Characterising the Härtig Quantifier model

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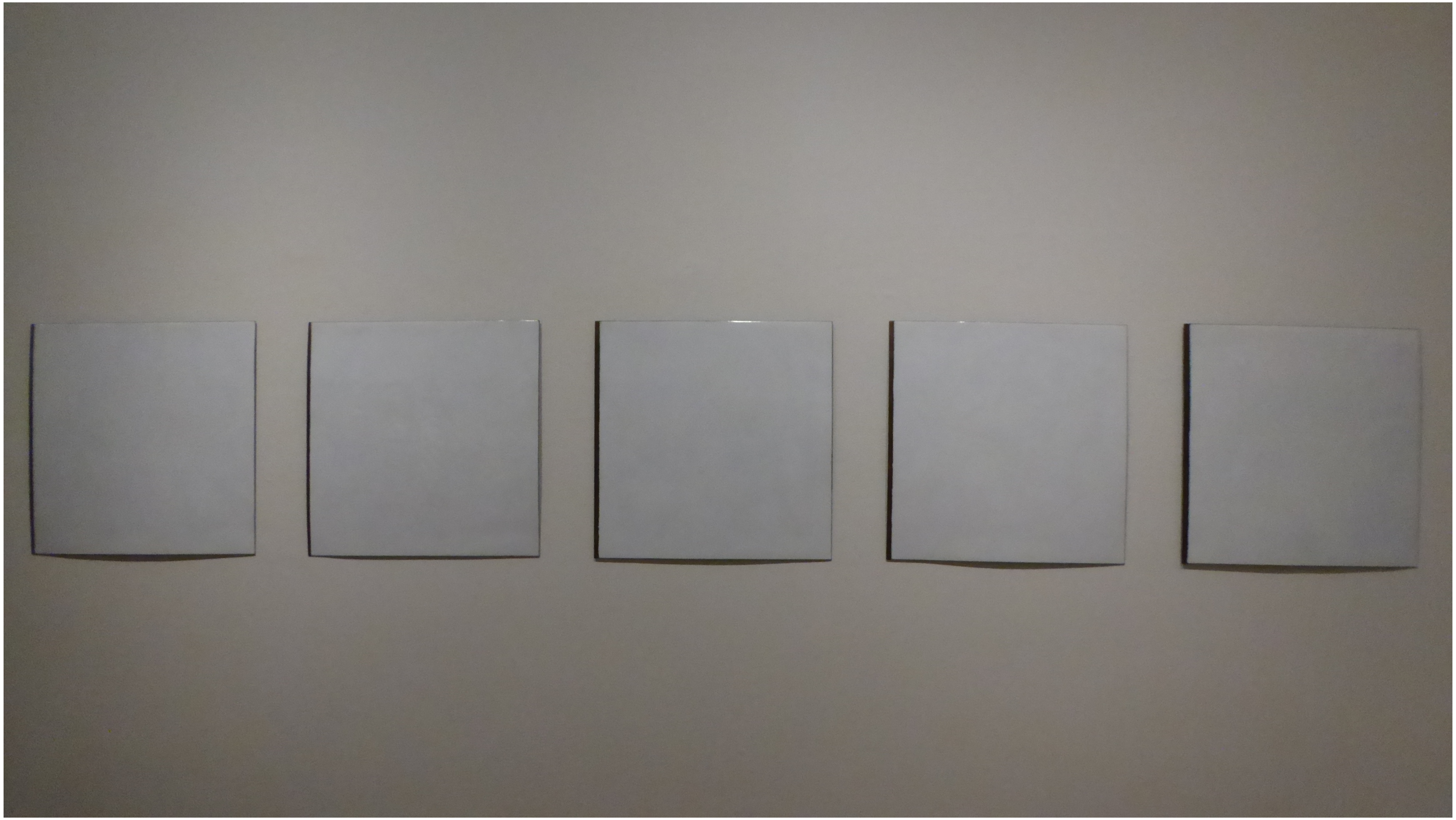

Reference: *Inner models from Extended Logics*, J. Kennedy, M.Magidor, & J. Väänänen. Isaac Newton Preprint Series. 2016.

Basic Idea: Replace first order definability in the construction of L by a notion of definability using a stronger notion of logic.

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Basic Idea: Replace first order definability in the construction of L by a notion of definability using a stronger notion of logic.

