Club isomorphisms on higher Aronszajn trees

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Outline

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The Suslin problem

For the rest of the talk, κ will denote a regular uncountable cardinal.

Definition

A κ-tree T is a κ-*Suslin tree* if it has no chain or antichain of size κ.

Recall that if a κ-tree T is normal, then T is Suslin iff it has no antichain of size κ.

Definition

The κ-*Suslin hypothesis* is the statement that there does not exist a κ-Suslin tree.

The ω_1-Suslin hypothesis

Theorem (Solovay-Tennenbaum)

The ω_1-Suslin hypothesis is consistent relative to ZFC.

Roughly speaking, this theorem follows from:

- Given an ω_1-Suslin tree, there is an ω_1-c.c. forcing poset for making it non-Suslin.
- Any finite support iteration of ω_1-c.c. forcings is ω_1-c.c.

The ω_1-Suslin hypothesis follows from Martin's axiom.

Aronszajn trees

Definition

A κ-tree T is a κ-*Aronszajn tree* if it has no chain of size κ (that is, no cofinal branch).

Definition

For an infinite cardinal μ, a μ^+-tree T is *special* if there is a function $f : T \to \mu$ such that $x <_T y$ implies $f(x) \neq f(y)$.

Any special μ^+-tree is Aronszajn but not Suslin.

Club isomorphisms

Given a κ-tree T and $A \subseteq \kappa$, let $T \restriction A := \{x \in T : \mathrm{ht}_T(x) \in A\}$.

Definition

Let T and U be κ-trees. We say that T and U are *club isomorphic* if there exists a club $C \subseteq \kappa$ such that the trees $T \restriction C$ and $U \restriction C$ are isomorphic.

For $\kappa = \mu^+$, a κ-Suslin tree cannot be club isomorphic to a special κ-tree.

The Abraham-Shelah property

Theorem (Abraham-Shelah)

The statement that any two normal ω_1-Aronszajn trees are club isomorphic is consistent relative to ZFC.

The Abraham-Shelah property implies the ω_1-Suslin hypothesis, because from ZFC there exists a special ω_1-tree.

Roughly speaking, this theorem follows from:

- Given two normal ω_1-Aronszajn trees, there exists a proper forcing of size ω_1 which makes them club isomorphic;
- (CH) Any countable support iteration of length ω_2 of proper forcings which have size ω_1 is proper and ω_2-c.c.

The Abraham-Shelah property also follows from PFA, but not from Martin's axiom.

The ω_2-Suslin hypothesis

Theorem (Laver-Shelah)

The ω_2-Suslin hypothesis together with CH is consistent relative to the existence of a weakly compact cardinal.

Note that in a model with the tree property on ω_2, there are no ω_2-Aronszajn trees, and hence no ω_2-Suslin trees.

A more natural generalization of Suslin's hypothesis to ω_2 is the existence of a special ω_2-Aronszajn tree together with the nonexistence of an ω_2-Suslin tree, since that replicates the situation on ω_1.

Since CH implies the existence of a special ω_2-tree, it provides a natural context to study the Suslin hypothesis on ω_2.

The Laver-Shelah construction

The idea of Laver-Shelah: Levy-collapse a weakly compact cardinal to become ω_2, and then iterate adding antichains of size ω_2 to ω_2-Suslin trees.

Two major difficulties to overcome which were comparatively easy in the case of ω_1:

- How to add an antichain of size ω_2 to an ω_2-Suslin tree with an ω_2-c.c. forcing?
- How to preserve ω_2 while iterating countably closed ω_2-c.c. forcings?

The weak compactness of κ together with some technical iterated forcing arguments are used to resolve these issues.

Outline

Question

These results suggest a natural question:

Question

Is it consistent with CH that any two normal ω_2-Aronszajn trees are club isomorphic?

Closure of levels provides an obstacle

Given a tree T and $\delta < \mathrm{ht}(T)$, we say that T is *closed at level* δ if every cofinal branch of $T \upharpoonright \delta$ has an upper bound in T.

- Under CH, there exists a normal ω_2-Aronszajn tree T_1 which is closed at levels of cofinality ω.
- Under CH, there exists a normal ω_2-Aronszajn tree T_2 which is not closed at any level.
- Club isomorphisms between ω_2-trees preserve the property of whether a level is closed or not, so T_1 and T_2 are not club isomorphic.

Revised question

This obstruction leads to a natural revision of the question.

A tree is *countably closed* if it is closed at every level of cofinality ω.

Question

Is it consistent with CH that any two countably closed normal ω_2-Aronszajn trees are club isomorphic?

