Rearrangements and Subseries

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Jörg Brendle Rearrangements and Subseries

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joint work with Andreas Blass, Will Brian, Joel Hamkins, Michael Hardy, Paul Larson, and Jonathan Verner

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 $\sum a_n$ series

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- $\sum a_n$ absolutely convergent $\iff \sum |a_n|$ converges

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 $\sum a_n \text{ conditionally convergent (c.c.)} \iff \sum a_n \text{ converges and} \\ \sum |a_n| = +\infty$

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<u>Notes:</u> (1) $\sum a_n$ convergent $\implies a_n \to 0$

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<u>Notes:</u> (1) $\sum a_n$ convergent $\implies a_n \rightarrow 0$ (2) If $\sum a_n$ is conditionally convergent then

$$\sum_{n\in P}a_n=+\infty \quad \text{and} \quad \sum_{n\in N}a_n=-\infty$$

where $P = \{n \in \omega : a_n > 0\}$ and $N = \{n \in \omega : a_n < 0\}$

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Riemann's Rearrangement Theorem

Suppose $\sum a_n$ is conditionally convergent and $r \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then there is a rearrangement $\pi \in \text{Sym}(\omega)$ such that $\sum a_{\pi(n)} = r$.

Riemann's Rearrangement Theorem

Suppose $\sum a_n$ is conditionally convergent and $r \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then there is a rearrangement $\pi \in \text{Sym}(\omega)$ such that $\sum a_{\pi(n)} = r$.

Also there is $\pi \in \text{Sym}(\omega)$ such that $\sum a_{\pi(n)}$ diverges by oscillation. ($\liminf_k \sum_{0}^k a_{\pi(n)} < \limsup_k \sum_{0}^k a_{\pi(n)}$)

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How many permutations do we need such that for every conditionally convergent series $\sum a_n$ there is a permutation π in our family such that $\sum a_{\pi(n)}$ no longer converges to the same limit?

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$$\mathfrak{rr} := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \neq \sum a_n)\} \text{ the rearrangement number}$$

 $\mathfrak{rr}_o := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \text{ diverges by oscillation})\}$

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$$\mathfrak{rr} := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \neq \sum a_n)\} \text{ the rearrangement number} \\ \mathfrak{rr}_o := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \text{ diverges by oscillation})\} \\ \mathfrak{rr}_i := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} = \pm \infty)\} \end{cases}$$

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$$\mathfrak{x} := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \neq \sum a_n)\} \text{ the rearrangement number}$$

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Theorem 1

 $\mathfrak{rr}_o = \mathfrak{rr}$

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$$\mathfrak{rr} := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \neq \sum a_n)\} \text{ the rearrangement number}$$
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Theorem 1 $\mathfrak{rr}_o = \mathfrak{rr}$

Fact:
$$\forall \pi \in \operatorname{Sym}(\omega) \exists \sigma_{\pi} \in \operatorname{Sym}(\omega)$$
 such that
• $\exists^{\infty} n (\sigma_{\pi}[\{0, ..., n-1\}] = \{0, ..., n-1\})$
• $\exists^{\infty} n (\sigma_{\pi}[\{0, ..., n-1\}] = \pi[\{0, ..., n-1\}])$

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Theorem 1 $\mathfrak{rr}_o = \mathfrak{rr}$

$$\underline{Fact:} \ \forall \pi \in \operatorname{Sym}(\omega) \ \exists \sigma_{\pi} \in \operatorname{Sym}(\omega) \ \text{such that} \\
 \bullet \ \exists^{\infty} n \ (\sigma_{\pi}[\{0, ..., n-1\}] = \{0, ..., n-1\}) \\
 \bullet \ \exists^{\infty} n \ (\sigma_{\pi}[\{0, ..., n-1\}] = \pi[\{0, ..., n-1\}])$$

 $\Pi \text{ witness for } \mathfrak{rr} \implies \Pi \cup \{\sigma_{\pi} : \pi \in \Pi\} \text{ witness for } \mathfrak{rr}_o$

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non(meager) is the least size of a non-meager set of reals

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Theorem 2 $\mathfrak{rr} \leq \operatorname{non}(meager)$

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 $\mathfrak{rr} \leq \mathsf{non}(\mathit{meager})$

<u>Proof:</u> $\sum a_n$ c.c. given. $K \in \omega$.

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 $\mathfrak{rr} \leq \operatorname{non}(meager)$

<u>Proof:</u> $\sum a_n$ c.c. given. $K \in \omega$. { $\pi \in Sym(\omega) : \exists n_0 (\sum_{n < n_0} a_{\pi(n)} > K)$ } open dense.

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<u>Proof:</u> $\sum a_n$ c.c. given. $K \in \omega$. { $\pi \in \text{Sym}(\omega) : \exists n_0 (\sum_{n < n_0} a_{\pi(n)} > K)$ } open dense. Similarly with < -K instead of > K.

