From Forcing to Realizability

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What is Realizability?

Establishes a correspondence between formulas provable in a logical system and programs interpreted in a model of computation. Then uses tools from computer science to extract information about proofs in the logical system.

A short history

Kleene 1945

Correspondence between formulas of Heyting arithmetic and (sets of indexes of) recursive functions.

Curry Howard 1958

Isomorphism between proofs in intuitionistic logic and simply typed lambda-terms.

Griffin 1990

Correspondence between classical logic and lambda-terms plus control operators.

Krivine 2000-2004

The programs-formulas correspondence is extended to any formula provable in ZF+DC. Krivine's technique generalizes Forcing: forcing models are special cases of realizability models.

Realizability models of set theory- spoiler alert!

The Axiom of Choice

Open problem: can we realize the Axiom of Choice?
Krivine 2004 –> Dependent Choice can be realized (by 'quote')

Realizability is not forcing... maybe

Krivine 2013 - Consistency (relatively to the consistency of ZF) of the theory:
ZF + DC + there is a sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ of infinite subsets of $\mathbb{R}$ such that:

- for $n \geq 2$, the sequence is 'strictly increasing', i.e. there is an injection but no surjection between X_n and X_{n+1}
- $X_m \times X_n$ is equipotent with X_{mn} for every $n, m \geq 2$

The λ-calculus

Syntax of λ-calculus

λ-terms: $M, N ::= x \mid MN \mid \lambda x.M$ $\quad$ ($\Lambda = \{$ all λ-terms $\}$)

β-reduction

$(\lambda x.M)N \to_\beta M[x := N]$.

Forcing vs. realizability

Forcing	**Realizability**
$\mathbb{P}$: set of conditions (Boolean algebra)	Λ: the 'programs' ; Π : the 'stacks'
$\wedge$ 'meet'	( ) 'application' ; **.** 'push' ; $\star$ 'process' k_π 'continuation'
$\leq$ partial order on $\mathbb{P}$	$\succ$ preorder on $\Lambda \star \Pi$
$\mathbb{1}$ maximal condition	$I,\ K,\ W,\ C,\ B,\ cc,\ \varsigma \in \Lambda$ 'instructions'
$\perp \subseteq \mathbb{P} \times \mathbb{P}$	$\perp \subseteq \Lambda \star \Pi$ final segment
V 'ground model' $V^{\mathbb{P}}$ the Boolean-valued model	$\mathcal{M}$ 'ground model' $\mathcal{N}$ 'realizability model'
$\lvert\lvert\varphi\rvert\rvert \in \mathbb{P}$	$\lvert\varphi\rvert \subseteq \Lambda$; $(\!\lvert\varphi\rvert\!) \subseteq \Pi$
$\{\mathbb{1}\}$	$\Lambda^* \subseteq \Lambda$: the 'proof-like programs'. Contains the instructions and it's closed by application.
$V^{\mathbb{P}} \models \varphi$ if $\lvert\lvert\varphi\rvert\rvert = \mathbb{1}$ $\mathbb{1} \Vdash \varphi$ reads "$\mathbb{1}$ forces φ"	$\mathcal{N} \models \varphi$ if $\exists\theta \in \Lambda^*\ (\theta \in \lvert\varphi\rvert)$ $\theta \Vdash \varphi$ reads "θ realizes φ"

Krivine's machine

$\succ$ is the least preorder on $\Lambda \star \Pi$ such that for all $\xi, \eta, \zeta \in \Lambda$ and $\pi, \sigma \in \Pi$,

- $\xi(\eta) \star \pi \succ \xi \star \eta \,.\, \pi$
- $I \star \xi \,.\, \pi \succ \xi \star \pi$
- $K \star \xi \,.\, \eta \,.\, \pi \succ \xi \star \pi$
- $E \star \xi \,.\, \eta \,.\, \pi \succ \xi(\eta) \star \pi$
- $W \star \xi \,.\, \eta \,.\, \pi \succ \xi \star \eta \,.\, \eta \,.\, \pi$
- $C \star \xi \,.\, \eta \,.\, \zeta \,.\, \pi \succ \xi \star \zeta \,.\, \eta \,.\, \pi$
- $B \star \xi \,.\, \eta \,.\, \zeta \,.\, \pi \succ \xi(\eta(\zeta)) \star \pi$
- $cc \star \xi \,.\, \pi \succ \xi \star k_\pi \,.\, \pi$
- $k_\pi \star \xi \,.\, \sigma \succ \xi \star \pi$

Krivine's machine

We call 'combinatory terms' or *c*-terms the programs which are written with variables, instructions and the application. Every lambda-term can be translated into a *c*-term.

