On Singular Stationarity

Omer Ben-Neria
University of California, Los Angeles

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Introduction

We discuss consistency results related to two notions of singular stationarity

1. Mutually stationary sets
2. Tightly stationary sets

These notions were introduced by Foreman and Magidor in their work on the non-saturation of generalized nonstationary ideals.

Conventions

- $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of regular cardinals with limit $\kappa_\omega = \cup_n \kappa_n$
- We focus on fixed-cofinality stationary sequences- $\vec{S} = \langle S_n \rangle_n$ of stationary sets $S_n \subseteq \kappa_n \cap \mathrm{Cof}(\mu)$ for some fixed regular cardinal μ
- An algebra $\mathfrak{A}$ is an expansion of $\langle H_\theta, \in, <_\theta \rangle$ by countably many finitary functions. $\kappa_\omega << \theta$ is regular
- A subalgebra or substructure of $\mathfrak{A}$ is an elementary substructure $M \prec \mathfrak{A}$

Definition

A sequence of stationary sets $\langle S_n \rangle_n$ is **mutually stationary** (MS) if for every algebra $\mathfrak{A}$ there exists $M \prec \mathfrak{A}$ such that $\sup(M \cap \kappa_n) \in S_n$ for all but finitely many $n < \omega$

- **(Foreman and Magidor)** For every $\vec{\kappa}$ and every stationary sequence $\vec{S}$ of $S_n \subseteq \kappa_n \cap$ $\mathrm{Cof}(\omega)$, $\vec{S}$ is MS

Different behavior in uncountable cofinality

- **(Foreman Magidor)** In L, there is a sequence $\vec{S}$ of stationary sets $S_n \subseteq \omega_n \cap$ $\mathrm{Cof}(\omega_1)$ which is not MS
- **(Schindler)** Similar non-MS examples exist in other canonical extender models $L[E]$
- **(Koepke and Welch)** If every $\vec{S}$ with $S_n \subseteq \omega_n \cap$ $\mathrm{Cof}(\omega_1)$ is MS then there is an inner model with stationarily measurable cardinals $\alpha < \omega_n$ of Mitchell order $o(\alpha) \geqslant \omega_{n-2}$.

Q:*Is it provable (in ZFC) that there exists a sequence of stationary sets $S_n \subseteq \omega_n$ of some fixed cofinality which is not MS?*

Theorem

It is consistent relative to the existence of infinitely many supercompact cardinals that every sequence of fixed-cofinality stationary sets $S_n \subseteq \omega_n$ is MS

- **(Cummings-Foreman-Magidor)** If $\vec{\kappa}$ is a Prikry generic sequence then eqvery stationary sequence $\vec{S}$ is MS
- **(Koepke)** It is possible to have $\kappa_n = \omega_{2n+1}$, in a model where every sequence of $S_n \subseteq \kappa_n \cap \mathrm{Cof}(\omega_1)$ is MS

Outline of proof (cofinality $\mu = \omega_2$)

We will work in a model of GCH.
Given a sequence $\langle S_n \rangle_n$ of stationary sets $S_n \subseteq \omega_n \cap \mathrm{Cof}(\omega_2)$ and an algebra $\mathfrak{A}$, we would like to construct $M \prec \mathfrak{A}$ which meets S_n for every $n \geqslant 4$.

Split the argument into three parts:

I The strategy (building M in ω-many steps)

II Zoom in on a step (ideal-based construction)

III Utilizing supercompactness

Part I - The strategy

- Given an algebra $\mathfrak{A}$, we start from $N \prec \mathfrak{A}$ with $|N| = \aleph_2$, $\vec{S} \in N$, and ${}^{<\omega_2}N \subseteq N$ (we say N is $< \omega_2$-closed), and build an ω-sequence of extensions

$$N = M_3 \prec M_4 \prec \ldots \prec M_n \prec \ldots \prec \mathfrak{A}$$

  such that
  1. $\sup(M_n \cap \omega_n) \in S_n$ (M_n meets S_n)
  2. $M_n \cap \omega_{n-1} = M_{n-1} \cap \omega_{n-1}$ (end extension property)
  3. $|M_n| = \aleph_2$ and is $< \omega_2$-closed
- $M_\omega = \cup_n M_n \prec \mathfrak{A}$ satisfies the desirable properties