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 $\mathfrak{rr} \leq \mathsf{non}(\mathit{meager})$

<u>Proof:</u> $\sum a_n$ c.c. given. $K \in \omega$. $\{\pi \in \text{Sym}(\omega) : \exists n_0 (\sum_{n < n_0} a_{\pi(n)} > K)\}$ open dense. Similarly with < -K instead of > K. $\{\pi \in \text{Sym}(\omega) : \sum a_{\pi(n)} \text{ diverges by oscillation}\}$ dense G_{δ} . Done!

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$\mathfrak{b} := \min\{|F| : F \subseteq \omega^{\omega} \text{ and } \forall g \in \omega^{\omega} \exists f \in F \exists^{\infty} n (g(n) < f(n))\}$ the unbounding number

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Theorem 3

 $\mathfrak{b} \leq \mathfrak{r}\mathfrak{r}$

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Theorem 3	
$\mathfrak{b} \leq \mathfrak{rr}$	

<u>Proof:</u> Π family of permutations, $|\Pi| < \mathfrak{b}$.

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Theorem 3 $\mathfrak{b} \leq \mathfrak{rr}$

<u>Proof:</u> Π family of permutations, $|\Pi| < \mathfrak{b}$. $\Pi \rightarrow \omega^{\omega} : \pi \mapsto f_{\pi}$ s.t. for all n

• $f_{\pi}(n) > n$

• $\forall m \leq n \quad \forall k \geq f_{\pi}(n) \quad (\pi(m) < \pi(k))$

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Theorem 3 $\mathfrak{b} \leq \mathfrak{r}\mathfrak{r}$

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 $\exists g \in \omega^{\omega} \text{ s.t. } g \geq^* f_{\pi} \text{ for all } \pi \in \Pi.$

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Theorem 3

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 $\begin{array}{l} \underline{\operatorname{Proof:}} \ \Pi \ \text{family of permutations, } |\Pi| < \mathfrak{b}. \\ \Pi \to \omega^{\omega} : \pi \mapsto f_{\pi} \ \text{s.t. for all } n \\ \bullet \ f_{\pi}(n) > n \\ \bullet \ \forall m \leq n \quad \forall k \geq f_{\pi}(n) \quad (\pi(m) < \pi(k)) \\ \exists g \in \omega^{\omega} \ \text{s.t. } g \geq^{*} f_{\pi} \ \text{for all } \pi \in \Pi. \\ \operatorname{Let} \ \{i_{n} : n \in \omega\} \subseteq \omega \ \text{s.t. } i_{n+1} \geq g(i_{n}). \end{array}$

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Theorem 3

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Theorem 3

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$$b_k = \left\{ egin{array}{cc} a_n & ext{if } k = i_n \ 0 & ext{otherwise} \end{array}
ight.$$

Theorem 3

 $\mathfrak{b} \leq \mathfrak{rr}$

<u>Proof:</u> Π family of permutations, $|\Pi| < \mathfrak{b}$. $\Pi \rightarrow \omega^{\omega} : \pi \mapsto f_{\pi}$ s.t. for all n• $f_{\pi}(n) > n$ • $\forall m \leq n \quad \forall k \geq f_{\pi}(n) \quad (\pi(m) < \pi(k))$ $\exists g \in \omega^{\omega}$ s.t. $g \geq^{*} f_{\pi}$ for all $\pi \in \Pi$. Let $\{i_{n} : n \in \omega\} \subseteq \omega$ s.t. $i_{n+1} \geq g(i_{n})$. $\sum a_{n}$ c.c. given. Define

$$b_k = \begin{cases} a_n & \text{if } k = i_n \\ 0 & \text{otherwise} \end{cases}$$

Then $\sum b_k = \sum a_n$ c.c. Also $\sum b_k = \sum b_{\pi(k)}$ for all $\pi \in \Pi$.

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Theorem 3

 $\mathfrak{b} \leq \mathfrak{rr}$

Proof: Π family of permutations, $|\Pi| < \mathfrak{b}$. $\Pi \to \omega^{\omega} : \pi \mapsto f_{\pi}$ s.t. for all n• $f_{\pi}(n) > n$ • $\forall m \le n \quad \forall k \ge f_{\pi}(n) \quad (\pi(m) < \pi(k))$ $\exists g \in \omega^{\omega}$ s.t. $g \ge^* f_{\pi}$ for all $\pi \in \Pi$. Let $\{i_n : n \in \omega\} \subseteq \omega$ s.t. $i_{n+1} \ge g(i_n)$. $\sum a_n$ c.c. given. Define

$$b_k = \begin{cases} a_n & \text{if } k = i_n \\ 0 & \text{otherwise} \end{cases}$$

Then $\sum b_k = \sum a_n$ c.c. Also $\sum b_k = \sum b_{\pi(k)}$ for all $\pi \in \Pi$. Why? Because $\forall^{\infty} n < m(\pi(i_n) < \pi(i_m))$. Done!
$$\mathfrak{rr}_i := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} = \pm \infty)\}$$

$$\mathfrak{rr}_f := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \text{ converges } \neq \sum a_n)\}$$

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 $\mathfrak{d} := \min\{|F| : F \subseteq \omega^{\omega} \text{ and } \forall g \in \omega^{\omega} \exists f \in F \ \forall^{\infty} n \ (g(n) < f(n))\}$ the dominating number

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Theorem 4

 $\mathfrak{d} \leq \mathfrak{rr}_i, \mathfrak{rr}_f$

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$$\begin{split} \mathfrak{rr}_i &:= \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \; \exists \pi \in \Pi \\ & (\sum a_{\pi(n)} = \pm \infty)\} \end{split}$$

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Theorem 4

 $\mathfrak{d} \leq \mathfrak{rr}_i, \mathfrak{rr}_f$

Proof similar to Theorem 3.