Execution theorem

Let $\theta[x_1, ..., x_n] \in \Lambda$ be a *c*-term, let $\xi_1, ..., \xi_n \in \Lambda$ and $\pi \in \Pi$, then

$$\lambda x_1 ... \lambda x_n.\theta \star \xi_1 \text{ . } ... \text{ . } \xi_n \text{ . } \pi \succ \theta[\xi_1/x_1, ..., \xi_n/x_n] \star \pi$$

Non extensional set theory ZF_ε

$\mathcal{L} = \{\varepsilon\ , \in, \subseteq\}$.
$x \simeq y$ is the formula $x \subseteq y \wedge y \subseteq x$

- Extensionality: $\forall x \forall y (x \in y \iff \exists z\ \varepsilon\ y (x \simeq z))$;
  $\forall x \forall y (x \subseteq y \iff \forall z\ \varepsilon\ x (z \in y))$
- Foundation:
  $\forall x_1 ... \forall x_n \forall a (\forall x (\forall y\ \varepsilon\ x F[y, x_1, ..., x_n] \Rightarrow F[x, x_1, ..., x_n]) \Rightarrow F[a, x_1, ..., x_n])$
- Pairing: $\forall a \forall b \exists x (a\ \varepsilon\ x \wedge b\ \varepsilon\ x)$
- Union: $\forall a \exists b \forall x\ \varepsilon\ a \forall y\ \varepsilon\ x (y\ \varepsilon\ b)$
- Powerset: $\forall a \exists b \forall x \exists y\ \varepsilon\ b \forall z (z\ \varepsilon\ y \iff (z\ \varepsilon\ a \wedge z\ \varepsilon\ x))$
- Replacement: $\forall x_1 ... \forall x_n \forall a \exists b \forall x\ \varepsilon\ a (\exists y F[x, y, x_1, ..., x_n] \Rightarrow (\exists y\ \varepsilon\ b F[x, y, x_1 ... x_n]))$
- Infinity $\forall x_1 ... x_n \forall a \exists b [a\ \varepsilon\ b \wedge \forall x\ \varepsilon\ b (\exists y F[x, y, x_1, ..., x_n] \Rightarrow \exists y\ \varepsilon\ b F[x, y, x_1, ..., x_n])]$

ZF_ε is a conservative extension of ZF.

The realizability relation

We define the two truth values $|\varphi| \subseteq \Lambda$ and $(\!|\varphi|\!) \subseteq \Pi$.

- $(\!|\top|\!) = \emptyset$, $(\!|\bot|\!) = \Pi$, $(\!|a \not\varepsilon b|\!) = \{\pi \in \Pi;\ (a, \pi) \in b\}$
- $(\!|a \subseteq b|\!) = \{\xi \, . \, \pi;\ \exists c (c, \pi) \in a \text{ and } \xi \Vdash c \notin b\}$
- $(\!|a \notin b|\!) = \{\xi \, . \, \xi' \, . \, \pi;\ \exists c (c, \pi) \in b \text{ and } \xi \Vdash a \subseteq c \text{ and } \xi' \Vdash c \subseteq a\}$
- $(\!|\varphi \Rightarrow \psi|\!) = \{\xi \, . \, \pi;\ \xi \Vdash \varphi \text{ and } \pi \in (\!|\psi|\!)\}$
- $(\!|\forall x \varphi|\!) = \bigcup_a (\!|\varphi[a/x]|\!)$

$\xi \in |\varphi| \iff \forall \pi \in (\!|\varphi|\!)(\xi \star \pi \in \bot)$

$\xi \Vdash \varphi$ means $\xi \in |\varphi|$

The excluded middle is realized

Theorem

$cc \Vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$

Lemma

If $\pi \in (\!|A|\!)$, *then* $k_\pi \Vdash A \Rightarrow B$

Proof.

Let $\xi \Vdash A$, then for every stack $\pi' \in (\!|B|\!)$, we have $k_\pi \star \xi \,.\, \pi' \succ \xi \star \pi \in \perp$ □

Proof of theorem

Let $\xi \Vdash (A \Rightarrow B) \Rightarrow A$ and $\pi \in (\!|A|\!)$. Then $cc \star \xi \,.\, \pi \succ \xi \star k_\pi \,.\, \pi$ which is in $\perp$, because $k_\pi \Vdash A \Rightarrow B$ by the above lemma.