Part II - Zoom in on a single step $M_{n-1} \mapsto M_n$

- Let $\kappa = \omega_n$, $\lambda = \omega_{n-1}$, $M_{n-1} = N$. We would like to construct $N^+ \prec \mathfrak{A}$ which meets $S = S_n$ and end extends N above λ
- Let I be a κ-complete ideal on κ and $\leqslant_I$ be the order on $I^+ = \mathcal{P}(\kappa) \setminus I$ defined by $A \leqslant_I B$ iff $A \setminus B \in I$.
- We say the ideal I is $(\omega_2 + 1)$-closed if I^+ has a dense subset D such that $\leqslant_I \restriction D$ is a $(\omega_2 + 1)$-closed poset
- Lemma: If $I \in N$ is a $(\omega_2 + 1)$-closed ideal with $S \in I^+$ then there is a $\subseteq$-decreasing sequence $\langle A_i \mid i \leqslant \omega_2 \rangle$ of I^+-subsets of S, such that $A_i \in N$ for all $i < \omega_2$, and for every $f : \kappa \to \lambda$ in N, $f \restriction A_i$ is constant for some $i < \omega_2$.

- Let $\bar{A} = A_{\omega_2} \subseteq S$. For every $\alpha \in \bar{A}$ and $f \in {}^{\kappa}\lambda \cap N$, $f(\alpha) \in N$.
  Consequently,
  $\overline{N} = \mathrm{SK}^{\mathfrak{A}}(N \cup \{\alpha\}) = \{f(\alpha) \mid f \in N, \mathrm{dom}(f) = \kappa\}$ is an end extension of N above λ. However, we want $\sup(\overline{N} \cap \kappa) = \alpha \in S$, but $\alpha \in \overline{N}$
- Let $\delta \in N$ be a ladder system- $\delta(\nu) = \langle \delta(\nu)_i \mid i < \omega_2 \rangle$ is cofinal in ν, for every $\nu \in \kappa \cap \mathrm{Cof}(\omega_2)$
- $$\text{Define } N^+ = \mathrm{SK}^{\mathfrak{A}}(N \cup \{\delta(\alpha) \restriction i \mid i < \omega_2\}).$$
- $N^+ \subseteq \overline{N}$ since $N \cup \{\delta, \alpha\} \subseteq \overline{N}$. Therefore, N^+ end extends N above λ.
  N^+ has size $|N^+| = \aleph_2$ and is $< \omega_2$-closed

Is $\sup(N^+ \cap \kappa) = \alpha$?

- $\sup(N^+ \cap \kappa) \geqslant \alpha$ since $\delta(\alpha) \subseteq N^+$
- $\sup(N^+ \cap \kappa) \leqslant \alpha$ if α belongs to every club $C \subseteq \kappa$ in N
  Because every $\gamma \in N^+ \cap \kappa$ is of the form $\gamma = f(\delta(\alpha) \restriction j)$ for some $j < \omega_2$ and $f \in N$, and N contains the club C of closure points of f
- To find such α in $\bar{A} = A_{\omega_2}$ it suffices for $\bar{A} \in I^+$ to be stationary
- Corollary: If $S \in I^+$ for some ideal I which is $(\omega_2 + 1)$-closed and nonstationary then N has an end extension N^+ above λ with $\sup(N^+ \cap \kappa) \in S$

Part III - Utilizing supercopmactness

- Suppose that $\langle \kappa_n \mid 1 \leqslant n < \omega \rangle$ is an increasing sequence of supercompact cardinals (also set $\kappa_0 = \omega$)
- Force with $\mathbb{P}$, a full support iteration of Levy collapse posets, $\mathrm{Coll}(\kappa_n, < \kappa_{n+1})$, $n < \omega$
- Fix $n \geqslant 4$ and let $j : V \to M$ be a κ_ω^+-supercompact embedding with $\mathrm{cp}(j) = \kappa_n$, in V
- It is well-known that the quotient $j(\mathbb{P})/\mathbb{P}$ is $(< \kappa_{n-1})$-closed, and in particular $(\omega_2 + 1)$-closed in $V[G]$. Moreover, the quotient $j(\mathbb{P})/\mathbb{P}$ has a master condition g (i.e., g extends $j(p)$ for every $p \in G$)