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rr versus cov(null)

cov(null) is the least size of a family of null sets covering the reals

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rr versus cov(null)

cov(null) is the least size of a family of null sets covering the reals

Theorem 5

 $cov(null) \leq \mathfrak{rr}$

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rr versus cov(null)

cov(null) is the least size of a family of null sets covering the reals

Theorem 5

 $cov(null) \leq \mathfrak{rr}$

proof based on:

Rademacher's Lemma

Let $(c_n : n \in \omega)$ be a sequence of reals. Set

$$A = \{f \in 2^{\omega} : \sum_{n} (-1)^{f(n)} c_n \text{ converges}\}$$

Then

$$\mu(A) = \begin{cases} 1 & \text{if } \sum_{n} c_n^2 \text{ converges} \\ 0 & \text{otherwise} \end{cases}$$

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Corollary 6 $CON (\mathfrak{d} < \mathfrak{rr})$

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 $\begin{array}{l} \text{Corollary 6} \\ \text{CON } (\mathfrak{d} < \mathfrak{rr}) \end{array}$

<u>Proof:</u> In the random model, $cov(null) > \mathfrak{d}$. So follows from Theorem 5.

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 $\frac{\text{Corollary 6}}{\text{CON } (\mathfrak{d} < \mathfrak{rr})}$

<u>Proof:</u> In the random model, $cov(null) > \mathfrak{d}$. So follows from Theorem 5.

Corollary 7CON (cov(null) < rr)

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<u>Proof:</u> In the random model, $cov(null) > \mathfrak{d}$. So follows from Theorem 5.

Corollary 7CON (cov(null) < rr)

<u>Proof:</u> In the Laver / Hechler model, $cov(null) < \mathfrak{b}$. So follows from Theorem 3.

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 $\frac{\text{Corollary 6}}{\text{CON } (\mathfrak{d} < \mathfrak{rr})}$

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Corollary 7CON (cov(null) < rr)

<u>Proof:</u> In the Laver / Hechler model, $cov(null) < \mathfrak{b}$. So follows from Theorem 3.

Question 1

 $\mathsf{CON} \ (\mathfrak{rr} < \mathsf{non}(\mathsf{meager}))?$

No known upper bounds for $\mathfrak{rr}_i, \mathfrak{rr}_f$

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No known upper bounds for $\mathfrak{rr}_i, \mathfrak{rr}_f$ $\mathfrak{rr}_i := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi$ $(\sum a_{\pi(n)} = \pm \infty)\}$

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No known upper bounds for $\mathfrak{rr}_i, \mathfrak{rr}_f$

$$\begin{split} \mathfrak{rr}_i &:= \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \; \exists \pi \in \Pi \\ & (\sum a_{\pi(n)} = \pm \infty)\} \end{split}$$

Theorem 8 $CON (rr_i < c)$

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No known upper bounds for $\mathfrak{rr}_i, \mathfrak{rr}_f$

$$\begin{split} \mathfrak{rr}_i &:= \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \; \exists \pi \in \Pi \\ & (\sum a_{\pi(n)} = \pm \infty)\} \end{split}$$

Theorem 8

 $CON (\mathfrak{rr}_i < \mathfrak{c})$

<u>Proof Idea:</u> Start with a model of $\mathfrak{c} > \omega_1$.

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No known upper bounds for $\mathfrak{rr}_i, \mathfrak{rr}_f$

$$\mathfrak{rr}_i := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} = \pm \infty)\}$$

Theorem 8

 $CON (\mathfrak{rr}_i < \mathfrak{c})$

<u>Proof Idea:</u> Start with a model of $\mathfrak{c} > \omega_1$.

Make a finite support iteration of σ -centered forcing of length ω_1 .

No known upper bounds for $\mathfrak{rr}_i, \mathfrak{rr}_f$

$$\begin{split} \mathfrak{rr}_i &:= \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \; \exists \pi \in \Pi \\ & (\sum a_{\pi(n)} = \pm \infty)\} \end{split}$$

Theorem 8

 $CON (\mathfrak{rr}_i < \mathfrak{c})$

<u>Proof Idea:</u> Start with a model of $\mathfrak{c} > \omega_1$.

Make a finite support iteration of σ -centered forcing of length ω_1 . At each stage we add a permutation π s.t. for all ground model c.c. $\sum a_n$, $\sum a_{\pi(n)}$ diverges to either $+\infty$ or $-\infty$.

