Generalized descriptive set theory and classification

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Main question

Given T, can one provide non-trivial lower/upper bounds for the spectrum function $I(\kappa, T)$?

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This gave birth to a beautiful branch of model theory, later called **stability theory**.

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So if the function $I(\kappa, T)$ is to have a non-trivial upper bound, then T must be (stable) superstable, NDOP and NOTOP: such theories are briefly called **classifiable**.

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The solution of the spectrum problem for classifiable theories depends upon a key construction which assigns to each model of size κ a skeleton of submodels. Each submodel has cardinality at most $2^{\aleph_0}$, and the skeleton is partially ordered by the natural tree order on a subset of $\kappa^{<\omega}$.

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The isomorphism type of a κ-sized model M of T depends only on $\mathcal{T}_M$, therefore it is enough to count how many such decomposition trees one can have to get an upper bound for the spectrum function.

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Remark: The upper bound in 1 may trivialize (e.g. when κ is a fixed point of the $\aleph$-function), but in general it is easy to find cardinals for which this is not the case: for example, under GCH there are unboundedly many κ for which such upper bound is $< 2^\kappa$, or even $\leq \kappa$.

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  - each E_α is an equivalence relation, and each E_0-class is infinite;
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  Then T^γ is classificable shallow of depth $\gamma + 1$.

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Here is where generalized Descriptive Set Theory enters the scene...

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The **generalized Cantor space** ${}^\kappa 2$ is the subspace of ${}^\kappa\kappa$ consisting of functions taking values in $\{0, 1\}$.

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If A is Borel, the smallest ordinal $\alpha < \kappa^+$ for which $A \in \mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$ is called the **Borel rank** of A, and denoted by $\mathrm{rk}_B(A)$.

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Given a countable complete first-order theory T in the vocabulary $\mathcal{L}$, let

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Thus $\mathrm{Mod}^{\kappa}_{T}$ is a Borel subset of ${}^{\kappa}2$, and $\cong^{\kappa}_{T}$ is a $\boldsymbol{\Sigma}^1_1$ equivalence relation on it.

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Within this framework, we can say that

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Does the Borelness (and/or the Borel rank) of $\cong^\kappa_T$ depend on both parameters, or it just depends on the theory T?

(In the latter case, we can regard T itself as “simple” if some/any $\cong^\kappa_T$ is Borel.)

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If T is classifiable shallow, what is the Borel rank of $\cong^{\kappa}_{T}$? Is it related to the depth of T?

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Thus in the "good" case $B(\kappa, T)$ is almost everywhere dominated by a constant function which (unlike Shelah's upper bound) depends only on $\mathrm{dp}(T)$ and not on the cardinal κ.

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Let T be a countable complete first order theory, and κ be such that $\kappa^{<\kappa} = \kappa > 2^{\aleph_0}$.

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Remark: This gap theorem, unlike Shelah's, is never trivial for the relevant κ's: in particular, under GCH the descriptive gap is non-trivial for every successor cardinal $\kappa \geq \aleph_2$.

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Example

For $\gamma < \omega_1$, consider again the theory T^γ of γ-many (coarser and coarser) equivalence relations, plus some extra conditions. It is not hard to see that $\cong^T_\gamma$ is Borel, and that its Borel rank increases with γ. Thus the T^γ's are classifiable shallow with depth increasing with γ.

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- There is also some freedom in the choice of the set-theoretic universe to work in (for example, forcing extensions preserving cardinals and the continuum should be fine).
- Computing the Borel rank of $\cong^\kappa_T$ seems in general simpler than directly computing the depth of the theory.

Scott height

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Definition

Let κ be an infinite cardinal. Given $M, N \in \mathrm{Mod}^{\kappa}_{T}$ and an ordinal β, set $M \equiv_{\beta} N$ if and only if M and N satisfy the same $\mathcal{L}_{\infty\kappa}$-formulæ with quantifier rank $< \beta$.