Answer

Theorem (K. 2017)

It is consistent that any two countably closed normal ω_2-Aronszajn trees are club isomorphic relative to the existence of an ineffable cardinal.

The ω_2-Suslin hypothesis

Proposition

(CH) The statement that any two countably closed normal ω_2-Aronszajn trees are club isomorphic implies the ω_2-Suslin hypothesis.

Assuming CH, if S is a normal ω_2-Suslin tree, it is possible to build a countably closed normal ω_2-Aronszajn tree U which contains S as a subtree, preserving heights of nodes.

CH implies the existence of a countably closed normal special ω_2-tree W. A club isomorphism between U and W would imply that for some club C, $S \restriction C$ is special, which is impossible.

Outline

Property of Baumgartner-Malitz-Reinhart

Consider the two facts:

- There exists an ω_1-c.c. forcing for killing an ω_1-Suslin tree;
- There exists a proper forcing which adds a club isomorphism between two normal ω_1-Aronszajn trees.

Both of these facts can be proven based on a well-known result of Baumgartner-Malitz-Reinhart.

Theorem

If $\{a_i : i < \omega_1\}$ is a family of pairwise disjoint finite subsets of an ω_1-Aronszajn tree T, then there are $i < j$ such that every node in a_i is incomparable in T with every node in a_j.

Generalization of Baumgartner-Malitz-Reinhart

Generalizing to ω_2 under CH:

- Does there exist a countably closed ω_2-c.c. forcing for killing an ω_2-Suslin tree?
- Does there exist a countably closed forcing for adding a club isomorphism between two countably closed normal ω_2-Aronszajn trees which is proper for stationarily many models of size ω_1?

The answer is yes in the model constructed by Laver-Shelah, which satisfies:

Property

(CH) If $\{a_i : i < \omega_2\}$ is a family of pairwise disjoint countable subsets of an ω_2-Aronszajn tree T, then there are $i < j$ such that every node in a_i is incomparable in T with every node in a_j.

A forcing poset for adding a club isomorphism

Consider T and U which are countably closed normal ω_2-Aronszajn trees. Define a countably closed forcing poset $\mathbb{P}(T, U)$ for adding a club isomorphism from T to U.

Conditions in $\mathbb{P}(T, U)$ are pairs (A, f) satisfying:

- $A \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$ is countable;
- f is an isomorphism between countable downwards closed normal subtrees of $T \restriction A$ and $U \restriction A$.

Let $(B, g) \leq (A, f)$ if $A \subseteq B$ and $f \subseteq g$.

Assume the property described in the previous slide holds. Suppose that $(A, f) \in \mathbb{P}(T, U)$, N is an elementary substructure of size ω_1, $N^\omega \subseteq N$, and $N \cap \omega_2 \in A$. Then (A, f) is $(N, \mathbb{P}(T, U))$-generic.

Iterating the forcings

There is no known general iteration theorem for iterating countably closed ω_2-c.c. forcings while preserving ω_2 which is applicable to the Laver-Shelah forcing iteration.

The Laver-Shelah proof involves a technical argument to show that the specific iteration under consideration is ω_2-c.c.

To establish the consistency of the Abraham-Shelah property on ω_2, we adapt the Laver-Shelah construction to prove that a specific countable support iteration of countably closed forcings which are ω_2-proper is ω_2-proper (on a stationary set of models).

Ineffability

Instead of working with a weakly compact cardinal as in the Laver-Shelah proof, we use an ineffable cardinal κ.

Let J be the ineffability ideal on κ. Then:

Lemma

Let $\langle N_i : i < \kappa \rangle$ be a $\subseteq$-increasing continuous sequence of elementary substructure of size less than κ. Let $\langle x_i : i \in S \rangle$ be such that $S \in J^+$ and each $x_i \subseteq N_i$. Then there is a set $X \subseteq \bigcup_i N_i$ and a stationary set $U \subseteq S$ such that for all $i \in U$, $X \cap N_i = x_i$.

The forcing iteration

Let $\mathbb{P}_\alpha$ denote our forcing iteration up to α. Conditions in $\mathbb{P}_\alpha$ are of the form

$$p = (a, X),$$

satisfying:

- a is a countable function with $\mathrm{dom}(a) \subseteq \alpha$ so that for all $\gamma \in \mathrm{dom}(a)$, $a(\gamma)$ is a $\mathbb{P}_\gamma$-name for a condition in the poset for adding a club isomorphism between two countably closed normal ω_2-Aronszajn trees $\dot{T}_\alpha$ and $\dot{U}_\alpha$;
- X is a countable function with $\mathrm{dom}(X) \subseteq \alpha + 1$ so that for each $\beta \in \mathrm{dom}(X)$, $X(\beta)$ is a countable subset of

$$\{M \in P_\kappa(\beta) : \mathrm{cf}(M \cap \kappa) > \omega\},$$

and if $M \in X(\beta)$ and $\gamma \in M \cap \mathrm{dom}(a)$, then $M \cap \kappa$ appears in the condition $a(\gamma)$.