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No known upper bounds for $\mathfrak{rr}_i, \mathfrak{rr}_f$

$$\begin{split} \mathfrak{rr}_i &:= \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \; \exists \pi \in \Pi \\ & (\sum a_{\pi(n)} = \pm \infty)\} \end{split}$$

Theorem 8

 $CON (\mathfrak{rr}_i < \mathfrak{c})$

<u>Proof Idea</u>: Start with a model of $c > \omega_1$. Make a finite support iteration of σ -centered forcing of length ω_1 . At each stage we add a permutation π s.t. for all ground model c.c. $\sum a_n$, $\sum a_{\pi(n)}$ diverges to either $+\infty$ or $-\infty$. (This needs some preliminary forcing.)

No known upper bounds for $\mathfrak{rr}_i, \mathfrak{rr}_f$

$$\begin{split} \mathfrak{rr}_i &:= \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \; \exists \pi \in \Pi \\ & (\sum a_{\pi(n)} = \pm \infty)\} \end{split}$$

Theorem 8

 $CON (\mathfrak{rr}_i < \mathfrak{c})$

<u>Proof Idea:</u> Start with a model of $\mathfrak{c} > \omega_1$.

Make a finite support iteration of σ -centered forcing of length ω_1 . At each stage we add a permutation π s.t. for all ground model c.c. $\sum a_n$, $\sum a_{\pi(n)}$ diverges to either $+\infty$ or $-\infty$. (This needs some preliminary forcing.) Thus the ω_1 permutations adjoined along the iteration witness $\mathfrak{rr}_i = \omega_1$.

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 $\mathfrak{rr}_f := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \text{ converges } \neq \sum a_n)\}$

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 $\mathfrak{rr}_f := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \text{ converges } \neq \sum a_n)\}$

Theorem 9

 $CON (\mathfrak{rr}_f < \mathfrak{c})$

Jörg Brendle Rearrangements and Subseries

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 $\mathfrak{rr}_f := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \text{ converges } \neq \sum a_n)\}$

Theorem 9

 $CON (\mathfrak{rr}_f < \mathfrak{c})$

<u>Proof Idea:</u> Start with a model of $\mathfrak{c} > \omega_1$.

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 $\mathfrak{rr}_f := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \text{ converges } \neq \sum a_n)\}$

Theorem 9

 $CON (\mathfrak{rr}_f < \mathfrak{c})$

<u>Proof Idea:</u> Start with a model of $\mathfrak{c} > \omega_1$.

Make a finite support iteration of σ -linked forcing of length ω_1 .

 $\mathfrak{rr}_f := \min\{|\Pi| : \Pi \subseteq \operatorname{Sym}(\omega) \text{ and } \forall \text{ c.c. } \sum a_n \exists \pi \in \Pi \\ (\sum a_{\pi(n)} \text{ converges } \neq \sum a_n)\}$

Theorem 9

 $CON (\mathfrak{rr}_f < \mathfrak{c})$

<u>Proof Idea:</u> Start with a model of $\mathfrak{c} > \omega_1$.

Make a finite support iteration of σ -linked forcing of length ω_1 . Use the Lévy-Steinitz Theorem (finite-dimensional version of Riemann's Theorem).

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Theorem 9	Т	heorem	9	
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 $CON (\mathfrak{rr}_f < \mathfrak{c})$

<u>Proof Idea:</u> Start with a model of $\mathfrak{c} > \omega_1$. Make a finite support iteration of σ -linked forcing of length ω_1 . Use the Lévy-Steinitz Theorem (finite-dimensional version of Riemann's Theorem).



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Theorem 9	
$CON \left(\mathfrak{rr}_f < \mathfrak{c} ight)$	

<u>Proof Idea</u>: Start with a model of $\mathfrak{c} > \omega_1$. Make a finite support iteration of σ -linked forcing of length ω_1 . Use the Lévy-Steinitz Theorem (finite-dimensional version of Riemann's Theorem).

Conjecture 1	
$\mathfrak{rr}_i \leq \mathfrak{rr}_f$	
Conjecture 2	
$CON\;(\mathfrak{rr}_i < \mathfrak{rr}_f)$	

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E.g.

$$\sum_{n\in P} a_n = +\infty$$
 and $\sum_{n\in N} a_n = -\infty$

where $P = \{n \in \omega : a_n > 0\}$ and $N = \{n \in \omega : a_n < 0\}$

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E.g.

$$\sum_{n\in P}a_n=+\infty$$
 and $\sum_{n\in N}a_n=-\infty$

where $P = \{n \in \omega : a_n > 0\}$ and $N = \{n \in \omega : a_n < 0\}$

How many subsets of ω do we need such that for every conditionally convergent series $\sum a_n$ there is a set X in our family such that $\sum_{n \in X} a_n$ diverges?

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E.g.

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How many subsets of ω do we need such that for every conditionally convergent series $\sum a_n$ there is a set X in our family such that $\sum_{n \in X} a_n$ diverges?