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The $\mathcal{L}_{\infty\kappa}$**-Scott height** of $M \in \mathrm{Mod}_T^\kappa$ is the smallest ordinal β, if it exists, such that $M \equiv_\beta N \Rightarrow M \cong N$ for every $N \in \mathrm{Mod}_T^\kappa$.

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Remark: Since $M, N \in \mathrm{Mod}^\kappa_T$ are $\mathcal{L}_{\infty\kappa}$-equivalent if and only if they are $\mathcal{L}_{\kappa^+\kappa}$ equivalent, the $\mathcal{L}_{\infty\kappa}$-Scott height of M is either $\leq \kappa^+$ or ∞.

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Definition

The $\mathcal{L}_{\infty\kappa}$-**Scott height** of a theory T is the sup of all the $\mathcal{L}_{\infty\kappa}$-Scott heights of the κ-sized models of T.

Scott height vs Borelness: the countable case

Theorem (Becker-Kechris)

For any $\mathcal{L}_{\omega_1 \omega}$-sentence φ, the following are equivalent:

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The intuitive meaning is

if $\cong^\omega_\varphi \leq_B \cong^\omega_\psi$, then $\cong^\omega_\varphi$ is not more complicated than $\cong^\omega_\psi$.

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How large is the $\leq_B$-gap between $\cong^\kappa_T$ and $\cong^\kappa_{T'}$? Is there any equivalence relation which lies strictly between the two?

Yet another Descriptive Main Gap Theorem

Along these lines, Hyttinen, Kulikov, and Moreno recently proved the following result.

Theorem (Hyttinen-Kulikov-Moreno)

Suppose that $\kappa^{<\kappa} = \kappa = \lambda^+$ with $2^\lambda > 2^{\aleph_0}$ and $\lambda^{<\lambda} = \kappa$. Then the following statement is consistent:

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Thus the isomorphism relation between the models of a classifiable theory T is way more simple than the isomorphism relation between the models of a non-classifiable theory T'.

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Since $\leq_B^\kappa$ provides a notion of complexity for isomorphism relations which is much finer that the simple distinction Borel vs non-Borel (possibly including Borel ranks), Hyttinen-Kulikov-Moreno theorem gives a much better and more informative gap among countable complete first-order theories.

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- it is just a *consistency result*, while all other gap theorems presented so far are ZFC theorems.

Back Skip

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Crucial fact

Given an ordinal α, let $\mathcal{T}_\alpha$ be the tree of strictly descending sequences of ordinals $< \alpha$ (such a tree is well-founded and has rank $\alpha + 1$).

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Given an ordinal α, let $\mathcal{T}_\alpha$ be the tree of strictly descending sequences of ordinals $< \alpha$ (such a tree is well-founded and has rank $\alpha + 1$). Then for any two κ-sized models M, N,

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Scott height vs Borelness: the (un)countable case

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Thus if a theory T has $\mathcal{L}_{\infty\kappa}$-Scott height β, then for every $M, N \in \mathrm{Mod}^\kappa_T$

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Given $\mathcal{T}$ and h as above, we define

$$B(\mathcal{T}, h) = \{x \in {}^{\kappa}\kappa \mid \mathrm{II} \text{ has a winning strategy in } G(\mathcal{T}, h, x)\},$$

and call the pair $(\mathcal{T}, h)$ a Borel* code for $B(\mathcal{T}, h)$.

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Fact (essentially Blackwell)

A set $A \subseteq {}^\kappa\kappa$ is Borel if and only if it admits a Borel* code $(\mathcal{T}, h)$ with $\mathcal{T}$ well-founded (we can even require that $\mathcal{T}$ be one of the $\mathcal{T}_\alpha$ described before, with $\alpha < \kappa^+$).

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A slightly more careful argument actually shows:

Lemma (Mangraviti-M.)

$A \in \mathbf{\Pi}^0_\alpha$ iff $A = B(\mathcal{T}, h)$ for some $\mathcal{T}$ well-founded of rank $\leq \alpha + 1$.

