frac{1}{2}$ and $T, R > 0$. Let $s = \sigma + it$ belong to the domain $\sigma \geq \epsilon + \frac{1}{2}$, $|t| \leq T$, and z and z' to the disc $\mathcal{D}_R$ of radius R. Then, assuming GRH for $L(f,s)$, we have

$$\lim_{N \to +\infty} \sum_{f \in B_k(N)} \omega(f) \overline{\mathfrak{g}(f,s,z)} \mathfrak{g}(f,s,z') = \sum_{n,m \in \mathbb{N}} n^{-\bar{s}} m^{-s} \sum_{x \geq 1} \overline{c_{z,x}(n)} c_{z',x}(m),$$

where $\omega(f)$ are the harmonic weights.

Remarks about the proofs

- Theorem 1 follows mostly from the work of Ihara and Matsumoto, where they define admissible families for which such results hold. To check that our families are admissible is straightforward thanks to numerous works on L-functions of cusp forms.

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- Theorem 2 is very tricky, but the main tool is an extension of the classical Jessen-Wintner theorem (to be discussed later).

- The proof of Theroem 3 is analogous to Ihara's proofs, except that we use the Petersson formula instead of the orthogonality of characters, and is much more technical.

The function $\tilde{M}_{s,p}$

- Let $\mathrm{Re}\, s = \sigma > 0$. Define the functions on $T_p = \mathbb{C}^1 = \{t \in \mathbb{C} \mid |t| = 1\}$ by

$$g_{s,p}(t) = \frac{-(\log p)\alpha(p)p^{-s}t}{1 - \alpha(p)p^{-s}t} + \frac{-(\log p)\beta(p)p^{-s}t}{1 - \beta(p)p^{-s}t},$$

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- We introduce the local factors $\tilde{M}_{s,p}(z_1, z_2)$ via

$$\tilde{M}_{s,p}(z_1, z_2) = \sum_{r=0}^{+\infty} \mathfrak{l}_{z_1}(p^r)\mathfrak{l}_{z_2}(p^r)p^{-2rs}.$$

The series is absolutely and uniformly conv. on compacts in $\mathrm{Re}\, s > 0$

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Outlines of the proof of theorem 2 in the L'/L-case

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$$\tilde{M}_{s,p}(z_1, z_2) = \int_{\mathbb{C}^1} \exp\left(\frac{i}{2}(z_1 g_{s,p}(t^{-1}) + z_2 g_{s,p}(t))\right) d^\times t,$$

$$\text{thus } \tilde{M}_{\sigma,p}(z_1, z_2) = \int_{\mathbb{C}^1} \psi_{z_1,z_2}(g_{\sigma,p}(t)) d^\times t,$$

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- The "trivial" bound $|\tilde{M}_{\sigma,p}(z)| \leq 1$ holds.

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- $\mathcal{F} M_{\sigma,P} = \tilde{M}_{\sigma,P}(z)$.
- There exists a set of primes $\mathcal{P}_f$ of positive density such that, for all $p \in \mathcal{P}_f$, $\tilde{M}_{\sigma,p}(z) \ll_{p,\sigma} (1 + |z|)^{-\frac{1}{2}}$.

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- Let P be a set of primes. If $|P \cap \mathcal{P}_f| > 4$, then $M_{\sigma,P}$ admits a continuous density (still denoted by $M_{\sigma,P}$) which is an L^1 function. The function $M_{\sigma,P}$ satisfies $M_{\sigma,P}(z) = M_{\sigma,P}(\bar{z}) \geq 0$.

Outlines of the proof of theorem 2 in the L'/L-case

The function $M_{\sigma,P}$

- There exists a unique positive measure $M_{\sigma,P}$ of compact support and mass 1 on $\mathbb{C} \simeq \mathbb{R}^2$ such that

$$M_{\sigma,P}(\Phi) = \int_{T_P} \Phi(g_{s,P}(t_P))d^\times t_P, \quad \text{for any cont. function } \Phi \text{ on } \mathbb{C},$$

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- $M_{\sigma,P}$ is of class $\mathcal{C}^r$ once $|P \cap \mathcal{P}_f| > 2(r+2)$.

From now on we assume that $\mathrm{Re}\, s = \sigma > \frac{1}{2}$, without mentioning it in each statement.

The function $\tilde{M}_s$

We define

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- $$\tilde{M}_\sigma(z_1, z_2) = \int_{\mathbb{C}} M_\sigma(w)\psi_{z_1,z_2}(w)|dw|.$$

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L-functions and varieties over finite and global fields,
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