... such that $\sum_{n \in X} a_n$ diverges either to $+\infty$ or $-\infty$?

$$\begin{array}{l} \beta := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ (\sum_{n \in X} a_n \text{ diverges})\} & \text{the subseries number} \end{array}$$

Jörg Brendle Rearrangements and Subseries

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$$\begin{split} \beta &:= \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ (\sum_{n \in X} a_n \text{ diverges})\} & \text{the subseries number} \\ \beta_o &:= \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ (\sum_{n \in X} a_n \text{ diverges by oscillation})\} \end{split}$$

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$$\begin{split} \beta &:= \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n \text{ diverges})\} & \text{the subseries number} \\ \beta_o &:= \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n \text{ diverges by oscillation})\} \\ \beta_i &:= \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n = \pm \infty)\} \end{split}$$

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$$\begin{split} &\beta := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n \text{ diverges})\} & \text{the subseries number} \\ &\beta_o := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n \text{ diverges by oscillation})\} \\ &\beta_i := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n = \pm \infty)\} \end{split}$$

Theorem 10

 $\beta_o \leq non(meager)$

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$$\begin{split} &\beta := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n \text{ diverges})\} & \text{the subseries number} \\ &\beta_o := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n \text{ diverges by oscillation})\} \\ &\beta_i := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n = \pm \infty)\} \end{split}$$

Theorem 10

 $\beta_o \leq non(meager)$

Theorem 11

 $cov(null) \leq \beta$

Jörg Brendle Rearrangements and Subseries

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Subseries numbers

$$\begin{split} &\beta := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n \text{ diverges})\} & \text{the subseries number} \\ &\beta_o := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n \text{ diverges by oscillation})\} \\ &\beta_i := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n = \pm \infty)\} \end{split}$$

Theorem 10

 $\beta_o \leq non(meager)$

Theorem 11

 $cov(null) \leq \beta$

Proofs: like for \mathfrak{rr}

Jörg Brendle Rearrangements and Subseries

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$$\begin{array}{l} \beta := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \; \exists X \in \mathcal{F} \\ (\sum_{n \in X} a_n \; \text{diverges})\} & \text{the subseries number} \end{array}$$

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ß versus s

$$\begin{split} &\beta := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall \text{ c.c. } \sum a_n \exists X \in \mathcal{F} \\ & (\sum_{n \in X} a_n \text{ diverges})\} & \text{the subseries number} \\ &\mathfrak{s} := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall Y \in [\omega]^{\omega} \exists X \in \mathcal{F} \\ & (|X \cap Y| = |Y \setminus X| = \omega)\} & \text{the splitting number} \end{split}$$

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ß versus s

$$eta := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} ext{ and } orall ext{ c.c. } \sum a_n \exists X \in \mathcal{F} \ (\sum_{n \in X} a_n ext{ diverges})\} ext{ the subseries number}$$

$$\mathfrak{s} := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ and } \forall Y \in [\omega]^{\omega} \exists X \in \mathcal{F} \\ (|X \cap Y| = |Y \setminus X| = \omega)\} \text{ the splitting number}$$

Theorem 12

 $\mathfrak{s} \leq \beta$

Jörg Brendle Rearrangements and Subseries

Theorem 12 $\mathfrak{s} \leq \mathfrak{B}$

<u>Proof:</u> $\mathcal{F} \subseteq \omega^{\omega}$, $|\mathcal{F}| < \mathfrak{s}$.

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Theorem 12 $\mathfrak{s} \leq \mathfrak{B}$

<u>Proof:</u> $\mathcal{F} \subseteq \omega^{\omega}$, $|\mathcal{F}| < \mathfrak{s}$. $\exists I = \{i_n : n \in \omega\} \subseteq \omega$ unsplit by members of \mathcal{F} .

Theorem 12 $\mathfrak{s} \leq \mathfrak{b}$

<u>Proof:</u> $\mathcal{F} \subseteq \omega^{\omega}$, $|\mathcal{F}| < \mathfrak{s}$. $\exists I = \{i_n : n \in \omega\} \subseteq \omega$ unsplit by members of \mathcal{F} . $\sum a_n$ c.c. given.

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 $\mathfrak{s} \leq \beta$

<u>Proof:</u> $\mathcal{F} \subseteq \omega^{\omega}$, $|\mathcal{F}| < \mathfrak{s}$. $\exists I = \{i_n : n \in \omega\} \subseteq \omega$ unsplit by members of \mathcal{F} . $\sum a_n$ c.c. given. Define

$$b_k = \begin{cases} a_n & \text{if } k = i_n \\ 0 & \text{otherwise} \end{cases}$$

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 $\mathfrak{s} \leq \mathfrak{b}$

<u>Proof:</u> $\mathcal{F} \subseteq \omega^{\omega}$, $|\mathcal{F}| < \mathfrak{s}$. $\exists I = \{i_n : n \in \omega\} \subseteq \omega$ unsplit by members of \mathcal{F} . $\sum a_n$ c.c. given. Define

$$b_k = \left\{egin{array}{cc} a_n & ext{if} \ k = i_n \ 0 & ext{otherwise} \end{array}
ight.$$

Then $\sum b_k = \sum a_n$ c.c.