Given any well-founded tree $\mathcal{T}$ on κ, let $\mathcal{T}^*$ be the (well-founded) tree generated by the sequences

$$\langle (p_0, A_0), f_0, \ldots, (p_n, A_n), f_n \rangle,$$

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Define also a labeling function h from the branches of $\mathcal{T}^*$ to the τ_b-clopen subsets of $(\mathrm{Mod}^\kappa_{\mathcal{L}})^2$ by setting

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Then for every $M, N \in \mathrm{Mod}^\kappa_{\mathcal{L}}$

$$\mathrm{II} \text{ wins } EF^\kappa_{\mathcal{T}}(M, N) \iff \mathrm{II} \text{ wins } G(\mathcal{T}^*, h, (M, N)).$$

Scott height vs Borelness: the (un)countable case

Theorem (S.D. Friedman-Hyttinen-Kulikov + Mangraviti-M.)

Let $\kappa^{<\kappa} = \kappa$. Suppose that T has $\mathcal{L}_{\infty\kappa}$-Scott height $\beta < \kappa^+$. Then $\cong^\kappa_T \in \mathbf{\Pi}^0_\delta$ with $\delta \leq 2\beta + 2 < \kappa^+$.

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Since $\mathcal{T}^*_\beta$ has rank $2\beta + 3$, we get $\cong^\kappa_T \in \mathbf{\Pi}^0_{2\beta+2}$. $\square$

Back Skip

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We are now ready to prove

Theorem (S.D. Friedman-Hyttinen-Kulikov + Mangraviti-M.) Proof

Let $\kappa^{<\kappa} = \kappa$. Suppose that $\cong^\kappa_T \in \mathbf{\Pi}^0_\delta$. Then T has $\mathcal{L}_{\infty\kappa}$-Scott height $\beta \leq \max\{3, \delta + 1\} < \kappa^+$.

Scott height vs Borelness: the (un)countable case

Enrich $\mathcal{L}$ with a new unary symbol P, and consider the set W of those $M \in \mathrm{Mod}^{\kappa}_{\mathcal{L} \cup \{P\}}$ such that $\left|P^M\right| = \left|\kappa \setminus P^M\right| = \kappa$.

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$$A = \{M \in W \mid M_1 \cong^{\kappa}_{T} M_2\},$$

being the preimage of $\cong^{\kappa}_{T}$, is in $\mathbf{\Pi}^0_{\delta'}$, where $\delta' = \max\{2, \delta\}$.

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By choice of N^+, N^-, player II wins both

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On the other hand, $M^0 \in A$ while $M^1 \notin A$, hence $M^0 \not\equiv_{\delta'+1} M^1$, as witnessed by φ, contradiction!

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Some questions

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This looks reasonable because $\mathcal{T}_M$ is a well-founded tree of rank $< \alpha$ (labelled with small structures, which give rise to a clopen condition when $\kappa > 2^{\aleph_0}$),

Some questions

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Question

In this situation, does α yield also a lower bound for δ and β?

Some questions

Problem

For every $\alpha < \omega_1$, find a "natural" example of a classifiable shallow theory T such that $\cong_T^\kappa$ has Borel rank $\geq \alpha$ for some (suitable) κ, possibly under additional set-theoretical assumptions or working in some specific model of ZFC.

The relevance of the problem lies in the fact that such theories would provides natural examples of classifiable shallow theories with larger and larger depth. Computing Borel ranks seems to be way more simpler than computing depths, at least to me.

Some questions

Recall that for some κ's, if T is classifiable shallow and T' is not, then

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$$\cong_T^\kappa <_B \cong_{T'}^\kappa.$$

Question

Under suitable assumptions on κ, how much large is the gap (w.r.t. $\leq_B$) between $\cong_T^\kappa$ and $\cong_{T'}^\kappa$, where T and T' are as above?

Some questions

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New arguments are needed, but e.g. Džamonja and Väänänen already developed a reasonable notion of Scott watershed in the context of **chainable models**, so the problem makes sense and it is quite intriguing.

The end

Thank you for your attention!