Adequacy lemma

Adequacy lemma

Let $A_1, ..., A_n, A$ be closed formulas of ZF_ε and suppose $x_1 : A_1, ..., x_n : A_n \vdash t : A$.
If $\xi_1 \Vdash A_1, ...\xi_n \Vdash A_n$, then $t[\xi_1/x_1, ..., \xi_n/x_n] \Vdash A$.

Corollary

If $\vdash t : A$, then $t \Vdash A$

The axioms of ZF_ε are realized

ZF_ε - Pairing axiom is realized:

Given two sets a and b, let $c = \{a, b\} \times \Pi$. We have $(\!| a \not\varepsilon c |\!) = (\!| b \not\varepsilon c |\!) = (\!| \bot |\!) = \Pi$, thus $I \Vdash a \varepsilon c$ and $I \Vdash b \varepsilon c$.

Remark

c may contain many other elements than a and b which have no name in $\mathcal{M}$.

Quote

Integers

Fix $\theta \mapsto n_\theta$ and enumeration of Λ
Inductively define for each $n \in \mathbb{N}$ an element $\underline{n} \in \Lambda$: let $\underline{0} = KI$, and $S = (BW)(BB)$; for each $n \in \mathbb{N}$, let $\underline{n+1} = S(\underline{n})$.

- $\varsigma \star \xi . \eta . \pi \succ \xi \star \underline{n}_\eta . \pi$

Non extensional choice

Theorem

For each formula $F[x,y]$ we can define a function symbol f such that:

$$\varsigma \Vdash \forall x \forall m^{\tilde{\mathbb{N}}} F[x, f(m,x)] \Rightarrow \forall y F[x,y]$$

Now, let $\varphi(x) = f(m,x)$ for the first m such that $\neg F[x, f(m,x)]$, if there is one; or else 0. Then

$$\mathcal{N} \models \forall x F[x, \varphi(x)] \Rightarrow \forall y F[x,y]$$

This implies Dependent Choice: indeed if A is a non empty set and R is an entire binary relation on A (i.e. for every $x \in A$, there is $y \in A$ such that $R(x,y)$) then we let $F[x,y]$ be $\neg R(x,y)$. By hypothesis $\forall x \exists y R(x,y)$, thus $\forall x \exists y \neg F[x,y]$. It follows from the statement above that $\neg F[x, \varphi(x)]$, i.e. $R(x, \varphi(x))$. Then fix any $a_0 \in A$, by letting $a_{n+1} = \varphi^{n+1}(a_0)$ we get the desired sequence.

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For each $m \in \mathbb{N}$ we let $P_m = \{\pi \in \Pi; \xi \star \underline{m} \,.\, \pi \notin \perp \text{ and } m = n_\xi\}$. For each individual x, we have $(\!|\forall x F[x, y]|\!) = \bigcup_a (\!|F[a, y]|\!)$. By means of the axiom of choice there is a function f such that given $m \in \mathbb{N}$ and y such that $P_m \cap (\!|\forall x F[x, y]|\!) \neq \emptyset$, we have $P_m \cap (\!|F[f(m, y), y]|\!) \neq \emptyset$. Now we want to show $\varsigma \Vdash \forall m^{\tilde{\mathbb{N}}} F[x, f(m, x)] \Rightarrow \forall y F[x, y]$ for every individuals x, y. Let $\xi \Vdash \forall m^{\tilde{\mathbb{N}}} F[f(m, y), y]$ and $\pi \in (\!|F[a, y]|\!)$. Suppose by contradiction that $\varsigma \star \xi \,.\, \pi \notin \perp$, then $\xi \star \underline{i} \,.\, \pi \notin \perp$ with $i = n_\xi$. It follows that $\pi \in P_i \cap (\!|F[a, y]|\!)$, thus there is $\pi' \in P_i \cap (\!|F[f(i, y), y]|\!)$. We have $\underline{i} \,.\, \pi' \in (\!|\forall m^{\tilde{\mathbb{N}}} F[f(m, y), y]|\!)$ and therefore, by hypothesis on ξ we have $\xi \star \underline{i} \,.\, \pi' \in \perp$, contradicting $\pi' \in P_i$. □

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Conclusions

The Axiom of Choice

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Is Realizability stronger than forcing?

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