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 $\mathfrak{s} \leq \mathfrak{b}$

<u>Proof:</u> $\mathcal{F} \subseteq \omega^{\omega}$, $|\mathcal{F}| < \mathfrak{s}$. $\exists I = \{i_n : n \in \omega\} \subseteq \omega$ unsplit by members of \mathcal{F} . $\sum a_n$ c.c. given. Define

$$b_k = \begin{cases} a_n & \text{if } k = i_n \\ 0 & \text{otherwise} \end{cases}$$

Then $\sum b_k = \sum a_n$ c.c. If $X \in \mathcal{F}$

• either $X \cap I$ finite and $\sum_{k \in X} b_k$ finite

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 $\mathfrak{s} \leq \mathfrak{b}$

<u>Proof:</u> $\mathcal{F} \subseteq \omega^{\omega}$, $|\mathcal{F}| < \mathfrak{s}$. $\exists I = \{i_n : n \in \omega\} \subseteq \omega$ unsplit by members of \mathcal{F} . $\sum a_n$ c.c. given. Define

$$b_k = \begin{cases} a_n & \text{if } k = i_n \\ 0 & \text{otherwise} \end{cases}$$

Then $\sum b_k = \sum a_n$ c.c. If $X \in \mathcal{F}$

• either $X \cap I$ finite and $\sum_{k \in X} b_k$ finite

• or $I \subseteq^* X$ and $\sum_{k \in X} b_k$ converges because $\sum b_k$ does.

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 $\mathfrak b$ lower bound of $\mathfrak r\mathfrak r$ (Theorem 3).

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 \mathfrak{b} lower bound of \mathfrak{rr} (Theorem 3).

Is \mathfrak{b} also lower bound of \mathfrak{B} ?

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 \mathfrak{b} lower bound of \mathfrak{rr} (Theorem 3).

Is \mathfrak{b} also lower bound of β ?

 $\overline{l} = (l_n : n \in \omega)$ sequence of finite subsets of ω with $\max(l_n) < \min(l_{n+1})$.

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 \mathfrak{b} lower bound of \mathfrak{rr} (Theorem 3).

Is \mathfrak{b} also lower bound of β ?

 $\overline{I} = (I_n : n \in \omega)$ sequence of finite subsets of ω with max $(I_n) < \min(I_{n+1})$. $B_n \subseteq I_n, \ C_n = I_n \setminus B_n, \ \overline{B} = (B_n : n \in \omega), \ \overline{C} = (C_n : n \in \omega).$

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 \mathfrak{b} lower bound of \mathfrak{rr} (Theorem 3).

Is \mathfrak{b} also lower bound of \mathfrak{B} ?

 $\bar{l} = (l_n : n \in \omega)$ sequence of finite subsets of ω with $\max(l_n) < \min(l_{n+1})$. $B_n \subseteq l_n, \ C_n = l_n \setminus B_n, \ \bar{B} = (B_n : n \in \omega), \ \bar{C} = (C_n : n \in \omega)$. $\bar{a} = (a_k : k \in \omega)$ sequence of reals with $0 \le a_k \le 1$.

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 \mathfrak{b} lower bound of \mathfrak{rr} (Theorem 3).

Is \mathfrak{b} also lower bound of \mathfrak{B} ?

$$\begin{split} \bar{I} &= (I_n : n \in \omega) \text{ sequence of finite subsets of } \omega \text{ with } \\ \max(I_n) &< \min(I_{n+1}). \\ B_n &\subseteq I_n, \ C_n &= I_n \setminus B_n, \ \bar{B} &= (B_n : n \in \omega), \ \bar{C} &= (C_n : n \in \omega). \\ \bar{a} &= (a_k : k \in \omega) \text{ sequence of reals with } 0 \leq a_k \leq 1. \\ r,s \geq 0. \ (\bar{I}, \bar{B}, \bar{a}) \text{ is } (r,s) \text{-sequence if, letting } b_n &= \sum_{k \in B_n} a_k \text{ and } \\ c_n &= \sum_{k \in C_n} a_k, \text{ we have that } \lim b_n &= r \text{ and } \lim c_n = s. \end{split}$$

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 $\bar{I} = (I_n : n \in \omega) \text{ sequence of finite subsets of } \omega \text{ with } \max(I_n) < \min(I_{n+1}).$ $B_n \subseteq I_n, \ C_n = I_n \setminus B_n, \ \bar{B} = (B_n : n \in \omega), \ \bar{C} = (C_n : n \in \omega).$ $\bar{a} = (a_k : k \in \omega) \text{ sequence of reals with } 0 \le a_k \le 1.$ $r, s \ge 0. \ (\bar{I}, \bar{B}, \bar{a}) \text{ is } (r, s) \text{-sequence if, letting } b_n = \sum_{k \in B_n} a_k \text{ and } c_n = \sum_{k \in C_n} a_k, \text{ we have that } \lim b_n = r \text{ and } \lim c_n = s.$ $D \in [\omega]^{\omega} \text{ almost splits } (r, s) \text{-sequence } (\bar{I}, \bar{B}, \bar{a}) \text{ if } \exists E \subseteq \omega \text{ s.t.}$

$$\lim_{n\in E}\left(\sum_{k\in B_n\cap D}a_k\right)=r \quad \text{and} \quad \lim_{n\in E}\left(\sum_{k\in C_n\cap D}a_k\right)=0.$$