- $\mathbb{P}$ adds κ_{n+1} new subsets to κ_n. Let $\vec{C} = \langle C_i \mid i < \kappa_{n+1} \rangle$ be sequence of $\mathbb{P}$-names in V which covers all possible clubs of κ_n in $V[G]$
- $C^* = \bigcap j``\vec{C}$ is in M, and $g \Vdash_{j(\mathbb{P})/\mathbb{P}} C^* \subseteq j(\kappa_n)$ is club.
- For every $\mathbb{P}$-name $\dot{S}$ of a stationary subset of κ_n,

$$g \Vdash_{j(\mathbb{P})/\mathbb{P}} (j(\dot{S}) \cap C^*) \text{ is stationary}$$

- We can find $\gamma < j(\kappa_n)$ and a condition r which extends g, such that $r \Vdash \gamma \in C^* \cap j(\dot{S})$
- Define in $V[G]$

$$I_{\gamma,r} = \{X \subseteq \kappa_n \mid r \Vdash \gamma \notin j(\dot{X})\}$$

- $I_{\gamma,r}$ is nonstationary, $(\omega_2 + 1)$-closed, and its dual filter contains S $\square$

Tight Stationarity

- Tight stationarity is a strengthening of mutual stationarity which satisfies versions of Solovay's Splitting Lemma and Fodor's Theorem
- The idea is to restrict the substructures $M \prec \mathfrak{A}$ to a family of "nicely behaved" structures

Definition

1. A substructure $M \prec \mathfrak{A}$ is tight for $\vec{\kappa} = \langle \kappa_n \rangle_n$ if for every function $f \in \prod_n (M \cap \kappa_n)$ there is some $g \in M \cap \prod_n \kappa_n$ such that $f(n) < g(n)$ for all $n < \omega$
2. A sequences $\langle S_n \rangle_n$ is tightly stationary if for every algebra $\mathfrak{A}$ there is a tight $M \prec \mathfrak{A}$ such that $\sup(M \cap \kappa_n) \in S_n$ for all $n < \omega$

- Foreman and Magidor raised the question of whether every mutually stationary sequence is tightly stationary
- Cummings-Foreman-Magidor, Foreman-Steprans, and Chen-Neeman constructed different models which contain mutually stationary sequences which are not tight
- Unfortunately, the substructure $M_\omega = \cup_n M_n$, from the last proof, is not

Theorem

If $\langle \kappa_n \rangle_n$ is an increasing sequence of 1-extendible cardinals then every sequence $\langle S_n \rangle_n$ of fixed-cofinality stationary sets $S_n \subseteq \kappa_n$ is tightly stationary in a generic extension

Theorem

It is consistent relative to the existence of a sequence $\langle \kappa_n \mid n < \omega \rangle$ of cardinals κ_n which are κ_n^{+n+3}-strong that there is a model with a subset $\{\omega_{s_n}\rangle_{n<\omega}$ of the ω_n's such that very fixed-cofinality sequence $\langle S_n \rangle_n$ is tightly stationary in a generic extension

- The theorems build on the results of Cummings, Foreman, Magidor, and Shelah which connect the existence of tight structures with certain properties with the existence of certain scales
- Using variations of Gitik's short-extenders forcing it is possible to add scales with desirable properties
  Specifically, we add a scale $\vec{f} = \langle f_\alpha \mid \alpha < \kappa_\omega^{++} \rangle$ such that there are stationarily many approachable and continuous ordinals $\delta < \kappa_\omega^{++}$ for which $f_\delta(n) \in S_n$ for almost all n

Thank You

preprints:

1. On Singular Stationarity I (mutual stationarity and ideal-based methods)
2. On Singular Stationarity II (tight stationarity and extenders-based methods)