$(N, \mathbb{P}_\alpha)$-generic conditions

Consider an elementary substructure N of size less than κ for which we would like to prove the existence of $(N, \mathbb{P}_\alpha)$-generic conditions.

Define a condition $p(N, \alpha)$ to be equal to $(\emptyset, X)$, where $\mathrm{dom}(X) = \{\alpha\}$ and $X(\alpha) = \{N \cap \alpha\}$.

Then whenever $(b, Y) \leq p(N, \alpha)$, then for all $\gamma \in N \cap \alpha \cap \mathrm{dom}(b)$, $N \cap \kappa$ appears in $b(\gamma)$.

Also, for all $p \in N \cap \mathbb{P}_\alpha$, p and $p(N, \alpha)$ are compatible.

κ-proper for stationarily many N

Proposition

Let $\langle N_i : i \in S \rangle$ be a $\subseteq$-increasing and continuous sequence of elementary substructures of size less than κ, where $S \in J^+$. Then there is $C \in J^*$ such that for all $i \in S \cap C$, $p(N_i, \alpha)$ is $(N_i, \mathbb{P}_\alpha)$-generic.

It follows that there are stationarily many N such that $p(N, \alpha)$ is $(N, \mathbb{P}_\alpha)$-generic. For example, consider a model N which satisfies that for all $D \in N \cap J^*$, $N \cap \kappa \in D$. Build a sequence as above inside N, with C as above. Then $N = N_{\kappa_N}$ (so to speak), and $N \cap \kappa \in C$ implies that $p(N, \alpha)$ is $(N, \mathbb{P}_\alpha)$-generic.

Intermediate extensions

The Laver-Shelah method for proving the κ-c.c. depended on being able to factor the iteration over a model of size less than κ (where κ is the weakly compact cardinal collapsed to become ω_2).

If $\mathbb{P}$ is the Laver-Shelah iteration, then $\mathbb{P}$ is κ-c.c. By the weak compactness of κ, using Π^1_1-reflection there exist models N such that $N \cap \kappa = \lambda$ is inaccessible, $N^{<\lambda} \subseteq N$, and $N \cap \mathbb{P}$ is λ-c.c.

It easily follows that $N \cap \mathbb{P}$ is a regular suborder of $\mathbb{P}$. Thus, if G is a generic filter for $\mathbb{P}$ then $V[N \cap G]$ is an intermediate extension of $V[G]$.

Strongly generic conditions

To adapt the Laver-Shelah method to the κ-proper context, we need to convert the $(N, \mathbb{P}_\alpha)$-genericity of the condition $p(N, \alpha)$ to strong $(N, \mathbb{P}_\alpha)$-genericity. By work of Mitchell, if a generic filter G contains a strongly $(N, \mathbb{P}_\alpha)$-generic condition, then $V[N \cap G]$ is an intermediate extension of $V[G]$.

Proposition

Let $\langle N_i : i \in S \rangle$ be $\subseteq$-increasing and continuous, where $S \in J^+$, such that for all $i \in S$, $p(N_i, \alpha)$ is $(N_i, \mathbb{P}_\alpha)$-generic. Then there is $C \in J^*$ such that for all $i \in S \cap C$, $p(N_i, \alpha)$ is strongly $(N_i, \mathbb{P}_\alpha)$-generic.

Strongly generic conditions

If not, then for a J-positive set of $i \in S$, there is a dense set $D_i \subseteq N_i \cap \mathbb{P}_\alpha$ such that D_i is not predense below $p(N_i, \alpha)$.

By ineffability, there is $D \subseteq \bigcup_i N_i$ and a stationary set $U \subseteq S$ such that for all $i \in U$, $D_i = D \cap N_i$. Easily D is a dense subset of $\mathbb{P}_\alpha$.

Consider the sequence $\langle M_i : i \in U \rangle$ where $M_i = Sk(N_i \cup \{D\})$. Then there are club many i such that $p(N_i, \alpha) = p(M_i, \alpha)$ and $N_i \cap \mathbb{P}_\alpha = M_i \cap \mathbb{P}_\alpha$. Now $p(M_i, \alpha)$ will be $(M_i, \mathbb{P}_\alpha)$-generic, and $D \in M_i$, so $D \cap M_i = D \cap N_i = D_i$ is predense below $p(M_i, \alpha) = p(N_i, \alpha)$, which is a contradiction.

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