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 $\overline{I} = (I_n : n \in \omega)$ sequence of finite subsets of ω with $\max(I_n) < \min(I_{n+1}).$ $B_n \subseteq I_n, \ C_n = I_n \setminus B_n, \ \overline{B} = (B_n : n \in \omega), \ \overline{C} = (C_n : n \in \omega).$ $\bar{a} = (a_k : k \in \omega)$ sequence of reals with $0 < a_k < 1$. $r, s \geq 0$. (I, B, \bar{a}) is (r, s)-sequence if, letting $b_n = \sum_{k \in B_n} a_k$ and $c_n = \sum_{k \in C_n} a_k$, we have that $\lim b_n = r$ and $\lim c_n = s$. $D \in [\omega]^{\omega}$ almost splits (r, s)-sequence $(\overline{I}, \overline{B}, \overline{a})$ if $\exists E \subseteq \omega$ s.t. $\lim_{n \in E} \left(\sum_{k \in R \cap D} a_k \right) = r \quad \text{and} \quad \lim_{n \in E} \left(\sum_{k \in C \cap D} a_k \right) = 0.$

 $\mathfrak{s}_{\mathsf{almost}} := \min\{|\mathcal{D}| : \mathcal{D} \subseteq [\omega]^{\omega} \text{ and } \forall (r^m, s^m) \text{-sequences} \\ (\bar{I}^m, \bar{B}^m, \bar{a}^m) \exists D \in \mathcal{D} \ (D \text{ almost splits all } (\bar{I}^m, \bar{B}^m, \bar{a}^m))\}$

 $\mathfrak{s}_{\mathsf{almost}} := \min\{ |\mathcal{D}| : \mathcal{D} \subseteq [\omega]^{\omega} \text{ and } \forall (r^m, s^m) \text{-sequences} \\ (\bar{I}^m, \bar{B}^m, \bar{a}^m) \exists D \in \mathcal{D} \ (D \text{ almost splits all } (\bar{I}^m, \bar{B}^m, \bar{a}^m)) \}$

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 $\mathfrak{s}_{\mathsf{almost}} := \min\{|\mathcal{D}| : \mathcal{D} \subseteq [\omega]^{\omega} \text{ and } \forall (r^m, s^m) \text{-sequences} \\ (\bar{I}^m, \bar{B}^m, \bar{a}^m) \exists D \in \mathcal{D} \ (D \text{ almost splits all } (\bar{I}^m, \bar{B}^m, \bar{a}^m)) \}$

<u>Fact:</u> $\beta \leq \mathfrak{s}_{\text{almost}}$ (even $\beta_o \leq \mathfrak{s}_{\text{almost}}$).

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<u>Fact:</u> $\beta \leq \mathfrak{s}_{almost}$ (even $\beta_o \leq \mathfrak{s}_{almost}$).

<u>Proof</u>: $\mathcal{D} \subseteq [\omega]^{\omega}$ almost splitting.

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 $\mathfrak{s}_{\mathsf{almost}} := \min\{ |\mathcal{D}| : \mathcal{D} \subseteq [\omega]^{\omega} \text{ and } \forall (r^m, s^m) \text{-sequences} \\ (\bar{I}^m, \bar{B}^m, \bar{a}^m) \exists \mathcal{D} \in \mathcal{D} \ (\mathcal{D} \text{ almost splits all } (\bar{I}^m, \bar{B}^m, \bar{a}^m)) \}$

<u>Fact</u>: $\beta \leq \mathfrak{s}_{\text{almost}}$ (even $\beta_o \leq \mathfrak{s}_{\text{almost}}$).

 $\frac{\text{Proof:}}{\sum_{k} x_{k}} \stackrel{\mathcal{D}}{\subset} [\omega]^{\omega} \text{ almost splitting.}$ $\sum_{k} x_{k} \text{ c.c. } a_{k} = |x_{k}|.$

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 $\mathfrak{s}_{\mathsf{almost}} := \min\{|\mathcal{D}| : \mathcal{D} \subseteq [\omega]^{\omega} \text{ and } \forall (r^m, s^m) \text{-sequences} \\ (\bar{I}^m, \bar{B}^m, \bar{a}^m) \exists \mathcal{D} \in \mathcal{D} \ (\mathcal{D} \text{ almost splits all } (\bar{I}^m, \bar{B}^m, \bar{a}^m)) \}$

<u>Fact:</u> $\beta \leq \mathfrak{s}_{\text{almost}}$ (even $\beta_o \leq \mathfrak{s}_{\text{almost}}$).