Example 1 Second order logic. We define:

$$L'_0 = \varnothing; \quad L'_{\alpha+1} = \mathrm{Def}_{SO}(\langle L'_\alpha, \in \rangle); \quad L'_\lambda = \bigcup_{\alpha<\lambda} L'_\alpha.$$

Theorem[Myhill-Scott] $L' =_{df} \bigcup_{\alpha<\infty} L'_\alpha = HOD$.

Examples 2,3 If we replace Def with $\mathrm{Def}_{\mathcal{L}_{\omega_1,\omega}}$ or $\mathrm{Def}_{\mathcal{L}_{\omega_1,\omega_1}}$ one gets $L(\mathbb{R})$ and the Chang model respectively.

Example 4 *Magidor-Malitz quantifier* $Q^{<\omega}_{\aleph_\alpha}$ (for $\alpha > 0$).

$$M \models Q^n_{\aleph_\alpha} x_1 \cdots x_n \varphi(x_1 \cdots x_n) \iff$$
$$\exists X \subseteq |M| \, [\, |X| \geq \aleph_\alpha \wedge \forall x_1, \cdots, x_n \in X(M \models \varphi(x_1, \cdots, x_n))].$$

Theorem [KMV] $0^\sharp \longrightarrow C(Q^{<\omega}_{\aleph_\alpha}) \;=\; L.$

Example 5 *(The cof_ω-quantifier Q^{cf}_ω)*

$$M \vDash Q^{cf}_\omega x, y\varphi(x, y, \vec{p})$$
$$\Longleftrightarrow \{(x, y) \mid M \vDash \varphi(x, y, \vec{p})\} \text{ codes a linear order of cofinality } \omega.$$

- $0^\sharp \longrightarrow C^* \neq L$.
- Uncountably many measurable cards. $\longrightarrow C^* \neq HOD$.
- L^μ exists $\longrightarrow L^\mu \subseteq C^*$.
- $\exists$ a proper class of Woodins $\longrightarrow$ all Regs. $> \aleph_1$ are Mahlo, indiscernible in C^*.
- $\exists \kappa < \lambda(\kappa$ Woodin , λ meas. $) \longrightarrow$ on a cone of x, $C^*(x) \vDash CH$.

The theme is thus to identify where possible the models $C(Q)$ for varying quantifiers Q.

Härtig quantifier model, $C(I)$

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- Let $C(I)$ be the resulting model.
- $C(I) = L[Card.]$.
- $0^\sharp$ exists $\rightarrow 0^\sharp \in C(I)$.

Theorem [KMV] $K^{DJ} \subseteq C(I)$; if L^μ exists, then $L^\mu \subseteq C(I)$; $V = L^\mu \longrightarrow V = C(I)$.

Theorem [KMV]
$Con(ZFC + \exists\kappa(\kappa$ supercompact $)) \Rightarrow Con(ZFC + \exists$ a supercompact $\wedge C(I) \neq HOD)$.

Characterizing $C(I) = L[Card.]$.

- In fact we characterise $L[P]$ where P is any proper class of cardinals which is either (i) closed or (ii) $P \subseteq$ Succ.Card.

- In Case (i) we let $P_0 = \langle \lambda_\alpha \mid \alpha \in On \rangle$ enumerate the successor elements of P. In Case (ii), let $P_0 = P$.

- For $\alpha \in Lim$, let $\lambda^*_\alpha =_{\mathrm{df}} sup_k \ \lambda_{\alpha+k}$.

Theorem 1 *Assume sufficiently large cardinals (e.g. 0^{sword}).*
Let $K^P = K^{L[P]} = L[E^P]$. Then (i):

$$K^P \models \lambda \textit{ is measurable } \longleftrightarrow \exists\alpha \in Lim(\lambda = \lambda^*_\alpha).$$

(ii) For $\alpha \in Lim$, if we set $c_\alpha = \langle \lambda_{\alpha+k} \mid k < \omega \rangle$ then the sequence $\langle c_\alpha \rangle_{\alpha \in Lim}$ is mutually Prikry generic over K^P and

$$L[P] = L[E^P][\langle c_\alpha \rangle].$$

Theorem 2 (0^{sword}) *Let P, Q be two cub classes of cardinals.*

$$\langle L[P], \in, P \rangle \equiv \langle L[Q], \in, Q \rangle.$$

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Cor. 3 (0^{sword}) $C(I) \neq HOD$; $C(I) \models GCH$.

- The former is because $0^{sword} \notin C(I)$.

The latter because again $C(I)$ is an $L[E][\langle c_\alpha\rangle_{\alpha\in Lim}]$-model.

- Thus assuming sufficiently large cardinals $C(I)$ contains Ramsey cardinals but no measurable.