 $\begin{array}{l} \underline{\text{Proof:}} \ \mathcal{D} \subseteq [\omega]^{\omega} \text{ almost splitting.} \\ \sum_{k} x_{k} \text{ c.c. } a_{k} = |x_{k}|. \\ P = \{k \in \omega : x_{k} \geq 0\}, \ N = \{k \in \omega : x_{k} < 0\}. \end{array}$

 $\mathfrak{s}_{\text{almost}} := \min\{|\mathcal{D}| : \mathcal{D} \subseteq [\omega]^{\omega} \text{ and } \forall (r^m, s^m) \text{-sequences} \}$ $(\bar{I}^m, \bar{B}^m, \bar{a}^m) \exists D \in \mathcal{D} (D \text{ almost splits all } (\bar{I}^m, \bar{B}^m, \bar{a}^m))\}$ <u>Fact:</u> $\beta \leq \mathfrak{s}_{\text{almost}}$ (even $\beta_o \leq \mathfrak{s}_{\text{almost}}$). Proof: $\mathcal{D} \subseteq [\omega]^{\omega}$ almost splitting. $\sum_{k} x_k$ c.c. $a_k = |x_k|$. $P = \{k \in \omega : x_k > 0\}, N = \{k \in \omega : x_k < 0\}.$ $\sum_{k} x_k$ c.c. $\Longrightarrow \exists \overline{I} = (I_n : n \in \omega)$ intervals with $\max(I_n) < \min(I_{n+1})$ s.t., letting $B_n = I_n \cap P$, $C_n = I_n \cap N$, $b_n = \sum_{k \in B_n} a_k$, and $c_n = \sum_{k \in C_n} a_k$, we have $\lim b_n = \lim c_n = 1$.

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 $\mathfrak{s}_{\text{almost}} := \min\{|\mathcal{D}| : \mathcal{D} \subseteq [\omega]^{\omega} \text{ and } \forall (r^m, s^m) \text{-sequences} \}$ $(\bar{I}^m, \bar{B}^m, \bar{a}^m) \exists D \in \mathcal{D} (D \text{ almost splits all } (\bar{I}^m, \bar{B}^m, \bar{a}^m))\}$ Fact: $\beta < \mathfrak{s}_{\text{almost}}$ (even $\beta_{\rho} < \mathfrak{s}_{\text{almost}}$). Proof: $\mathcal{D} \subseteq [\omega]^{\omega}$ almost splitting. $\sum_{k} x_k$ c.c. $a_k = |x_k|$. $P = \{k \in \omega : x_k \ge 0\}, N = \{k \in \omega : x_k < 0\}.$ $\sum_{k} x_k$ c.c. $\Longrightarrow \exists \overline{I} = (I_n : n \in \omega)$ intervals with $\max(I_n) < \min(I_{n+1})$ s.t., letting $B_n = I_n \cap P$, $C_n = I_n \cap N$, $b_n = \sum_{k \in B_n} a_k$, and $c_n = \sum_{k \in C_n} a_k$, we have $\lim b_n = \lim c_n = 1$. Thus, $(\overline{I}, \overline{B}, \overline{a})$ is (1, 1)-sequence.

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 $\mathfrak{s}_{\mathsf{almost}} := \min\{|\mathcal{D}| : \mathcal{D} \subseteq [\omega]^{\omega} \text{ and } \forall (r^m, s^m) \text{-sequences}\}$ $(\bar{I}^m, \bar{B}^m, \bar{a}^m) \exists D \in \mathcal{D} (D \text{ almost splits all } (\bar{I}^m, \bar{B}^m, \bar{a}^m))\}$ <u>Fact:</u> $\beta \leq \mathfrak{s}_{\text{almost}}$ (even $\beta_o \leq \mathfrak{s}_{\text{almost}}$). Proof: $\mathcal{D} \subseteq [\omega]^{\omega}$ almost splitting. $\sum_{k} x_k$ c.c. $a_k = |x_k|$. $P = \{k \in \omega : x_k > 0\}, N = \{k \in \omega : x_k < 0\}.$ $\sum_{k} x_k$ c.c. $\Longrightarrow \exists \overline{I} = (I_n : n \in \omega)$ intervals with $\max(I_n) < \min(I_{n+1})$ s.t., letting $B_n = I_n \cap P$, $C_n = I_n \cap N$, $b_n = \sum_{k \in B_n} a_k$, and $c_n = \sum_{k \in C_n} a_k$, we have $\lim b_n = \lim c_n = 1$. Thus, $(\overline{I}, \overline{B}, \overline{a})$ is (1, 1)-sequence. $D \in \mathcal{D}$ almost splits $(\overline{I}, \overline{B}, \overline{a}) \Longrightarrow \sum_{k \in D} x_k$ diverges.

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 $\mathfrak{s}_{\mathsf{almost}} = \omega_1$ in Laver model.

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 $\mathfrak{s}_{\mathsf{almost}} = \omega_1$ in Laver model.

 $\label{eq:corollary 14} CON(\beta < \mathfrak{b}); \mbox{ so also } CON(\beta < \mathfrak{rr})$

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 $\mathfrak{s}_{\mathsf{almost}} = \omega_1$ in Laver model.

Corollary 14 $CON(\beta < \mathfrak{b})$; so also $CON(\beta < \mathfrak{rr})$

Question 2

 $CON(\mathfrak{rr} < \beta)$? Even $CON(\mathfrak{rr} < \mathfrak{s})$?

Question 3

 $\beta = \beta_o?$

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Question	3

 $\beta = \beta_o?$

Theorem 15

 $\beta_i > cov(meager)$

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$$\beta = \beta_o?$$

 $\beta_i > cov(meager)$

Question 4

 $CON(\beta_i < \mathfrak{c})?$

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